WEAKLY COMPACT OPERATORS ON OPERATOR ALGEBRAS

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Let K be a compact space and $C(K)$ be the commutative B^*-algebra of all complex valued continuous functions on K, then Grothendieck [3] (also we can see other proofs in [2]) proved the following remarkable properties:

(I) An arbitrary bounded operator of $C(K)$ into a weakly sequentially complete Banach space is weakly compact.

(II) If T is a weakly compact operator of $C(K)$ into a Banach space, then T maps weakly fundamental sequences into strongly convergent sequences.

On the other hand, let M be a W^*-algebra and M_* be the associated space of M (namely, the dual of M_* is M (cf. [8])) then the author [7] noticed that the Banach space M_* is weakly sequentially complete. Therefore, the above Grothendieck's theorems are applicable in the theory of operator algebras.

In this note, we shall show some applications, and state some related problems.

PROPOSITION 1. Let A be a B^*-algebra, E an abstract L-space, T be a bounded operator of A into E, then T is weakly compact.

Proof. Let T^* be the dual of T, then T^* is a bounded operator on the dual E^* of E to the dual A^* of A; E^* is a Banach space of type $C(K)$ (cf. [5]) and the second dual A^{**} of A is a W^*-algebra (cf. [9]), so that A^* is the associated space of a W^*-algebra; hence A^* is weakly sequentially complete; therefore T^* is weakly compact, so that by the well-known theorem, T is weakly compact. This completes the proof.

Now we shall show some applications.

1. Let G be a locally compact group, $L^1(G)$ be the Banach space of all complex valued integrable functions on G with respect to a left, invariant Haar measure μ and $L^2(G)$ be the Banach space of all complex valued square integrable functions on G with respect to μ. Under the convolutions (denoted by "$*$"), $L^1(G)$ is a Banach algebra.

On the other hand, for $f \in L^1(G)$ and $g \in L^2(G)$, put $L_f g = f * g$,

Received March 18, 1963. This research was partially supported by the National Science Foundation Grant 19041.

then L_f is a bounded operator on $L^2(G)$; we shall denote the uniform norm of L_f by $\| L_f \|$.

Now, let T be an operator on $L^1(G)$. T is said to be spectrally continuous, if it satisfies $\| Th \|_1 \leqq r \| L_h \|$ for all $h \in L^1(G)$, where $\|\cdot\|_1$ is the L^1-norm and r is a fixed number.

Using the generalized Planchrel's theorem and the structure theorem of connected locally compact groups, Helgason [4] proved the following: Let G be a separable unimodular locally compact, non-compact, connected group, then a spectrally continuous operator on $L^1(G)$ commuting with all right translations is identically 0.

In his review for the Helgason's paper, Mautner [6] asked whether these restrictions on the group G can be dropped.

Now we shall show

THEOREM 1. *Let G be a locally compact, non-compact group, then a spectrally continuous operator T on $L^1(G)$ commuting with all right translations is identically zero.*

Proof. Let $R(G)$ be the uniform closure of the set $\{L_f \mid f \in L^1(G)\}$ in the B^*-algebra B of all bounded operators on $L^2(G)$, then $R(G)$ is a B^*-algebra; since T is spectrally continuous, it can be uniquely extended to a bounded operator $\tilde{T}$ of $R(G)$ into $L^1(G)$; by Proposition 1, $\tilde{T}$ is weakly compact; let S be the unit sphere of $R(G)$; since $\| L_f \| \leqq \| f \|_1$ for $f \in L^1(G)$, $L^1(G) \cap S$ contains the unit sphere of $L^1(G)$; therefore the set $\{Th \mid h \in L^1(G),\ \| h \|_1 \leqq 1\}$ is relatively weakly compact in $L^1(G)$; this implies that T is weakly compact as an operator on $L^1(G)$.

Since T commutes with all right translations, by the theorem of Wendel [10], there is a bounded Radon measure ν such that $Th = \nu * h$ for $h \in L^1(G)$; let f be an element of $L^1(G)$, then the mapping $h \to (f*\nu)^* * (f*\nu) * h$ on $L^1(G)$ is weakly compact, where $(f*\nu)^*(x) = \rho(x)\overline{f*\nu(x^{-1})}$, and $d\mu(x^{-1}) = \rho(x)d\mu(x)$ for $x \in G$; hence the mapping $h \to \{(f*\nu)^* * (f*\nu)\} * \{(f*\nu)^* * (f*\nu)\} * h$ is strongly compact (cf. Cor 3.7 in [2]).

Put $g = \{(f*\nu)^* * (f*\nu)\} * \{(f*\nu)^* * (f*\nu)\}$, then g belongs to $L^1(G)$. Let S_1 be the unit sphere of $L^1(G)$, then $g * S_1$ is relatively strongly compact in $L^1(G)$, so that the set $\{(g*f)^* = f^* * g^* \mid f \in S_1\}$ is also so; hence $S_1 * g^*$ is relatively strongly compact; let $\{v_\alpha\}_{\alpha \in \Pi}$ be a fundamental family of compact neighborhoods at a point s of G and let $\{f_\alpha\}_{\alpha \in \Pi}$ be a family of continuous positive functions on G such that the support of f_α is contained in v_α and $\int_G f_\alpha(x)dx = 1$, then the directed set $\{f_\alpha * g^*\}$ converges to sg^* in the L^1-norm, where $sg^*(x) = g^*(s^{-1}x)$; therefore the set $\{sg^* \mid s \in G\}$ is relatively strongly compact.

Now suppose that $\| g^* \|_1 \neq 0$, then it is enough to assume that

$\| g^* \|_1 = 1$. There is a finite set $\{s_1 g^*, s_2 g^*, \cdots, s_n g^*\}$ where $s_i \in G$ $(i = 1, 2, \cdots, n)$ such that $\inf_{1 \leqq i \leqq n} \| sg^* - s_i g^* \|_1 < 1/2$ for all $s \in G$.

On the other hand, let C be a compact subset of G such that

$$\int_{G-C} | g^*(x) | \, d\mu(x) < \frac{1}{10} \quad \text{and} \quad \int_{G-C} | g^*(s_i^{-1}x) | \, d\mu(x) < \frac{1}{10}$$

for $i = 1, 2, \cdots, n$, and s be an element of G such that $s \notin CC^{-1}$, then $s^{-1}C \cap C = (\phi)$; therefore

$$\begin{aligned} &\| sg^* - s_i g^* \|_1 \\ &\quad = \int_C | (sg^* - s_i g)(x) | \, d\mu(x) + \int_{G-C} | (sg^* - s_i g^*)(x) | \, d\mu(x) \\ &\quad \geqq \int_C | (s_i g^*)(x) | \, d\mu(x) - \int_C | (sg^*)(x) | \, d\mu(x) \\ &\qquad + \int_{G-C} | (sg^*)(x) | \, d\mu(x) - \int_{G-C} | (s_i g^*)(x) | \, d\mu(x) \\ &\quad \geqq \left(1 - \frac{1}{10}\right) - \int_{s^{-1}C} | g^*(x) | \, d\mu(x) + \left(1 - \frac{1}{10}\right) - \frac{1}{10} \\ &\quad \geqq \left(1 - \frac{1}{10}\right) - \frac{1}{10} + \left(1 - \frac{1}{10}\right) - \frac{1}{10} = \frac{8}{5} \qquad \text{for all } i . \end{aligned}$$

This is a contradiction; hence $g^* = 0$, so that $g = f * \nu = 0$; since f is an arbitrary element of $L^1(G)$, $\nu = 0$, so that $T = 0$. This completes the proof.

2. At first we shall show

PROPOSITION 2. Let A be a weakly sequentially complete B^*-algebra, then A is finite dimensional.

Proof. It is enough to assume that A has unit. Let C be a maximal abelian *-subalgebra of A, then C is a Banach space of type $C(K)$ and weakly sequentially complete; by the Grothendieck's theorem, the identity mapping T on C is weakly compact, so that $T^2 = T$ is strongly compact on C (cf. Cor 3.7 in [2]); hence C is finite-dimensional. Therefore there is a finite family of mutually orthogonal projections $(e_1, e_2, \cdots, e_n)$ by which C is linearly spanned; by the maximality of C, $e_i A e_i$ $(i = 1, 2, \cdots, n)$ is one-dimensional.

For any $x, y \in A$, there is a complex number $\lambda_i(x, y)$ such that $e_i y^* x e_i = \lambda_i(x, y) e_i$; clearly $\lambda_i(x, x) \geqq 0$, and if $\lambda_i(x, x) = 0$, $xe_i = 0$; moreover $\| xe_i \| = \| e_i x^* x e_i \|^{1/2} = \lambda_i(x, x)^{1/2}$; therefore a Banach subspace Ae_i of A is a hilbert space; since $A = \sum_{i=1}^n Ae_i$, A is reflexive, so that A is a reflexive W^*-algebra; since all irreducible *-representations of

A are σ-continuous, A is of type I; since the center of A is finite-dimensional, A is a direct sum of a finite family of type I-factors $(A_1, A_2, \cdots, A_m)$; since A_j can be considered the algebra of all bounded operators on a hilbert space $\mathfrak{h}_j$ for $j = 1, 2, \cdots, m$, the reflexivity of A_j implies the finite-dimensionality of $\mathfrak{h}_j$ and so the finite-dimensionality of A_j; hence A is finite-dimensional. This completes the proof.

COROLLARY 1. *Let A be an infinite dimensional B^*-algebra and E be a Banach space of type L^p $(1 \leqq p < +\infty)$ or the associated space of a W^*-algebra, then the Banach space A is not topologically isomorphic to E.*

Since B^*-algebras are Banach spaces which have many analogous properties with $C(K)$; therefore it is very natural to ask whether the theorems of Grothendieck are positive in B^*-algebras.

We have no solution for the property (I); here we shall show that the property (II) is negative, and show an application.

A negative example. Let $B(\mathfrak{h})$ be the B^*-algebra of all bounded operators on an infinite dimensional hilbert space $\mathfrak{h}$, and e be an one-dimensional projection on $\mathfrak{h}$, then the Banach subspace $B(\mathfrak{h})e$ of $B(\mathfrak{h})$ is isometric to $\mathfrak{h}$ [cf. [7]]; therefore the mapping $x \xrightarrow{T} xe$ of $B(\mathfrak{h})$ into $B(\mathfrak{h})e$ is weakly compact; the unit sphere S of $B(\mathfrak{h})e$ is weakly compact in $B(\mathfrak{h})$; therefore if $B(\mathfrak{h})$ satisfies the property (II), $TS = S$ is strongly compact, so that $B(\mathfrak{h})e$ is finite-dimensional, a contradiction.

Concerning the property (I), we can notice that many operators satisfy the property (I).

For instance, let A be a B^*-algebra, A^* the dual of A. For x, $a \in A$ and $f \in A^*$, put $(L_af)(x) = f(ax)$ and $(R_af)(x) = f(xa)$; we can consider bounded operators $a \xrightarrow{T} R_af$, $a \xrightarrow{S} L_af$ of A into A^*, then T and S are weakly compact (cf. [8]).

Finally we shall show an application.

THEOREM 2. *Let A be a B^*-algebra having an infinite dimensional irreducible $*$-representation, and E be a Banach space of type L^p $(1 \leqq p \leqq +\infty)$ or type $C(\Omega)$, where Ω is a locally compact space and $C(\Omega)$ is the Banach space of all continuous functions vanishing at infinity, or the associated space of a W^*-algebra, then the Banach space A is not topologically isomorphic to E.*

Proof. It is enough to show that A is not topologically isomorphic to $C(\Omega)$. Suppose that A is topologically isomorphic to $C(\Omega)$, then there is an isomorphism T of A onto $C(\Omega)$. Take the second dual T^{**} of T, the T^{**} gives an isomorphism of A^{**} onto $C(\Omega)^{**}$; A^{**} is a W^*-algebra and $C(\Omega)^{**}$ is a Banach space of type $C(K)$; since a $*$-representation of A can be uniquely extended to a σ-continuous $*$-representation of A^{**} (cf. [8]), A^{**} has an infinite dimensional irreducible W^*-representation; hence there is a central projection z of A^{**} such that $A^{**}z$ is a factor of type I_∞; from the above negative example, $A^{**}z$ has not the property (II); on the other hand, since $A^{**} = A^{**}z \oplus A^{**}(1-z)$, where 1 is the unit of A^{**}, $C(\Omega)^{**} = T^{**}(A^{**}z) + T^{**}(A^{**}(1-z))$; since $T^{**}(A^{**}z)$ has the closed complement subspace in $C(\Omega)^{**}$, $T^{**}(A^{**}z)$ has the property (II), so that $A^{**}z$ has the property (II). This is a contradiction, and completes the proof.

COROLLARY 2. *Let F be the associated space of a W^*-algebra without a type I_n part $(n < +\infty)$, and E be a Banach space of type L^p or $C(\Omega)$ then F is not topologically isomorphic to E.*

Proof. Suppose that F is topologically isomorphic to E, then F^* is topologically isomorphic to E^*. This is a contradiction.

REMARK. Theorem 2 and Corollary 2 imply that the above mentioned B^*-algebras or associated spaces (for instance, the B^*-algebra $\mathscr{C}$ of all compact operators on an infinite dimensional hilbert space, the B^*-algebra $B(\mathfrak{h})$ of all bounded operators on an infinite dimensional hilbert space $\mathfrak{h}$, the B^*-algebra $R(G)$ corresponding to all non-almost periodic locally compact groups, and all W^*-factor with an exception of type $I_n (n < +\infty)$ and their associated spaces) are not topologically contained in the classes of the so-called classical Banach spaces $((M), (m), (C), (c), (C^{(p)})_{p\geqq 1}, (L^p)_{1\leqq p\leqq+\infty}, (l^p)_{1\leqq p\leqq +\infty})$ mentioned by Banach (cf. [1]); therefore it is very meaningful to examine whether many unsolved problems concerning Banach spaces are positive in these examples.

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