B^* ALGEBRA UNIT BALL EXTREMAL POINTS

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Results of Kadison [3] and Jacobson [2] are combined to show that the points described by the title are unitaries, left shifts, right shifts, or sums of these. The extremality property is preserved by homomorphisms; conversely, when range and domain are AW^* algebras, every extremal point of the range has an extremal point in its preimage. Exact formulations of these results and of a few simple consequences are given in section one; proofs follow in section two.

In what follows, A will be a self-adjoint subalgebra of some B^* algebra; "x is extremal (A)" will mean that x is an extremal point of the unit ball of A with respect to the B^* norm indicated by the context; "weak topology" will mean the weak operator topology with respect to the representation of A by bounded operators on a Hilbert space which is indicated by the context.

1. Theorems. Our starting point is a formula due to Kadison ([3], Theorem 1). In a mildly generalized form, his result is:

THEOREM 1. *Let A be a self-adjoint subalgebra of some B^* algebra B. Then x is extremal (A) if and only if*

$$(1 - x^*x)A(1 - xx^*) = \{0\} .$$

Here "1" stands for the identity of A if there is one; otherwise the meaning of the equation is to be found by performing the indicated multiplications for each $y \in A$. It turns out (Theorem 2) that the existence of any element extremal (A) implies that A has an identity.[1]

An obvious consequence of this formula is the perseverance of extremality. Calling "reasonable" any linear topology making involution continuous, and multiplication continuous in each variable separately, we have:

COROLLARY (i) *If $\bar{A}$ is the closure of A in B with respect to a reasonable topology, and if x is in A, then x is extremal (A) if and only if x is extremal $(\bar{A})$.*

(ii) *If ϕ is a *-homomorphism of A into a B^* algebra B_1, then x extremal (A) implies that ϕx is extremal (ϕA).*

Using the methods of [2], one can draw substantial information about the form of an individual extremal element from Theorem 1.

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[1] This has already been proved by Sakai [5, p. 1.3]

THEOREM 2. *Let A be a self-adjoint subalgebra of the algebra $\mathscr{B}(H)$ of all bounded operators on the Hilbert space H. Let x be extremal (A). Then*

(i) *A has an identity*[1], *which we now take to be the identity operator on H—thus possibly changing the meaning of H.*

(ii) *x satisfies one of the following*

(a) *x is unitary*

(b) *x is semi-unitary—i.e., exactly one of xx^*, x^*x is the identity*

(c) *There is a projection p such that px and $(1-p)x$ are semi-unitary* on pH and $(1-p)H$ respectively. *Further, p can be taken from the center of the weak closure of A or, if A is AW^*, from the center of A.*

(iii) *If x is a semi-unitary with $xx^* = 1$, then*

(a) *$H = \sum_0^\infty \oplus H_i$ where x is an isometry of H_0 onto H_0 and of H_{i+1} onto H_i $(i \geqq 1)$, and maps H_1 onto zero.*

(b) *Let X be the left shift on unilateral l_2. The map taking a polynomial in x and x^* into the same polynomial in X and X^* induces a *-isomorphism from the uniformly closed subalgebra of $\mathscr{B}(H)$ generated by x and x^* onto the uniformly closed subalgebra of $\mathscr{B}(l_2)$ generated by X and X^*.*

(c) *The weakly closed subalgebra W of $\mathscr{B}(H)$ generated by x and x^* is naturally *-isomorphic to $\mathscr{B}(l_2) \oplus Z$, where Z is the weakly closed subalgebra of $\mathscr{B}(H)$ generated by x and x^* restricted to H_0. $Z + 1$ is the center of W.*

Clearly there is a restatement of (iii) applying to semi-unitary operators with $x^*x = 1$; in it X is the right shift on l_2, and x maps H_i onto H_{i+1} for all $i \geqq 1$. It is also clear that (iii) can be applied separately to the components px and $(1-p)x$ of an element satisfying (iic). For example, the uniformly closed algebra generated by such an element is *-isomorphic with the uniformly closed subalgebra of $\mathscr{B}(l_2 \oplus l_2)$ generated by $u_1 + u_2^*$, where u_1 is the left shift on the first l_2, zero on the second, and u_2 is zero on the first l_2, the left shift on the second.

Part (iii) gives us three ways of looking at a semi-unitary element. With regard to the uniformly closed, self-adjoint subalgebras they generate, all semi-unitaries are the same. From the standpoint of weakly closed algebras, semi-unitaries differ only in their unitary parts. Viewed spatially—i.e., as representing a similarity class—a semi-unitary is determined by its unitary part and the dimension of its (or its adjoint's) null space. In the light of (iiia), Putnam's result that similar normal operators are unitarily equivalent is easily seen to imply that similar semi-unitary operators are unitarily equivalent. Another method of classifying extremal elements is considered in [3].

Calling a projection p infinite (A) if there is a partial isometry u in A with $uu^* = p$, $u^*u \neq p$, $u^*u \leqq p$, we see that each projection p which is infinite (A) gives rise to at least one semi-unitary, viz. $u + (1 - p)$. Conversely, the existence of a semi-unitary implies the existence of a projection infinite (A), viz 1. Elements of type (iic) are similarly related to projections p in the weak closure of A having the property that p and $1 - p$ are both infinite (weak closure A).

Clearly the study of extremal points will be most rewarding when they exist in substantial number. We have seen that when an identity is lacking, there are no extremal elements. It is well known that if A is a B^* algebra with identity, there are enough unitaries so that every element of A is a linear combination of four of them. But much more can be asked—namely, that the unit ball be the (somehow) closed convex hull of its extremal points. This fails to happen for the general B^*-algebra. An exercise in Bourbaki shows that if A is a weakly closed subalgebra of $\mathscr{B}(H)$, then A is the weakly closed convex hull of its extremal points; the proof may be written "Alaoglu: Krein-Milman."

The weakly-closed, or similar, situation has another useful feature; restating an argument of Calkin ([1], proofs of Theorems 2.4 and 2.5) we obtain:

THEOREM 3. *If A is an AW^* algebra, ϕ a*-homomorphism of A into a B^* algebra, and y a point extremal (ϕA), then there exists an x extremal (A) with $\phi x = y$.*

As an application of this theorem, we consider how the type (in the sense of [4]) of an AW^* algebra determines the type of an AW^* homomorphic image.

THEOREM 4. *Let A, B be AW^* algebras, with B the image of A under some non-trivial *-homomorphism. Then*

(i) *A of type I_n implies B of type I_n*

(ii) *A of type II_1 implies B of type II_1*

(iii) *A of type II_∞ implies B of type II_∞ or III*

(iv) *A of type III implies B of type III*

(v) *A of type I_∞ implies B of type I_∞, II_∞, or III.*

It is likely that another attack would produce a substantially improved theorem in this direction.

2. Proofs.

Proof of Theorem 1. The proof of [3] may be modified to apply in the case where A is not closed, nor known to have identity. In fact, let x be extremal (A). Letting $h = x^*x$, we observe that $0 \leqq \sigma(h) \leqq 1$, where $\sigma(h)$ is the spectrum of h in the closure of A. Let C be the intersection with A of the uniformly closed subalgebra generated by h. Then C is isometrically*-isomorphic with an algebra of continuous, complex valued functions on $\sigma(h)$. Further, C contains $s = h(1-h)^2$.

We desire to show the inequality

$$||(1 \pm s)h(1 \pm s)|| \leqq 1 .$$

In view of the identification of C with a function algebra, this reduces to showing that for real t between zero and one,

$$0 \leqq t[1 \pm t(1-t)^2]^2 \leqq 1 .$$

This is obvious when the ambiguous sign is minus; when it is plus, the expression in t may be expanded as a convex combination of points obviously in $[0, 1]$.

We thus have $||x(1 \pm s)|| \leqq 1$. Writing $x = (1/2)[(x + xs) + (x - xs)]$ and using the extremality of x, we have $xs = 0$ and so $sx^*xs = 0$— i.e., $h^4(1-h)^4 = 0$. Again viewing C as a function algebra, we conclude from the last equation that the function h assumes only the values zero and one, so h is a projection and x a partial isometry.

Thus if $y \in (1 - x^*x)A(1 - xx^*)$, then $y = (1 - x^*x)y(1 - xx^*)$ and so $xy = yx = 0$. It follows that $||x \pm y^*||^2 = ||x^*x + yy^*|| = \max(||x^*x||, ||yy^*||)$. Assuming that $||y|| \leqq 1$, we have $||x \pm y^*|| = 1$ and so, by the extremality of x, $y^* = 0$.

The converse, that an x giving $(1 - x^*x)A(1 - xx^*) = 0$ is extremal (A), is proved in [3]; it also follows from Theorem 2, which is based entirely on the equation $(1 - x^*x)A(1 - xx^*) = 0$.

Proof of Theorem 2. Suppose x satisfies

$$(1) \qquad (1 - x^*x)A(1 - xx^*) = 0 ;$$

then, since A is self-adjoint, x also satisfies

$$(2) \qquad (1 - xx^*)A(1 - x^*x) = 0 .$$

For each y in A we have, by (1),

$$(1 - x^*x)(1 - xx^*)yy^*(1 - xx^*)(1 - x^*x) = 0 ,$$

and so $(1 - x^*x)(1 - xx^*)y = 0$. Performing the indicated multiplications, we obtain

$$(x^*x + xx^* - x^*x^2x^*)y = y .$$

The same argument may be made with x permuted with x^* and y permuted with y^*; the result is that $x^*x + xx^* - x^*x^2x^*$ is also a right identity for A. As previously agreed, we consider this element to be the identity operator on H, and denote it by "1".

We must now show that (1) implies part (ii) of the theorem. Observe first that (1) implies

$$0 = (1 - x^*x)x^*(1 - xx^*)x = (x^* - x^*xx^*)(x - xx^*x) ,$$

and so, that x is a partial isometry.

We next show:

$$(3) \qquad \begin{aligned} x^k x^{*m}(1 - x^*x) &= x^{*m-k}(1 - x^*x) \\ x^{*k}x^m(1 - xx^*) &= x^{m-k}(1 - xx^*) \end{aligned} \qquad 0 \leqq k \leqq m .$$

The first line of (3) may be rewritten as

$$(1 - x^k x^{*k})x^{*n}(1 - x^*x) = 0 \qquad k \geqq 0,\ n \geqq 0$$

and this equation established by induction on k. It clearly holds for $k=0$ and, by (2) for $k=1$. Writing $1-x^{k+1}x^{*k+1}$ as $(1 - x^k x^{*k}) + x^k(1 - xx^*)x^{*k}$, we see that the induction hypothesis reduces the previous equation to:

$$x^k(1 - xx^*)x^{*k+n}(1 - x^*x) = 0 ;$$

but this is already true by (2). The second line of (3) is proved in the same way, using (1) in place of (2).

That x is a partial isometry, together with (3), gives

$$(4) \qquad x^k x^{*m}(1 - x^*x) = x^{*k}x^m(1 - xx^*) = 0 \qquad 0 \leqq m \leqq k .$$

We can now copy the argument of [2]. Define e_i, f_i by

$$\begin{aligned} e_i &= x^{*i-1}x^{i-1} - x^{*i}x^i \\ f_i &= x^{i-1}x^{*i-1} - x^i x^{*i} \end{aligned} \qquad i \geqq 1 .$$

It follows that, for all $i, j \geqq 1$,

$$e_i e_j = \delta_{ij} e_j , \qquad f_i f_j = \delta_{ij} f_j$$
$$e_i = 0 \text{ if and only if } e_1 = 0 , \qquad f_i = 0 \text{ if and only if } f_1 = 0$$
$$e_i f_j = 0 .$$

These relations are immediate consequences of (2), (3), (4), and the fact that x is a partial isometry. We suppose for a moment that A is an AW^* algebra. Let

$$q = \vee_1^\infty e_i , \qquad r = \vee_1^\infty f_i ,$$

the supremum being that given by the AW^* character of A. Then for any y in A, $ye_i = 0$ for all i implies $yq = 0$, and similarly for yf_i and yp. For any i and j,

$$f_i A e_j = x^{i-1}(1 - xx^*)x^{*i-1}Ax^{*j-1}(1 - x^*x)x^{j-1} = 0$$

by (2). From what we have just said, this implies $rAq = 0$.

Now consider the left annihilator of Aq. Since A is AW^*, the annihilator can be written as Ap for some projection p in A. It is easily shown that, since Aq is a left ideal, p is central in A. We thus have

$$(1 - p)q = q(1 - p) = q$$

and, since $rAq = 0$, r is in Ap—i.e.,

$$pr = rp = r \ .$$

By definition, $rf_1 = f_1 r = f_1$; thus, $pf_1 = p(rf_1) = (pr)f_1 = f_1$, and so,

$$p(1 - xx^*) = 1 - xx^* = (1 - xx^*)p \ .$$

Rearranging terms,

$$xx^*(1 - p) = (1 - p)xx^* = 1 - p \ .$$

On the other hand, $e_1 = qe_1$, so $(1 - p)e_1 = (1 - p)qe_1 = e_1$, and

$$(1 - p)x^*x = (1 - p) - e_1 \ .$$

Thus $(1 - p)x$ is semi-unitary in $(1 - p)A$, and unitary just in case $e_1 = 0$.

In the same way, we show

$$x^*xp = px^*x = p$$
$$p - xx^*p = f_1 \ ,$$

so px is semi-unitary in pA, unitary just in case $f_1 = 0$.

This proves (ii) when A is AW^*; the statements about the case where A is a self-adjoint subalgebra of $\mathscr{B}(H)$ follow on observing that the weak closure of A is AW^*.

To obtain (iii), observe that if x is semi-unitary with $xx^* = 1$, then x satisfies (1), and we may define elements e_i in terms of x as above (the f_i are of course all zero in this case). Define the spaces H_i of (a) by

$$H_i = \begin{cases} (1 - q)H & i = 0 \\ e_i H & i > 0 \ . \end{cases}$$

We observe from (3) that $xe_1 = 0$, $xe_i = e_{i-1}x$ for $i > 1$—so $x^*xe_i = e_i$ for $i > 1$. Consequently for any $\xi, \eta \in H$, $i > 1$

$$(xe_i\xi, xe_i\eta) = (x^*xe_i\xi, e_i\eta) = (e_i\xi, e_i\eta) .$$

Since $H = xx^*H \subseteqq xH$, $x(e_iH) = e_{i-1}(xH) = e_{i-1}H$.

Further,

$$(x(1-q)\xi, x(1-q)\eta) = ((1-e_1)(1-q)\xi, (1-q)\eta) = ((1-q)\xi, (1-q)\eta)$$

and

$$(x(1-q)\xi, e_i\eta) = (x^*(1-q)\xi, e_i\eta) = 0 ,$$

so x and x^* take H_0 into itself—and so, since x^* is never zero, x takes H_0 onto itself. Part (a) is now established.

To show (*b*), identify l_2 with a subspace of H by picking some ξ in H_1 with $\|\xi\| = 1$ and identifying the sequence $\{\eta_i\}$ in l_2 with $\Sigma\eta_i x^{*i}\xi$. The restriction map is now clearly a*-isomorphism, from the algebra of polynomials in x and x^* onto the algebra of polynomials in X and X^*; it remains only to prove that this map is an isometry. We know from [7] §2, that certain algebras with involution have a unique B^* norm: a sufficient condition is that the algebra have a faithful *-representation on some Hilbert space, and that for each z in the algebra there is a real k such that $f(z^*z) \leqq kf(1)$ for each functional f on the algebra which is positive on all y^*y.

The algebra of polynomials in X and X^* has been defined as being represented on l_2, and so satisfies the first part of this condition. Further, $X^{*k}X^k - X^{*k+1}X^{k+1}$ is a projection for each $k \geqq 0$. Thus if f is any positive functional,

$$f(1) \geqq f(X^*X) \geqq f(X^{*2}X^2) \geqq \cdots .$$

It is readily shown that any Y^*Y in the polynomial algebra can be written as $\Sigma a_k X^{*k}X^k$—so

$$f(Y^*Y) = \Sigma a_k f(X^{*k}X^k) \leqq (\Sigma\,|\,a_k\,|)f(1)$$

for any positive functional f, and the second part of the condition is also satisfied. Thus there is only one B^* norm on the algebra generated by X and X^*, and the norm this algebra inherits from $\mathscr{B}(l_2)$ is the same it gets from $\mathscr{B}(H)$ via the restriction map.

The isomorphism between the polynomial algebras can be obtained without considering x to be represented on any space; this is done in [2]. The isomorphism can be shown isometric by showing that a polynomial in X and X^* has the same norm as the same polynomial in U and U^*, U being the (unitary) left shift on bi-lateral l_2.

To prove (*c*), let $\{\xi_\alpha^1\}$ be an ortho-normal basis for H_1, and let $\xi_\alpha^k = x^{*k}\xi_\alpha^1$; then $\{\xi_\alpha^k\}$ is an ortho-normal basis for H_k. Let p_α be the projection on the closed linear span of $\{\xi_\alpha^k\colon k = 1, 2, 3, \cdots\}$, and $\tau_{\alpha,\beta}$ the isometry of $p_\alpha H$ onto $p_\beta H$ which, for each k takes ξ_α^k onto ξ_β^k. Observe that p_α and $\tau_{\alpha,\beta}$ commute with x and x^*. Consequently, if $w \in W$, then w commutes with all $\tau_{\alpha,\beta}$. It follows that there exist scalars $\nu_{i,j}$ such that, for any α,

$$w\xi_\alpha^i = \Sigma_j \nu_{ij}\xi_\alpha^j\ . \tag{5}$$

Suppose z commutes with x and x^*. The equations

$$(z\xi_\alpha^i, \xi_\beta^j) = (zx^{*i-1}\xi_\alpha^1, x^{*j-1}\xi_\beta^1) = (zx^{j-1}x^{*i-1}\xi_\alpha^1, \xi_\beta^1) = (z\xi_\alpha^1, x^{i-1}x^{*j-1}\xi_\beta^1)$$

show that

$$(z\xi_\alpha^i, \xi_\beta^j) = \begin{cases} 0 & i \neq j \\ (z\xi_\alpha^1, \xi_\beta^1) & i = j\ . \end{cases}$$

In other words, there exist scalars $\lambda_{\alpha,\beta}$ such that for each i,

$$z\xi_\alpha^i = \Sigma_\beta \lambda_{\alpha,\beta}\xi_\beta^i\ . \tag{6}$$

Further, if p_0 is the projection on H_0, then z commutes with p_0.

Now, given any w commuting with p_0 and satisfying (5), it follows from (6) and the fact that the ξ_α^i span $H_0^\perp$ that $(1 - p_0)w$ commutes with every z which commutes with x and x^*. Since every element of W commutes with p_0, we have $(1 - p_0)W$ isomorphic to $\mathscr{B}(l_2)$ under the correspondence obtained naturally via (5). Clearly $p_0 W$ is isomorphic to Z, and the proof of (*c*) complete.

Proof of Theorem 3. The first step is to show that, under the conditions of the hypothesis, the pre-image of a partial isometry contains a partial isometry. The proof follows an argument of Calkin [1, Theorems 2.4 and 2.5].

Let y be a partial isometry of B, and let v be any element of $\phi^{-1}(y)$. Since A is AW^*, v has a polar decomposition in A—i.e., there are elements u and h in A such that

$$\begin{aligned} &u \text{ is a partial isometry} \\ &h = (v^*v)^{1/2} = u^*uh = hu^*u \\ &v = uh \end{aligned}$$

(see [6], Lemma 2.1).

Since $\phi(h^2)$ is a projection, zero and one are in the spectrum of $\phi(h^2)$. Since ϕ is a homomorphism, the spectrum of $\phi(h^2)$ is contained in the spectrum of h^2. Since $h \geqq 0$, this implies that zero and one are in the spectrum of h. Let p_λ be the resolution of the identity for h given by the spectral theorem, let $0 < \alpha < 1$, and let $q = 1 - p_\alpha$. Then $0 \neq q \neq 1$.

Let C be some maximal commutative, self-adjoint subalgebra of A containing 1 and h—and so q as well. Let $x \to \hat{x}$ be the Gelfand representation of C on $C(\Omega)$, the algebra of all continuous, complex-valued functions on Ω, the (compact, Hausdorff) maximal ideal space of C. By definition,

$$\hat{q}(\omega) = 0 \text{ implies } (1 - \hat{h})(\omega) \geqq 1 - \alpha$$
$$\hat{q}(\omega) = 1 \text{ implies } (1 - \hat{h})(\omega) \leqq 1 - \alpha \, .$$

We now assert that there exist self-adjoint r, s, and t in C such that

$$hrq = q \, , \qquad h(1 + h)sq = q$$
$$(1 - h^2)t(1 - q) = 1 - q \, .$$

Since the Gelfand representation is a *-isomorphism, this is equivalent to asserting that there are real valued functions $\hat{r}$, $\hat{s}$, and $\hat{t}$ in $C(\Omega)$ such that

$$\hat{r}(\omega) = \hat{h}^{-1}(\omega) \text{ when } \hat{q}(\omega) \neq 0 \, ,$$
$$\hat{s}(\omega) = [\hat{h}(1 + \hat{h})]^{-1} \text{ when } \hat{q}(\omega) \neq 0 \, ,$$
$$\hat{t}(\omega) = [1 - \hat{h}^2]^{-1}(\omega) \text{ when } (1 - \hat{q})(\omega) \neq 0 \, .$$

But $\hat{h}^{-1}$ and $[\hat{h}(1 + \hat{h})]^{-1}$ are bounded and continuous on the closed set $\{\omega : \hat{h}(\omega) \geqq \alpha\}$, which contains the set $(\omega : \hat{q}(\omega) \neq 0\}$, and $[1 - \hat{h}^2]^{-1}$ is bounded and continuous on $\{\omega : (1 - \hat{h})(\omega) \geqq 1 - \alpha\}$, which contains the set $\{\omega : (1 - \hat{q})(\omega) \neq 0\}$. The existence of $\hat{r}$, $\hat{s}$, and $\hat{t}$ is therefore guaranteed by the Tietze extension theorem.

$$qu^*uq = qrhu^*uhrq = qrh^2rq = q \, ,$$

so uq is a partial isometry. Since ϕv is a partial isometry, $uh - uh^3$ is in kernel ϕ. Therefore

$$uh(1 - q) = u(1 - h)h(1 + h)t(1 - q)$$
$$= u(h - h^3)t(1 - q) \, ,$$

which is in kernel ϕ. Also,

$$u(1 - h)q = u(1 - h)h(1 + h)sq$$
$$= u(h - h^3)sq$$

which is in kernel ϕ. Therefore $uh - uq = uh(1 - q) - u(1 - h)q$ is in kernel ϕ, which gives the desired result.

We can now show that if y is extremal (B), then $\phi^{-1}(y)$ contains an element extremal (A). For let v be any partial isometry in $\phi^{-1}(y)$. The fundamental comparability theorem for AW^* algebras—e.g., [4], Theorem 5.6—says that there exist central projections e_1, e_2 in A such that

$$e_1(1 - v^*v) \leqq e_1(1 - vv^*)$$
$$e_2(1 - v^*v) \geqq e_2(1 - vv^*)$$
$$e_1 + e_2 = 1 .$$

In other words, there are partial isometries w and z in A such that

$$w^*w = e_1(1 - v^*v) , \qquad ww^* \leqq e_1(1 - vv^*)$$
$$z^*z = e_2(1 - vv^*) , \qquad zz^* \leqq e_2(1 - v^*v) .$$

The first equation implies $0 = vw^*w$, and so $vw^* = wv^* = 0$; it also gives $w^*w(1 - e_1) = 0$, and so $we_1 = e_1w = w$. The first inequality gives $0 = e_1v^*(1 - vv^*)ww^* = v^*ww^*$, and so $v^*w = w^*v = 0$. Similarly, $vz = zv = 0$ and $e_2z = ze_2 = z$. Define u_1 and u_2 by

$$u_1 = e_1v + w , \qquad u_2 = e_2v + z^* .$$

We have at once from the preceding equations that

$$u_1{}^*u_1 = e_1 , \qquad u_1e_1 = e_1u_1 = e_1$$
$$u_2u_2{}^* = e_2 , \qquad u_2e_2 = e_2u_2 = e_2 .$$

Since $\phi(v)$ is extremal $(\phi(A))$, $(1 - v^*v)A(1 - vv^*)$ is contained in kernel ϕ; in particular, $(1 - v^*v)w(1 - vv^*)$ is in kernel ϕ, but $(1 - v^*v)w(1 - vv^*) = w = u_1 - e_1v$. Thus $u_1 - e_1v$ is in kernel ϕ. Similarly $u_2 - e_2v$ is in kernel ϕ. Consequently $(u_1 + u_2) - v$ is in kernel ϕ. We have already seen that $u_1 + u_2$ is an extremal point of A.

Proof of Theorem 4. All algebras mentioned are assumed to be AW^*. The terminology is taken from [4].

The case where A is of type I_n follows at once from the definitions—i.e., A is of type I_n if and only if it has matrix units e_{ij}, $1 \leqq i, j \leqq n$, with all e_{ii} being Abelian projections. But the properties of being a set of matrix units, or an Abelian projection are both preserved by homomorphisms.

Note that if p is a projection infinite (A), and φ a * homomorphism of A into a B^* algebra, then $\varphi(p)$ is either zero or infinite $\varphi(A)$. For if $p = p_1 + p_2$ with $p \backsim p_1 \sim p_2$ and $\varphi(p) \neq 0$, we have one of $\varphi(p_1), \varphi(p_2) \neq 0$; say $\varphi(p_1)$. Then $\varphi(p_2) < \varphi(p)$, $\varphi(p_2) \sim \varphi(p)$, and so

$\varphi(p)$ is infinite. (We thank the referee for supplying this argument to replace one which was somewhat grandiose.) In consequence, the homomorphic image of an algebra of infinite type is again of infinite type or else zero. An easy consequence of the first part of the proof of Theorem 3 is that each projection in the image algebra comes from a projection in the pre-image. This, with the previous remark, shows that the image of an algebra of Type III is again of Type III.

Conversely, the image of an algebra of finite type is again of finite type; for an AW^* algebra is of finite type if and only if all its extremal elements are unitary. By Theorem 3, the latter property must be inherited by any homomorphic image.

LEMMA. *If q is an abelian projection of B, there is an abelian projection p of A with $\phi p = q$.*

Proof. As we have noted, the proof of Theorem 3 can be used to find a projection p_0 of A with $\phi p_0 = q$. Consider the AW^* algebra p_0Ap_0. We know from [4], that any AW^* algebra can be written as a direct sum of two ideals, the first of which is a 2×2 matrix algebra, and the second of which is commutative. Thus we have

$$p_0Ap_0 = A_1 \oplus A_2$$

A_1 a 2×2 matrix algebra, A_2 commutative. Thus $p_0 = p_1 + p_2$, $p_i \in A_i$. Since p_2Ap_2 is contained in A_2, it is commutative—i.e., p_2 is an abelian projection.

We observe that a homomorphism of a $n \times n$ matrix algebra $(n \geqq 2)$ into a commutative ring must be zero; for if e_{ij} are matrix units, $\phi e_{ii} = \phi e_{ii}e_{ij}e_{ji} = \phi e_{ij}\phi e_{ii}\phi e_{ji} = 0$. Since ϕ is a homomorphism from p_0Ap_0 to qBq, this means that $\phi p_0 = \phi p_2$, as desired.

With these observations, the implications of Theorem 4 may be read on at once.

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