ON FIXED POINTS OF AUTOMORPHISMS OF CLASSICAL LIE ALGEBRAS

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1. Introduction. Let A be the automorphism group of a semi-simple Lie algebra $\mathfrak{L}$ over an algebraically closed field of characteristic zero. Let $n(A_i)$ denote the minimal multiplicity of 1 as characteristic root for elements of a connected (algebraic) component A_i of A, and let $m(A_i)$ denote the minimal dimension of fixed point spaces for elements of A_i. Jacobson showed in [3] that $n(A_i) = m(A_i)$, and determined these numbers. It is the purpose of this paper to extend these results to automorphisms of classical Lie algebras over essentially arbitrary fields, using the method of Chevalley [1], as extended by Steinberg [10], for associating such algebras with semi-simple complex Lie algebras.

Throughout the paper fields of characteristics 2 and 3 will be excluded without further mention. The results obtained here are valid in some cases in characteristics 2 and 3, but exclusion of these cases permits considerable simplification of the exposition. All vector spaces in this paper are finite dimensional.

2. Lie algebras and automorphism groups. Let $\mathfrak{L}_C$ be a semi-simple Lie algebra over the complex field C. Let $\mathfrak{H}_C$ be a Cartan subalgebra of $\mathfrak{L}_C$, and let e_i, f_i, h_i $(1 \leqq i \leqq l)$ be a canonical set of generators; i.e. the h_i form a basis for $\mathfrak{H}_C$, and

$$(1)\qquad \begin{aligned} [h_ih_j] &= 0\ , \\ [e_if_j] &= \delta_{ij}h_i\ , \\ [e_ih_j] &= A_{ji}e_i\ , \\ [f_ih_j] &= -A_{ji}f_i\ , \end{aligned}$$

where (A_{ij}) is the Cartan matrix of $\mathfrak{L}_C$. Let $\alpha_i(h_j) = A_{ji}$ for $i, j = 1, 2, \cdots, l$. Then $\pi \equiv \{\alpha_1, \alpha_2, \cdots, \alpha_l\}$ is a fundamental system of roots (of $\mathfrak{L}_C$ with respect to $\mathfrak{H}_C$), and the e_i (respectively, f_i) are root vectors for the α_i (respectively, $-\alpha_i$).

For each (nonzero) root α, let $\mathfrak{L}_\alpha$ denote the root space of α, and let h_α be the unique element of $[\mathfrak{L}_\alpha, \mathfrak{L}_{-\alpha}]$ such that $\alpha(h_\alpha) = 2$. In particular, $h_{\alpha_i} = h_i$, $1 \leqq i \leqq l$.

THEOREM (Chevalley [1]). *$\mathfrak{L}_C$ contains a complete set $\{e_\alpha\}$ of root vectors for the* (nonzero) *roots α such that*

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$$(2)\qquad [e_\alpha e_{-\alpha}] = h_\alpha \qquad \textit{for all } \alpha\,;$$

$$(3)\qquad [e_\alpha e_\beta] = \pm(r+1)e_{\alpha+\beta}\,,$$

for all roots α, β such that $\alpha + \beta$ is a root, where r is the largest integer q such that $\beta - q\alpha$ is a root.

It is easily seen from Chevalley's proof of this theorem that the set $\{e_\alpha\}$ may be taken to contain the e_i and f_i, $1 \leqq i \leqq l$. Furthermore, the h_α are integral linear combinations of the h_i, and the roots are integral linear combinations of the α_i, so the set $\{h_i \mid 1 \leqq i \leqq l\} \cup \{e_\alpha \mid \alpha$ a nonzero root$\}$ is a basis for $\mathfrak{L}_C$ with an *integral* multiplication table contained in (1)–(3) and the relations

$$(4)\qquad [e_\alpha h_i] = \alpha(h_i)e_\alpha\,.$$

Such a basis $\{h_i, e_\alpha\}$ (containing the e_i and f_i) will be called a *Chevalley basis* for $\mathfrak{L}_C$. Henceforth a particular Chevalley basis will be assumed fixed. When it is convenient to do so, linear transformations in $\mathfrak{L}_C$ will be identified with their matrices relative to this basis.

Let K be an arbitrary field, and form a Lie algebra $\mathfrak{L}$ over K, related to $\mathfrak{L}_C$, as in [1]: $\mathfrak{L}$ is the tensor product (over the integers) of the additive group of K with the additive group generated by the Chevalley basis $\{h_i, e_\alpha\}$ of $\mathfrak{L}_C$; $\mathfrak{L}$ is equipped with the multiplication table (1)–(4) after identifying $1_K \otimes e_\alpha$ with e_α, etc. Thus the h_α, e_α, etc., are now thought of as elements of $\mathfrak{L}$, but observe that the subscripts still refer to roots of $\mathfrak{L}_C$.

Let $\mathfrak{H} = \sum_1^l Kh_i$. $\mathfrak{H}$ is an abelian subalgebra of $\mathfrak{L}$, and the roots of $\mathfrak{L}$ relative to $\mathfrak{H}$ are the linear functions $\bar{\alpha}$ defined by $\bar{\alpha}(h_\beta) =$ the class modulo the characteristic of K of $\alpha(h_\beta)$.

We follow the approach of Steinberg [10] in relating the Lie algebras $\mathfrak{L}$ of Chevalley with the Lie algebras of classical type of Mills and Seligman [4]. First let $\mathfrak{L}_C$ be simple. Then we have [10, 2.6]:
(a) No h_α is in the center $\mathfrak{Z}$ of $\mathfrak{L}$.
(b) $\mathfrak{Z} = \{h \in \mathfrak{H} \mid \bar{\alpha}(h) = 0$ for all roots α of $\mathfrak{L}_C\}$.
(c) If $\bar{\mathfrak{L}} = \mathfrak{L}/\mathfrak{Z}$, and $\bar{\mathfrak{H}} = \mathfrak{H}/\mathfrak{Z}$, then $\bar{\mathfrak{L}}$ is simple and $\bar{\mathfrak{H}}$ is a Cartan subalgebra of $\bar{\mathfrak{L}}$.

More generally, if $\mathfrak{L}_C$ is only semi-simple, then $\mathfrak{L}_C = \mathfrak{L}_{1,C} \oplus \cdots \oplus \mathfrak{L}_{r,C}$, where the $\mathfrak{L}_{i,C}$ are (non-abelian) simple ideals in $\mathfrak{L}_C$. Thus $\mathfrak{L} = \mathfrak{L}_1 \oplus \cdots \oplus \mathfrak{L}_r$, where the $\mathfrak{L}_i$ are the Lie algebras of Chevalley corresponding to the $\mathfrak{L}_{i,C}$, and are non-abelian ideals in $\mathfrak{L}$. The center $\mathfrak{Z}_i$ of $\mathfrak{L}_i$ is as described in (b), and the center $\mathfrak{Z}$ of $\mathfrak{L}$ is $\mathfrak{Z}_1 \oplus \cdots \oplus \mathfrak{Z}_r$. Furthermore, $\mathfrak{L}/\mathfrak{Z} \cong (\mathfrak{L}_1/\mathfrak{Z}_1) \oplus \cdots \oplus (\mathfrak{L}_r/\mathfrak{Z}_r)$. Every such algebra $\bar{\mathfrak{L}} = \mathfrak{L}/\mathfrak{Z}$ will be called a *classical Lie algebra*. (These are essentially the Lie algebras of classical type of Mills and Seligman, although some additional algebras over fields of characteristics 2 and 3 can be obtained by the process described here.)

If $\mathfrak{L}_C$ is simple, $\mathfrak{Z} \neq 0$ if and only if $\mathfrak{L}_C$ is of type A_l and the characteristic p of K divides $l+1$. In this case, $\mathfrak{Z}$ is one-dimensional [8, § 1].

Let A_C denote the automorphism group of $\mathfrak{L}_C$. As an algebraic group, A_C has a decomposition

$$A_C = A_0 \cup A_1 \cup \cdots \cup A_{r-1} \tag{5}$$

into connected (algebraic) components, where A_0 is the component of the identity automorphism. (The terminology of algebraic groups will be seen to be more natural here than that of topological groups.)

An *automorphism of the Cartan matrix* (A_{ij}) of $\mathfrak{L}_C$ is a permutation s of the numbers $1, 2, \cdots, l$ such that $A_{ij} = A_{s(i), s(j)}$ for all i, j. Associated with such a permutation s is a unique automorphism σ of $\mathfrak{L}_C$ such that $e_i^\sigma = e_{s(i)}, f_i^\sigma = f_{s(i)}$, $i = 1, 2, \cdots, l$ [2, p. 280]. Following Steinberg, we call σ a *graph* automorphism of $\mathfrak{L}_C$. The set F of graph automorphisms is a finite group, and the elements of $F = \{1, \sigma_1, \cdots, \sigma_{r-1}\}$ form a system of coset representatives of A_0 in A_C [2, Chapter IX; 3, Corollary to Theorem 6]:

$$A_C = A_0 \cup \sigma_1 A_0 \cup \cdots \cup \sigma_{r-1} A_0 \, . \tag{6}$$

This decomposition coincides with (5), and the number r of algebraic components is also the order of F.

For each root α and each complex number t, let $x_\alpha(t)$ denote the automorphism $\exp(t \operatorname{ad} e_\alpha)$ of $\mathfrak{L}_C$. The significance of the Chevalley basis for automorphisms is that the matrix of every $x_\alpha(t)$ has entries which are polynomials in t with integer coefficients [1]. Let $x_\alpha(\xi)$ denote the matrix obtained from $x_\alpha(t)$ by replacing the complex parameter t by an indeterminate ξ. We can then replace ξ by an arbitrary element t of K to obtain a matrix over K, again denoted $x_\alpha(t)$. Considered as a linear transformation of $\mathfrak{L}$ relative to the Chevalley basis, $x_\alpha(t)$ is an automorphism.

We also introduce certain diagonal (relative to the Chevalley basis) automorphisms of $\mathfrak{L}$. Let k be any homomorphism of the additive group generated by the roots of $\mathfrak{L}_C$ into the multiplicative group K^*. We associate with k the automorphism $\eta(k)$ of $\mathfrak{L}$ defined by $h\eta(k) = h$ for $h \in \mathfrak{H}$, $e_\alpha \eta(k) = k(\alpha) e_\alpha$ for α a root of $\mathfrak{L}_C$. In particular, we can associate a homomorphism k with each $t \in K^*$ and each root α of $\mathfrak{L}_C$ by defining $k(\beta) = t^{\beta(h_\alpha)}$ for each root β. The corresponding automorphism will be denoted $z_\alpha(t)$.

Next we associate automorphisms of $\mathfrak{L}$ with the graph automorphisms of $\mathfrak{L}_C$. Let σ be a graph automorphism with associated permutation s. We have $h_i^\sigma = [e_i^\sigma, f_i^\sigma] = [e_{s(i)}, f_{s(i)}] = h_{s(i)}$, so σ permutes the h_i's. For an arbitrary root $\gamma = \sum k_i \alpha_i$, let $\gamma' = \sum k_i \alpha_{s(i)}$. γ' is a root [2, p. 122, XVI] and one can show that $e_\gamma^\sigma = \pm e_{\gamma'}$. This is done by induction on the level (i.e. $\sum |k_i|$) of γ. Hence, relative to the Chevalley basis, the matrix of σ has only the numbers 0, 1, -1 as entries (and in fact, exactly one nonzero entry in each row and column). Thus the matrix of σ defines an automorphism σ of $\mathfrak{L}$ over K. These automorphisms will also be called graph automorphisms.

The automorphism group of $\bar{\mathfrak{L}}$ is isomorphic to the automorphism group of $\mathfrak{L}$ [10, p. 1122]. We will therefore identify automorphisms of $\mathfrak{L}$ with their induced automorphisms in $\bar{\mathfrak{L}}$, but all references to matrices will mean relative to the Chevalley basis in $\mathfrak{L}$.

The group G of Chevalley is the group of automorphisms of $\mathfrak{L}$ (or $\bar{\mathfrak{L}}$) generated by the $x_\alpha(t)$ for all roots α and $t \in K$ and the $\eta(k)$ for all homomorphisms k of the additive group generated by the roots into K^*.

THEOREM (Steinberg). *If A is the automorphism group of $\mathfrak{L}$ (or $\bar{\mathfrak{L}}$), G the Chevalley group, and $F = \{1, \sigma_1, \cdots, \sigma_{r-1}\}$ the group of graph automorphisms, then G is normal in A, and*

$$A = G \cup \sigma_1 G \cup \cdots \cup \sigma_{r-1} G \tag{7}$$

is the coset decomposition of A over G.

Steinberg proves this theorem in [10] only for the case of $\mathfrak{L}_C$ simple, but the extension to the semi-simple case is straightforward if one considers the action of A in $\bar{\mathfrak{L}}$. The analogy between equations (7) and (6) is clear; in fact, they coincide if K is an algebraically closed field of characteristic zero. However (7) is also analogous to (5) by the following result.

THEOREM (Ono [5, Theorem 3]). *If K is infinite, and the Killing form of $\mathfrak{L}_C$ is nondegenerate modulo the characteristic of K, then G is the algebraic component of* 1 *in A, and* (7) *is the decomposition of A into connected algebraic components.*

3. **Indices of automorphism groups.** For each component (or coset) A_i of A_C define the index $n(A_i)$ to be the minimal multiplicity of the characteristic root 1 for elements of A_i. For each $\eta \in A_C$, let $\mathfrak{F}(\eta)$ denote the subspace of $\mathfrak{L}_C$ of η-fixed points. Define another index $m(A_i)$ to be the minimal $\dim \mathfrak{F}(\eta)$, $\eta \in A_i$. We have [3, Theorem 6 and Corollary, Theorem 10]:

THEOREM (Jacobson). *Let σ_i be the unique element of F in A_i,* *and let s_i be the associated automorphism of the Cartan matrix. Then $n(A_i) = m(A_i) =$ the number of cycles in the decomposition of s_i into disjoint cycles.*

COROLLARY. *$n(A_0) = l = \dim \mathfrak{H}_C$, and $0 < n(A_i) < l$ if $i \neq 0$.*

In view of Steinberg's theorem in the previous section, it is reasonable to ask for the relationship between $n(A_i)$ and both the minimal multiplicity $n(\sigma_i G)$ of 1 as characteristic root and the minimal dimension $m(\sigma_i G)$ of fixed point spaces for elements of $\sigma_i G$ in the automorphism group A of $\mathfrak{L}$. (Obviously a distinction between $\mathfrak{L}$ and $\mathfrak{L}$ must be maintained here; we will consider $\bar{\mathfrak{L}}$ in § 4.)

In the sequel we will make use of the subgroup G' of G generated by the automorphisms $x_\alpha(t)$ for α a root of $\mathfrak{L}_C$ and $t \in K$. For each root α and each $t \in K^*$, $z_\alpha(t) \in G'$, and if K is algebraically closed, $G' = G$ [1, § IV].

THEOREM 1. *Let $\mathfrak{L}_C$, A_C, A_i, K, $\mathfrak{L}$, A, G, and σ_i be as defined above. Then $n(\sigma_i G) \geqq m(\sigma_i G) \geqq n(A_i)$.*

Proof. The first inequality is clear. We first assume K is algebraically closed, so that G is generated by the $x_\alpha(t)$. We have seen that an arbitrary element η of A can be written as a product of exactly one $\sigma_i \in F$ and certain $x_\alpha(t_j)$'s in some order. Thinking now of matrices, η is then a specialization of a corresponding product $\eta(\xi)$ of matrices σ_i, $x_\alpha(\xi_j)$, where the ξ's are indeterminates, one for each x-type factor. Since the entries of $x_\alpha(\xi_j)$ are polynomials in ξ_j with integer coefficients, $\eta(\xi)$ is a matrix whose entries are polynomials in certain indeterminates $\xi_1, \xi_2, \cdots, \xi_m$ with integer coefficients.

The number m of indeterminates appearing in a matrix $\eta(\xi)$ depends not only on the automorphism η but on the choice of a representation of η as a product of the generators; this number plays no special role here, but it must not be assumed to be constant.

The integer coefficients of the polynomial entries of $\eta(\xi)$ may be chosen so that specialization of the ξ_j to complex numbers t_j gives an element $\eta(t)$ of A_C, and the choice of σ_i determines the component in which $\eta(t)$ lies.

Let σ_i be fixed, and let $l_i = n(A_i)$. The fact that $l_i \leqq \dim \mathfrak{F}(\eta)$ for $\eta \in A_i$ can be expressed as follows: for every specialization $\xi_j \to t_j \in C$, rank $(\eta(t) - I) \leqq n - l_i$, where $n = \dim \mathfrak{L}_C = \dim \mathfrak{L}$. A similar statement can be made for $\eta(\xi)$, for if $\eta(\xi) - I$ had a nonzero minor of size $> n - l_i$, that minor would be a polynomial and would remain nonzero under some specialization $\xi_j \to t_j \in C$. Hence we see that for every $\eta(\xi)$ corresponding to σ_i (i.e. for every element $\eta \in \sigma_i G$ and for every representation of η as a product of σ_i and certain of the other generators) we have rank $(\eta(\xi)-I) \leqq n - l_i$. But then specializing $\xi_j \to t_j \in K$, the rank of such a matrix certainly cannot increase. Hence rank $(\eta - I) \leqq n - l_i$ for every $\eta \in \sigma_i G$, or in other words $m(\sigma_1 G) \geqq l_i$.

Now drop the assumption of algebraic closure on K, and let Ω be the algebraic closure of K. If η is an arbitrary element of $\sigma_i G$, then the extension of η to an automorphism of $\mathfrak{L}_\Omega$ is still in the component of $A(\mathfrak{L}_\Omega)$ corresponding to σ_i. This is clear, because $\eta = \sigma_i \tau$, $\tau \in G$, and τ can be expressed as a product of the generators of G, whose extensions to $\mathfrak{L}_\Omega$ are elements of $G(\mathfrak{L}_\Omega)$. Hence $\dim \mathfrak{F}(\eta) = \dim \mathfrak{F}(\eta_\Omega) \geqq l_i$ for $\eta \in \sigma_i G$. This completes the proof of Theorem 1.

THEOREM 2. *Let $\mathfrak{L}_C$, A_C, A_i, K, $\mathfrak{L}$, A, G, and σ_i be as in Theorem 1, and suppose further that K is infinite. Then $m(\sigma_i G) = m(A_i)$. For $i = 0$, $n(G) = n(A_0) = l$. If, in addition, the characteristic of K does not divide the length of any cycle in the permutation associated with σ_i, then $n(\sigma_i G) = n(A_i)$. In particular, this is the case if $\mathfrak{L}_C$ is simple.*

Proof. For the Chevalley group itself, we consider the diagonal automorphisms (or matrices) $z_\alpha(t) = \operatorname{diag}\{1, 1, \cdots, 1, \cdots, t^{\beta(h_\alpha)}, \cdots\}$, where each of the first l elements is 1, and the following entries are of the form $t^{\beta(h_\alpha)}$ where β runs through all the roots of $\mathfrak{L}_C$. For some selection of $t_1, t_2, \cdots, t_l \in K$, to be determined presently, let $\eta = \prod_1^l z_{\alpha_i}(t_i)$. The diagonal entries of η after the lth one are of the form $\prod_1^l t_i^{\beta(h_i)}$. For each root β, some $\beta(h_i) \neq 0$. Thus each of these entries is a rational expression in the t_i which is not identically 1. Since K is infinite, we can choose $t_1, \cdots, t_l$ so that none of the diagonal entries of η after the lth one is 1. (This can be expressed as a polynomial condition of degree $\leqq 3(n - l)$, where $n = \dim \mathfrak{L}$, since $|\beta(h_i)| \leqq 3$.) Thus η is an element of G for which $l = \dim \mathfrak{F}(\eta) =$ the multiplicity of 1 as characteristic root.

Now consider an element $\sigma \neq 1$ in F. σ maps $\mathfrak{H}$ into itself, and also maps the subspace $\mathfrak{S}$ spanned by the root vectors $\{e_\beta\}$ into itself. In $\mathfrak{H}$, σ acts as a permutation of the h_i, and in $\mathfrak{S}$ (as noted above) the matrix of σ has only 0, ± 1 as entries, and exactly one nonzero entry in each row and column. If η is chosen as in the previous paragraph, we have $\sigma\eta \,|\, \mathfrak{H} = \sigma \,|\, \mathfrak{H}$ (where the bar denotes restriction), and $\sigma\eta \,|\, \mathfrak{S}$ has nonzero entries where $\sigma \,|\, \mathfrak{S}$ does and each of these entries will be $\pm$ one of the entries of $\eta \,|\, \mathfrak{S}$. If K is infinite, then the t_i selected to define η can be chosen to satisfy not only the conditions imposed above, but also the condition that 1 not be a characteristic root of $\sigma\eta \,|\, \mathfrak{S}$.

Next consider the permutation matrix $\sigma \,|\, \mathfrak{H}$. For a suitable arrangement of the basis $h_1, \cdots, h_l$ of $\mathfrak{H}$, this matrix consists of diagonal blocks, where each block is the matrix of a cyclic permutation. Let T be a linear transformation in a k-dimensional space which cyclically permutes a basis $u_1, u_2, \cdots, u_k$. Then the fixed point space of T is spanned by $u_1 + u_2 + \cdots + u_k$. The characteristic polynomial of T (up to sign) is $(\lambda - 1)(\lambda^{k-1} + \lambda^{k-2} + \cdots + \lambda + 1)$. 1_K is a root of the second factor if and only if $k \cdot 1_K = 0$. Thus the multiplicity of 1 as characteristic root of T is 1 if and only if the characteristic of K does not divide k.

We have demonstrated that each cycle of s contributes exactly one dimension to the fixed point space of $\sigma \mid \mathfrak{H}$, and, if the characteristic does not divide the length of the cycle, exactly 1 to the multiplicity of 1 as characteristic root. If $\mathfrak{L}_C$ is simple, only cycles of lengths $\leqq 3$ occur, which completes the proof of Theorem 2.

COROLLARY. *Let $\mathfrak{L}$ be a split semi-simple Lie algebra over an arbitrary field of characteristic zero, and let $A = G \cup \sigma_1 G \cup \cdots \cup \sigma_{r-1} G$ be the automorphism group of $\mathfrak{L}$. Then $m(\sigma_i G) = n(\sigma_i G) =$ the number l_i of cycles in the decomposition of the permutation s_i. For G itself, $l_0 = l$, the dimension of a Cartan subalgebra, and for $i \neq 0$, $0 < l_i < l$.*

REMARKS. (a) The corollary extends the results of Jacobson [3] beyond the algebraically closed case. Part of this is essentially contained in [3] in remarks following Theorem 10.

(b) The decomposition of A in the corollary is also the decomposition into connected algebraic components, by Ono's theorem in § 2.

We will consider in the remaining sections the extent to which the exclusion of small fields is necessary to obtain the conclusions of Theorem 2. In particular, we will answer this explicity for the Chevalley group for algebras of types A, B, C, and D.

There is also the question of how these results may be extended to the algebras $\bar{\mathfrak{L}}$, in the case where one or more components are of type A_l, $p \mid l+1$. In the following section we will obtain explicit results in the case where $\mathfrak{L}_C$ itself is simple of type A_l, $p \mid l+1$.

4. Algebras of type A. Let $\mathfrak{L}_C$ be simple of type A_l. Then $\mathfrak{L}$ can be taken to be the Lie algebra of all $(l+1) \times (l+1)$ matrices of trace 0 over K. If A is any nonsingular $(l+1) \times (l+1)$ matrix, then the mapping $X \to A^{-1}XA$ is an automorphism η of $\mathfrak{L}$. This automorphism is in G, by [9, § 2] and the last paragraph of the proof of Theorem 1.

THEOREM 3. *If $\mathfrak{L}_C$ is of type A_l and K is any field (of charac-*

teristic $\neq 2, 3$), *then* $m(G) = l$. *If* $|K| > l + 1$, *then* $n(G) = l$.

Proof. Let η be an automorphism given by conjugation by a cyclic matrix A. The space of all matrices commuting with A (i.e. all polynomials in A) has dimension $l + 1$, since the minimum polynomial of A has degree $l + 1$. $\mathfrak{F}(\eta)$ is the intersection of this space with $\mathfrak{L}$, and has dimension l.

An alternate approach to selecting an $\eta \in G$ gives a slightly weaker result, but also gives an automorphism having 1 as characteristic root with multiplicity l. Let $\eta : X \to A^{-1}XA$ where $A = \operatorname{diag}\{a_1, a_2, \cdots, a_{l+1}\}$, the a_i being all distinct and all different from 0. This requires $|K| > l + 1$. Take as basis for $\mathfrak{L}$ the matrix units e_{ij}, $i \neq j$, and the diagonal matrices $h_i = e_{i+1,i+1} - e_{ii}$, $1 \leqq i \leqq l$. Then $h_i^\eta = h_i$, and $e_{ij}^\eta = a_i^{-1}a_je_{ij}$. Since $a_i^{-1}a_j \neq 1$ for $i \neq j$, we have $l = \dim \mathfrak{F}(\eta) =$ the multiplicity of 1 as characteristic root, which completes the proof.

Now suppose the characteristic p of K divides $l + 1$. Then $\mathfrak{L}$ has one-dimensional center $\mathfrak{Z}$ consisting of scalar multiples of the identity matrix. A more convenient basis than the one listed above is obtained by replacing h_l by $I = lh_1 + (l-1)h_2 + \cdots + 2h_{l-1} + h_l$, and taking this to be the first basis vector. The cosets of the remaining basis vectors then form a basis for $\bar{\mathfrak{L}} = \mathfrak{L}/\mathfrak{Z}$.

Since $l > 1$, we have one nontrivial graph automorphism σ with associated permutation $(1, l)(2, l-1)\cdots$, in which the number of cycles is $[(l+1)/2]$. We will denote by $\bar{n}(G)$ the minimal multiplicity of 1 as characteristic root for elements of G acting in $\bar{\mathfrak{L}}$, and similarly define $\bar{n}(\sigma G)$, $\bar{m}(G)$, $\bar{m}(\sigma G)$.

THEOREM 4. *Let $\bar{\mathfrak{L}}$ be a (simple) classical Lie algebra of type A_l over a field K of characteristic p, where $p \mid l + 1$. Let $A = G \cup \sigma G$ be the automorphism group of $\bar{\mathfrak{L}}$. Then $\bar{n}(G) \geqq \bar{m}(G) \geqq l - 1$, and $\bar{n}(\sigma G) \geqq \bar{m}(\sigma G) \geqq [(l+1)/2]$. If $|K| > l + 1$, then $\bar{n}(G) = \bar{m}(G) = l - 1$, and if K is infinite, then $\bar{n}(\sigma G) = \bar{m}(\sigma G) = [(l+1)/2]$.*

Proof. We observe first that $I^\sigma = (lh_1 + (l-1)h_2 + \cdots + 2h_{l-1} + h_l)^\sigma = lh_l + (l-1)h_{l-1} + \cdots + 2h_2 + h_1 = -I$. Every element of the subgroup G' of G acts by a conjugation in $\mathfrak{L}$ [6, (3.5)], so I is a fixed point of every element of G'. G is generated by G' and certain automorphisms leaving $\mathfrak{H} = \sum Kh_i$ pointwise fixed, so I is fixed under every element of G. On the other hand, if $\eta = \sigma\tau$, $\tau \in G$, then $I^\eta = (-I)^\tau = -I$, so I is not fixed under η.

Relative to the bases chosen above for $\mathfrak{L}$ and $\bar{\mathfrak{L}}$, every automorphism η of $\mathfrak{L}$ has a matrix of the form

$$A = \left[\begin{array}{c|c} a_1 & 0 \cdots 0 \\ \hline a_2 & \\ \cdots & B \\ a_n & \end{array}\right] \tag{8}$$

where B is the matrix of the induced automorphism $\bar{\eta}$ in $\bar{\mathfrak{L}}$. We have just seen that $a_1 = 1$ if $\eta \in G$ and $a_1 = -1$ if $\eta \in \sigma G$. For any η, the characteristic polynomial of A is

$$f(\lambda\,;\eta) = (\lambda - a_1)f(\lambda\,;\bar{\eta})\,, \tag{9}$$

where $f(\lambda\,;\bar{\eta})$ is the characteristic polynomial of B. Thus for $\eta \in G$, the multiplicity of 1 as characteristic root of $\bar{\eta}$ is exactly 1 less than that for η. In particular, if $|K| > l + 1$, $\bar{n}(G) \leqq l - 1$.

Now for $\eta \in G$, $\mathfrak{Z} \subseteqq \mathfrak{F}(\eta)$, hence $\dim \overline{\mathfrak{F}(\eta)} = \dim \mathfrak{F}(\eta) - 1$ (where the bar denotes image under $\mathfrak{L} \to \bar{\mathfrak{L}}$). Clearly $\overline{\mathfrak{F}(\eta)} \subseteqq \mathfrak{F}(\bar{\eta})$, so $l - 1 \leqq \bar{m}(G) \leqq \bar{n}(G)$. Again, if $|K| > l + 1$, $\bar{n}(G) = l - 1$.

On the other hand, if $\eta \in \sigma G$, $\mathfrak{Z} \cap \mathfrak{F}(\eta) = 0$, so $\dim \mathfrak{F}(\eta) = \dim \overline{\mathfrak{F}(\eta)} \leqq \dim \mathfrak{F}(\bar{\eta})$, and $\bar{m}(\sigma G) \geqq [(l + 1)/2]$. By (9), the multiplicity of 1 as characteristic root must be the same for η and $\bar{\eta}$. Hence if K is infinite, then $\bar{n}(\sigma G) = \bar{m}(\sigma G) = [(l + 1)/2]$.

5. Simple algebras of types B, C, D.

Let $\mathfrak{L}_G$ be simple of type B_l, C_l, or D_l. Then $\mathfrak{L}$ can be taken to be the Lie algebra of $n \times n$ matrices X over K ($n = 2l$ or $2l + 1$) such that $X = -S^{-1}X'S$, where X' is the transpose of X, and S is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & I_l \\ 0 & I_l & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & I_l \\ -I_l & 0 \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} 0 & I_l \\ I_l & 0 \end{bmatrix}$$

in the respective cases B, C, or D. If A is any matrix such that $ASA' = S$, then $X \to A^{-1}XA$ is an automorphism of $\mathfrak{L}$, and, as for type A_l, is in the Chevalley group. We will select in each case a diagonal matrix A which defines an automorphism of $\mathfrak{L}$ having l-dimensional fixed point space, after discarding a suitable number of small fields. The orthogonality condition requires that A be of the form diag $\{a_1, a_2, \cdots, a_l, a_1^{-1}, a_2^{-1}, \cdots, a_l^{-1}\}$ in cases C and D and of the form diag $\{1, a_2, a_3, \cdots, a_{l+1}, a_2^{-1}, \cdots, a_{l+1}^{-1}\}$ in case B.

Theorem 5. *Let $\mathfrak{L}$ be a simple classical Lie algebra of type B_l, C_l, or D_l over a field K, and let G be its Chevalley group. Then $n(G) = m(G) = l$ if $|K| > 2l$, $2l + 1$, or $2l - 1$ in the respective cases B_l, C_l, D_l.*

Proof. First consider case C. Denoting matrix units by e_{ij}, a basis for $\mathfrak{L}$ [7, § XVII] is

$$\begin{aligned}
h_i &= e_{ii} - e_{i+l,\,i+l} & ; \\
e_{(-i,\,j)} &= e_{ij} - e_{j+l,\,i+l}\,, & i \neq j\,; \\
e_{(-i,\,-j)} &= e_{i,\,j+l} + e_{j,\,i+l}\,, & i < j\,; \\
e_{(i,\,j)} &= e_{i+l,\,j} + e_{j+l,\,i}\,, & i < j\,; \\
e_{(-2i)} &= e_{i,\,i+l} & ; \\
e_{(2i)} &= e_{i+l,\,i} & ;
\end{aligned}$$

where in all cases $i, j = 1, 2, \cdots, l$. If we choose A as above, then conjugation by A acts diagonally, leaving the h_i fixed, and the remaining diagonal elements have the forms $a_i^{-1}a_j$, $a_i^{-1}a_j^{-1}$, a_ia_j $(i \neq j)$, a_i^{-2}, a_i^2. Hence we wish to choose the a_i so that no a_i is 0, 1, -1, or $a_j^{\pm 1}$ for $j \neq i$; in other words, so that

$$\textstyle\prod_1^l a_i(a_i^2 - 1) \prod_{i<j}(a_i - a_j)(a_ia_j - 1) \neq 0\,.$$

The left-hand side of this inequality is a polynomial of degree $2l + 1$ in each of the a_i. Thus there exist such elements in K if $|K| > 2l + 1$.

The details for types B and D are similar, and appropriate bases are given in [7, § XVII]. For type B the same conditions are obtained except that some a_i may be -1. Hence $|K| > 2l$ suffices. For type D, both 1 and -1 are allowed, so $|K| > 2l - 1$ suffices.

REMARK. Professor G. B. Seligman has communicated to the author a proof that $m(G) = l$ when $\mathfrak{L}$ is of type B_l, C_l, or D_l, over any field K of characteristic $\neq 2$ or 3. His proof is a natural analog of the first part of the proof of Theorem 3, although the details are naturally more complicated. As in Theorem 3, this approach does not yield $n(G) = l$.

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