

ON THE LOCATION OF THE ZEROS OF SOME INFRAPOLYNOMIALS WITH PRESCRIBED COEFFICIENTS

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1. Various results have been obtained regarding the zeros of infrapolynomials with prescribed coefficients. (See e.g. [Walsh, 1958], [Walsh and Zedek, 1956], [Fekete and Walsh, 1957], [Shisha and Walsh, 1961, 1963], and [Shisha, 1962]). Our purpose in the present note is twofold:

(i) to contribute more deeply to that study, making use of some properties of polynomials and rational functions, and

(ii) conversely, further to show how results concerning infrapolynomials can be used in the investigation of some rational functions and in particular some combinations of a polynomial and its derivative.

2. We repeat here the underlying definition. Let n and q be natural numbers ($q \leq n$), n_1, n_2, \dots, n_q integers such that $0 \leq n_1 < n_2 < \dots < n_q \leq n$, and S a pointset in the (open) complex plane. An n th infrapolynomial on S with respect to (n_1, n_2, \dots, n_q) is a polynomial $A(z) \equiv \sum_{\nu=0}^n a_\nu z^\nu$ having the property: There does not exist a polynomial $B(z) \equiv \sum_{\nu=0}^n b_\nu z^\nu$ such that $B(z) \not\equiv A(z)$, $b_{n_\nu} = a_{n_\nu}$ ($\nu = 1, 2, \dots, q$), $|B(z)| < |A(z)|$ whenever $z \in S$ and $A(z) \neq 0$, and $B(z) = 0$ whenever $z \in S$ and $A(z) = 0$.

3. Of special importance among the above sequences (n_1, n_2, \dots, n_q) , are "simple n -sequences" [Shisha and Walsh, 1961]. Given a natural number n , we define a "simple n -sequence" to be a sequence having one of the forms $(0, 1, \dots, k, n-l, n-l+1, \dots, n)$ [$k \geq 0, l \geq 0, k+l+2 \leq n$]; $(0, 1, \dots, k)$ [$0 \leq k < n$]; $(n-l, n-l+1, \dots, n)$ [$0 \leq l < n$]. We shall consider n th infrapolynomials on some special sets S with respect to simple n -sequences σ . The sets S will consist of $n-s+2$ points, where s is the number of elements of σ , and S will be required not to contain the origin, in case σ contains zero. As explained in the Introduction to the last mentioned paper, this particular situation is of special importance, as the general case is to a large extent reducible to it, and as these particular n th infrapolynomials are closely related to certain combinations of a polynomial and its derivative. Numerous results on such combinations exist in the literature.

Received May 2, 1963. This research was supported (in part) by the U.S. Air Force through the Air Force Office of Scientific Research.

4. **THEOREM.** Let n be a natural number, σ a simple n -sequence, s its number of elements. Let $S = \{z_1, z_2, \dots, z_{n-s+2}\}$ be a set of $n - s + 2$ (distinct) points of the (open) complex plane, and set $g(z) \equiv \prod_{\nu=1}^{n-s+2} (z - z_\nu)$. In case $\sigma = (0, 1, \dots, k)$ or $\sigma = (0, 1, \dots, k, n - l, n - l + 1, \dots, n)$ set $K = k + 1$. In case $0 \notin \sigma$, set $K = 0$. (Thus $K = \min[\nu, \nu \notin \sigma, \nu = 0, 1, 2, \dots]$). Also, in case $0 \in \sigma$, assume $0 \notin S$. Let $A(z) \equiv \sum_{\nu=0}^n a_\nu z^\nu$ be an n th infrapolynomial on S with respect to σ .

Then [by Theorem 1, Shisha and Walsh, 1961] one can set

$$(1) \quad A(z) \equiv P(z)g(z) + \alpha z^K \sum_{\nu=1}^{n-s+2} \lambda_\nu g(z)/(z - z_\nu).$$

Here $\lambda_1, \lambda_2, \dots, \lambda_{n-s+2}$ are nonnegative reals with $\sum_{\nu=1}^{n-s+2} \lambda_\nu = 1$, α is a complex number, and $P(z)$ is a polynomial of degree¹ $\leq s - 1$ such that $P(z)g(z) + \alpha z^{K+n-s+1}$ is of degree $\leq n$.²

I. Let S be contained in a disc $C: |z - c| \leq r$. Then every zero $\zeta (\notin C)$ of $A(z)$ satisfies

$$(2) \quad |P(\zeta) + \overline{(\zeta - c)} \alpha \zeta^K / \{| \zeta - c |^2 - r^2\}| \leq r |\alpha \zeta^K| / \{| \zeta - c |^2 - r^2\}.$$

If $K = 0$, and if a zero ζ of $A(z)$ satisfies $r < \rho_1 \leq |\zeta - c| \leq \rho_2$, then $|\alpha|/|\rho_2 + r| \leq |P(\zeta)| \leq |\alpha|/(\rho_1 - r)$ i.e. (in case $\alpha \neq 0$ and $P(z)$ is not a constant) ζ lies in the closed interior of the lemniscate $|P(z)| = |\alpha|/(\rho_1 - r)$, and in the closed exterior of the lemniscate $|P(z)| = |\alpha|/|\rho_2 + r|$.

II. Let $P(z) \equiv \beta z^t + \gamma z^{t-1} + \dots$ ($t \geq 0, \beta \neq 0$), and suppose that S and all the zeros of $P(z)$ lie in some closed disc C , and that $\alpha \neq 0, K = 0$. Let w_1, w_2, \dots, w_{t+1} be distinct solutions of $w^{t+1} = -\alpha/\beta$. Then every zero ($\notin C$) of $A(z)$ lies in $\bigcup_{\nu=1}^{t+1} (w_\nu + C)$.³

III. Suppose that $A(z)$ is a real polynomial,⁴ and that $\alpha \neq 0$. Assume, furthermore, that $P(z)/(\alpha z^K)$ is of the form $A + \sum_{\nu=1}^p A_\nu z^\nu + \sum_{\nu=1}^q B_\nu z^{-\nu}$ with all $Re(A_\nu) \leq 0$ and all $Re(B_\nu) \geq 0$. Let z_0 be a non-real zero of $A(z)$ satisfying $|\arg z_0| \leq \min(\pi/p, \pi/q)$.⁵ Then z_0 belongs to at least one (Jensen) disc

$$(3) \quad \left| z - \frac{1}{2}(z_\nu + \bar{z}_\nu) \right| \leq \frac{1}{2} |z_\nu - \bar{z}_\nu|.$$

¹ Degree of a polynomial means its exact degree. The polynomial 0 is assigned the degree -1 .

² One can show that α and $P(z)$ are uniquely determined, and in case $\alpha \neq 0$, so are $\lambda_1, \lambda_2, \dots, \lambda_{n-s+2}$.

³ $w_\nu + C$ denotes the closed disc consisting of all points $w_\nu + z, z \in C$.

⁴ i.e. the coefficients of $A(z)$ are real.

⁵ \arg denotes the principal value of the argument.

In particular, if $p = q = 1$, every non-real zero of $A(z)$ belongs to at least one of these discs.

IV. Suppose that $A(z)$ is a real polynomial, $\alpha \neq 0$, and that $P(z)/(\alpha z^k)$ is of the form $\sum_{\nu=0}^p A_\nu z^\nu + \sum_{\nu=1}^q B_\nu z^{-\nu}$ ($p \geq 0, q \geq 2$) with all $\operatorname{Re}(A_\nu) \leq 0$ and all $\operatorname{Re}(B_\nu) \geq 0$. Suppose furthermore that $\lambda_\nu > 0$ implies $\operatorname{Re}(z_\nu) > 0$ ($\nu = 1, 2, \dots, n - s + 2$). Let z_0 be a non-real zero of $A(z)$ satisfying $|\arg z_0| \leq \min \{\pi/(p + 1), \pi/(q - 1)\}$. Then:

A. There exists a $\nu, 1 \leq \nu \leq n - s + 2, \operatorname{Im}(z_\nu) \neq 0$, such that z_0 belongs to the closed interior of the circle passing through z_ν and \bar{z}_ν , and tangent to the line Oz_ν .

B. If neither z_0 nor \bar{z}_0 belongs to S , one can choose ν so that $\lambda_\nu > 0$, and therefore $\operatorname{Re}(z_0) > 0$.

V. Suppose that S is a real set contained in a finite interval $J: x_1 \leq x \leq x_2$, that $A(z)$ is a real polynomial, and that $K = 0$. Suppose $P(z)$ is of the form $\beta z^t + \gamma z^{t-1} + \dots$ ($t \geq 0, \beta \neq 0$), and that all zeros of $P(z)$ lie in the above interval. Then every real zero ($\notin J$) of $A(z)$ is of the form $\xi + \omega$ where $\xi \in J$ and ω is a real number satisfying $\omega^{t+1} = -\alpha/\beta$. Thus, if t is odd and $\alpha\beta > 0$, all real zeros of $A(z)$ lie in J .

5. Proof of Part I. Let $\zeta (\notin C)$ be a zero of $A(z)$. Then by (1),

$$P(\zeta) + \alpha \zeta^k \sum_{\nu=1}^{n-s+2} \lambda_\nu / (\zeta - z_\nu) = 0 .$$

By a result due to Walsh [cf. 1950, § 1.5.1, Lemma 1]

$$(4) \quad \sum_{\nu=1}^{n-s+2} \lambda_\nu / (\zeta - z_\nu) = 1 / (\zeta - z') , \quad z' \in C .$$

By an elementary mapping property of the function $1/z$ we have

$$|1/(\zeta - z') - \overline{(\zeta - c)} / \{|\zeta - c|^2 - r^2\}| \leq r / \{|\zeta - c|^2 - r^2\} ,$$

from which (2) follows. The rest of part I is easily obtained from (2).

Proof of Part II. Let $\zeta (\notin C)$ be a zero of $A(z)$. Again we have a relation (4), which implies $P(\zeta)(\zeta - z') = -\alpha$. Furthermore, the last left hand side can be written [Walsh, 1922] $\beta(\zeta - \eta)^{t+1}$ with $\eta \in C$. Hence $\zeta \in \bigcup_{\nu=1}^{t+1} (w_\nu + C)$.

Proof of Part III. We may assume $g(z_0) \neq 0, g(\bar{z}_0) \neq 0$. Since $\overline{A(\bar{z}_0)} = A(z_0) = 0$, we have by (1),

$$\begin{aligned}
 0 &= \overline{P(z_0)} / (\alpha \overline{z_0^K}) + \sum_{\nu=1}^{n-s+2} \lambda_\nu / (z_0 - \overline{z_\nu}) \\
 &= \overline{A} + \sum_{\nu=1}^p \overline{A_\nu} z_0^\nu + \sum_{\nu=1}^q \overline{B_\nu} z_0^{-\nu} + \sum_{\nu=1}^{n-s+2} \lambda_\nu / (z_0 - \overline{z_\nu}) \\
 &= A + \sum_{\nu=1}^p A_\nu z_0^\nu + \sum_{\nu=1}^q B_\nu z_0^{-\nu} + \sum_{\nu=1}^{n-s+2} \lambda_\nu / (z_0 - z_\nu) \\
 &= 2\operatorname{Re}(A) + \sum_{\nu=1}^p 2\operatorname{Re}(A_\nu) z_0^\nu + \sum_{\nu=1}^q 2\operatorname{Re}(B_\nu) z_0^{-\nu} \\
 &\quad + \sum_{\nu=1}^{n-s+2} \lambda_\nu \{ (z_0 - z_\nu)^{-1} + (z_0 - \overline{z_\nu})^{-1} \}.
 \end{aligned}$$

By theorem 21 [Shisha and Walsh, 1961], there exists a ν (with $\lambda_\nu > 0$)] such that z_0 lies in (3).

Similarly, using Theorem 22 [loc. cit.] one proves Part IV.⁶

Proof of Part V. Let ζ ($\notin J$) be a real zero of $A(z)$. Then $P(\zeta) + \alpha \sum_{\nu=1}^{n-s+2} \lambda_\nu / (\zeta - z_\nu) = 0$. Now, $\sum_{\nu=1}^{n-s+2} \lambda_\nu / (\zeta - z_\nu)$ can be written as $1/(\zeta - x')$, $x' \in J$. Also, since all zeros of $P(z)$ lie in J , one can set $P(\zeta)(\zeta - x') = \beta(\zeta - \xi)^{t+1}$, $\xi \in J$. Setting $\omega = \zeta - \xi$, we have $\zeta = \xi + \omega$, $\omega^{t+1} = -\alpha/\beta$.

6. We apply now our results to some special cases. We continue to assume the contents of the first paragraph of the Theorem. Thus, the contents of the second paragraph of the Theorem hold, too.

(a) Suppose $\sigma = (n)$. If $a_n = 0$ then $A(z) \equiv 0$, for otherwise the polynomial $B(z) \equiv 0$ would fulfill the properties stated at the end of § 2. We thus assume that $a_n \neq 0$. Then $a_n^{-1}A(z)$ is an infrapolynomial (“Extremalpolynom”) on S in the sense of Fekete and von Neumann [1922]. Also one easily sees that $P(z) \equiv 0$, $\alpha = a_n$. By a known result [loc. cit., p. 138, cf. also Fejér 1922] all zeros of $A(z)$ belong to the convex hull of S . Thus Parts I, II and V of the Theorem do not apply. Parts III and IV do apply; but they can be derived from known results [Fekete and von Neumann 1922 p. 138, and Walsh 1958 p. 305]. Thus, if z_0 is a non-real zero of $A(z)$, and if $A(z)$ is a real polynomial, then z_0 belongs to at least one of the discs (1). If, in addition, $\lambda_\nu > 0$ implies $\operatorname{Re}(z_\nu) > 0$ ($\nu = 1, 2, \dots, n + 1$), then A and B of Part IV hold.

(b) Suppose $\sigma = (n - 1, n)$. Then $s = 2$, $K = 0$ and [Shisha and Walsh 1961, p. 146]

⁶ Observe that if (i) $A(z)$ is a real polynomial, (ii) $\alpha \neq 0$, and (iii) S is symmetric in the axis of reals, then (i) α is real, (ii) $\lambda_\nu = \lambda_\mu$ if $z_\nu = \overline{z_\mu}$, and (iii) $g(z)$ and $P(z)$ are real polynomials. Indeed, suppose $z_\nu = \overline{z_\mu}$. Then (1) yields $\alpha z_\nu^K \lambda_\nu g'(z_\nu) = A(z_\nu) = \overline{A(z_\mu)} = \overline{\alpha z_\mu^K \lambda_\mu g'(z_\mu)}$. Thus, if α is real, $\lambda_\nu = \lambda_\mu$. To prove that α is real, choose ν_0, μ_0 so that $\lambda_{\nu_0} > 0$ and $z_{\nu_0} = \overline{z_{\mu_0}}$. Then $(\lambda_{\nu_0} + \lambda_{\mu_0})\operatorname{Im}(\alpha) = 0$, and therefore $\operatorname{Im}(\alpha) = 0$. From (1) we see now that $P(z)g(z)$ is a real polynomial; therefore, so is $P(z)$.

$$A(z) \equiv a_n g(z) + \left(a_{n-1} + a_n \sum_{\nu=1}^n z_\nu \right) \sum_{\nu=1}^n \lambda_\nu g(z)/(z - z_\nu),$$

$\lambda_\nu \geq 0, \sum_{\nu=1}^n \lambda_\nu = 1$. Thus, $P(z) \equiv a_n$ and $\alpha = a_{n-1} + a_n \sum_{\nu=1}^n z_\nu$. One can apply Part I. Part II implies that if $a_n \neq 0$ and if S lies in a closed disc C , then every zero ($\notin C$) of $A(z)$ lies in $-(\alpha/a_n) + C$. This, however, is a known result [loc. cit. Theorem 14, cf. also Walsh 1922 Theorem VI]. Again, the information we obtain from Part III follows from known results [Fekete and Walsh 1957 Theorem X, Fekete and von Neumann 1922 p. 138]. Assume that $A(z)$ is a real polynomial, $\alpha \neq 0, \lambda_\nu > 0$ implies $Re(z_\nu) > 0 (\nu = 1, 2, \dots, n)$, and $Re(a_n/\alpha) \leq 0$ (i.e. if $a_n \neq 0$ then $\sum_{\nu=1}^n Re(z_\nu) \leq -a_{n-1}/a_n$). By Part IV if z_0 is an arbitrary non-real zero of $A(z)$, then the conclusions *A* and *B* there hold. Finally, one can apply also Part V.

(c) Suppose $\sigma = (n - 2, n - 1, n)$. Then $s = 3, K = 0$. We set $P(z) \equiv \rho + \tau z$, so that (1) yields

$$a_n = \tau, \quad a_{n-1} = \rho - \tau \sum_{\nu=1}^{n-1} z_\nu, \quad a_{n-2} = -\rho \sum_{\nu=1}^{n-1} z_\nu + \tau \sum_{1 \leq j < k \leq n-1} z_j z_k + \alpha.$$

Thus, setting $\sigma_1 = \sum_{\nu=1}^{n-1} z_\nu, \sigma_2 = \sum_{1 \leq j < k \leq n-1} z_j z_k$, we have

$$A(z) \equiv (\rho + \tau z)g(z) + \alpha \sum_{\nu=1}^{n-1} \lambda_\nu g(z)/(z - z_\nu),$$

where

$$\rho = a_n \sigma_1 + a_{n-1}, \quad \tau = a_n, \quad \alpha = a_{n-2} + \sigma_1(a_n \sigma_1 + a_{n-1}) - a_n \sigma_2.$$

We may apply Parts I-V. For example, suppose that $A(z)$ is a real polynomial, that $\alpha \neq 0$, and that either $a_n = 0$, or $a_n \neq 0$ and

$$(a_{n-2}/a_n) + (a_{n-1}/a_n) Re(\sigma_1) + Re(\sigma_1^2 - \sigma_2) \leq 0.$$

Then $Re(\tau/\alpha) \leq 0$, and therefore, by Part III, every non-real zero of $A(z)$ belongs to at least one of the discs (3).

(d) Suppose $\sigma = (n - 3, n - 2, n - 1, n)$. Here $s = 4, K = 0$. We set $P(z) \equiv \rho + \sigma_0 z + \tau z^2$, and from (1) we get

$$a_n = \tau, \quad a_{n-1} = \sigma_0 - \tau \sum_{\nu=1}^{n-2} z_\nu, \quad a_{n-2} = \rho - \sigma_0 \left(\sum_{\nu=1}^{n-2} z_\nu \right) + \tau \sum_{1 \leq j < k \leq n-2} z_j z_k,$$

$$a_{n-3} = -\rho \left(\sum_{\nu=1}^{n-2} z_\nu \right) + \sigma_0 \left(\sum_{1 \leq j < k \leq n-2} z_j z_k \right) - \tau \left(\sum_{1 \leq j < k < m \leq n-2} z_j z_k z_m \right) + \alpha.$$

Thus, setting $\sigma_1 = \sum_{\nu=1}^{n-2} z_\nu, \sigma_2 = \sum_{1 \leq j < k \leq n-2} z_j z_k, \sigma_3 = \sum_{1 \leq j < k < m \leq n-2} z_j z_k z_m$, we have⁷

$$A(z) \equiv (\rho + \sigma_0 z + \tau z^2)g(z) + \alpha \sum_{\nu=1}^{n-2} \lambda_\nu g(z)/(z - z_\nu),$$

⁷ Observe that if $n = 4, \sum_{1 \leq j < k < m \leq n-r} z_j z_k z_m$ is zero, being an empty sum.

where

$$\begin{aligned} \tau &= a_n, & \sigma_0 &= a_{n-1} + a_n\sigma_1, & \rho &= a_{n-2} + (a_{n-1} + a_n\sigma_1)\sigma_1 - a_n\sigma_2, \\ \alpha &= a_{n-3} + (a_{n-2} + a_{n-1}\sigma_1 + a_n\sigma_1^2 - 2a_n\sigma_2)\sigma_1 - a_{n-1}\sigma_2 + a_n\sigma_3. \end{aligned}$$

Here again we can use I-V of the Theorem. For example, suppose S is contained in a disc $C: |z - c| \leq r$. By I, if a zero ζ of $A(z)$ satisfies $r < \rho_1 \leq |\zeta - c| \leq \rho_2$, then

$$|\alpha|/(\rho_2 + r) \leq |\rho + \sigma_0z + \tau z^2| \leq |\alpha|/(\rho_1 - r).$$

By II, if $\alpha\tau \neq 0$, if C contains also the zeros of $P(z) \equiv \rho + \sigma_0z + \tau z^2$, and if w_1, w_2, w_3 are distinct zeros of $w^2 + \alpha/\tau$, then every zero ($\notin C$) of $A(z)$ lies in $\bigcup_{v=1}^3 (w_v + C)$.

7. The following theorem is due to Marden [contained in his Theorem (1, 1), 1949]. *Let z_1, z_2, \dots, z_m be (distinct) points of the (open) complex plane, let $\mu_1, \mu_2, \dots, \mu_m$ be positive numbers, and let A_0, A_1, \dots, A_{p-1} ($p \geq 1$) be arbitrary complex numbers. Let*

$$F(z) \equiv \sum_{v=0}^{p-1} A_v z^v + \sum_{v=1}^m \mu_v / (z - z_v),$$

and set $S = \{z_1, z_2, \dots, z_m\}$. Let T be the set of those zeros of $F(z)$ at which S subtends an angle $< \pi/(p + 1)$. Then the number of points of T (each counted according to its multiplicity) is $\leq p$. From this follows a result on the zeros of combinations of the form $Q(z) \equiv P(z)f(z) + f'(z)$ where $f(z)$ and $P(z)$ are polynomials. (See loc. cit. Theorem (4.3)).

Using known results on infrapolynomials, we can derive Marden's theorem very easily. For the theorem is obviously true if all the A_v are zero. Furthermore, one obviously may assume that $A_{p-1} \neq 0, m > 1$. Set $g(z) \equiv \prod_{v=1}^m (z - z_v), \mu = \sum_{v=1}^m \mu_v, \lambda_v = \mu_v/\mu$ ($v = 1, 2, \dots, m$). Consider the polynomial

$$\begin{aligned} A(z) &\equiv A_{p-1}^{-1} g(z)F(z) \equiv \left(\sum_{v=0}^{p-1} (A_v/A_{p-1})z^v \right) g(z) + \mu A_{p-1}^{-1} \sum_{v=1}^m \lambda_v g(z)/(z - z_v) \\ &\equiv z^{m+p-1} + \dots \end{aligned}$$

which by Theorem 1 of [Shisha and Walsh, 1961] is an $(m + p - 1)$ th infrapolynomial on S with respect to $(m - 1, m, \dots, m + p - 1)$. By a theorem due to Zedek [cf. Zedek 1955, Walsh and Zedek 1956, and Fekete and Walsh 1957] the number of points of T (which is the number of zeros of $A(z)$, multiplicities taken into account, at which S subtends an angle $< \pi/(p + 1)$) is $\leq p$.

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