A CHARACTERIZATION OF WEAK* CONVERGENCE

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1. Introduction. Let X be a locally compact, Hausdorff space and $\{\mu_i ; i \in D\}$ be a net of Radon measures on X (in the sense of Caratheodory). The weak* or vague limit of this net is the Radon measure ν such that

$$\lim_i \int f d\mu_i = \int f d\nu$$

for every continuous function f vanishing outside some compact set. In this paper, we construct in § 3 a Radon measure φ^* from a given base $\mathscr{B}$ for the topology of X and $\liminf_i \mu_i$ and then, in § 4, we give necessary and sufficient conditions for φ^* to be the weak* limit of the μ_i. In particular, if the latter exists then it is the φ^* generated when $\mathscr{B}$ is the family of all open sets.

The measure φ^* is obtained from another measure φ by a standard regularizing process. The definition of φ easily extends to abstract spaces but that of φ^* makes essential use of the topology. Thus, it is of some importance to know when $\varphi = \varphi^*$, that is, when a measure constructed through an abstract process from the μ_i turns out to be, in the topological situation, the weak* limit of the μ_i. In Theorem 3.3 we give a condition for $\varphi = \varphi^*$ and in § 5 we give an example to show that the condition cannot be eliminated.

We refer to standard texts such as Halmos [1], Kelley [2], and Munroe [3] for the elementary properties and concepts of topology and measure theory used in this paper.

2. Notation.

2.1 ω denotes the set of natural numbers.

2.2 0 denotes both the empty set and the smallest number in ω.

2.3 μ is a Caratheodory (outer) measure on X if and only if μ is a function on the family of all subsets of X such that $\mu 0 = 0$ and

$$0 \leqq \mu A \leqq \sum_{n \in \omega} \mu B_n \leqq \infty \quad \text{whenever } A \subset \bigcup_{n \in \omega} B_n \subset X .$$

2.4 For μ *a* Caratheodory measure on X, A is μ-measurable if and only if $A \subset X$ and for evey $T \subset X$

$$\mu T = \mu(T \cap A) + \mu(T - A) .$$

2.5 For X a topological space, μ is a Radon measure on X if and

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only if μ is a Caratheodory measure on X such that:

(i) open sets are μ-measurable,

(ii) if C is compact then $\mu C < \infty$,

(iii) if α is open then $\mu\alpha = \sup\{\mu C ; C \text{ compact, } C \subset \alpha\}$,

(iv) if $A \subset X$ then $\mu A = \inf\{\mu\alpha ; \alpha \text{ open, } A \subset \alpha\}$.

2.6 For X a topological space, $C_0(X)$ is the family of all real-valued continuous functions on X vanishing outside some compact set.

2.7 $(D, <)$ is a directed set if and only if $D \neq 0$, D is partially ordered by $<$ so that for any $i, j \in D$ there exists $k \in D$ with $i < k$ and $j < k$.

2.8 A net is a function on a directed set.

2.9 $\bar{A}$ denotes the closure of A.

3. **The lim inf measure.** Let X be a regular topological space; $\mathscr{B}$ be a base for the topology of X, closed under finite unions and intersections; $(D, <)$ be a directed set and, for each $i \in D$, μ_i be a Radon measure on X.

For each $\alpha \in \mathscr{B}$, let

$$g\alpha = \varliminf_{i \in D} \mu_i\alpha \; (= \sup_{j \in D} \inf_{\substack{i \in D \\ j < i}} \mu_i\alpha) < \infty$$

and let φ be the Caratheodory measure on X generated by g and $\mathscr{B}$ (see method I of Munroe [3]), i.e. for each $A \subset X$,

$$\varphi A = \inf\left\{\sum_{\alpha \in H} g\alpha \,;\; H \text{ countable, } H \subset \mathscr{B},\; A \subset \bigcup_{\alpha \in H} \alpha\right\}.$$

As we show in § 5, φ need not be a Radon measure even when X is compact and Hausdorff. For this reason, for any $A \subset X$ let

$$\varphi^* A = \inf_{\substack{\alpha \text{ open} \\ A \subset \alpha}} \; \sup_{\substack{C \text{ compact} \\ C \subset \alpha}} \varphi C .$$

We then have the following:

3.1 THEOREM. *φ is a Caratheodory measure on X such that:*

(i) *if A and B are disjoint, closed, compact sets then $\varphi(A \cup B) = \varphi A + \varphi B$.*

(ii) *if $A \subset X$ then $\varphi A = \inf\{\varphi\alpha;\ \alpha \text{ open, } A \subset \alpha\}$.*

(iii) *if C is compact and for every $\alpha \in \mathscr{B}$, $g\alpha = \lim_i \mu_i\alpha$ then*

$$\varphi C = \inf\{g\alpha;\ \alpha \in \mathscr{B},\ C \subset \alpha\} .$$

3.2 THEOREM. *φ^* is a Radon measure on X such that:*

(i) $\varphi^* \leqq \varphi$.

(ii) *if C is compact then $\varphi^* C = \varphi C$.*

3.3 THEOREM. *If every open set in X is the countable union of compacta then $\varphi^* = \varphi$.*

Proofs

Proof of 3.1

(i) Let A, B be closed, compact and $A \cap B = 0$. Since X is regular and $\mathscr{B}$ is closed to finite unions, there exist $\alpha, \beta \in \mathscr{B}$ such that $A \subset \alpha$, $B \subset \beta$ and $\alpha \cap \beta = 0$. Given $\varepsilon > 0$, choose $\gamma_n \in \mathscr{B}$ for $n \in \omega$ so that $A \cup B \subset \bigcup_{n \in \omega} \gamma_n$ and

$$\sum_{n \in \omega} g\gamma_n \leqq \varphi(A \cup B) + \varepsilon .$$

Let $\gamma'_n = \gamma_n \cap \alpha$ and $\gamma''_n = \gamma_n \cap \beta$. Then $\gamma'_n, \gamma''_n \in \mathscr{B}$, $A \subset \bigcup_{n \in \omega} \gamma'_n$, $B \subset \bigcup_{n \in \omega} \gamma''_n$ and hence

$$\varphi A + \varphi B \leqq \sum_{n \in \omega} (g\gamma'_n + g\gamma''_n) \leqq \sum_{n \in \omega} g\gamma_n \leqq \varphi(A \cup B) + \varepsilon .$$

Since ε is arbitrary and φ is a Caratheodory measure we have $\varphi(A \cup B) = \varphi A + \varphi B$.

(ii) Let $A \subset X$. If $\varphi A = \infty$ then the conclusion is trivial. So, let $\varphi A < \infty$ and $\varepsilon > 0$. Then there exists a countable $H \subset \mathscr{B}$ such that $A \subset \bigcup_{\alpha \in H} \alpha$ and

$$\sum_{\alpha \in H} g\alpha \leqq \varphi A + \varepsilon$$

and therefore

$$\varphi(\bigcup_{\alpha \in H} \alpha) \leqq \sum_{\alpha \in H} \varphi\alpha \leqq \sum_{\alpha \in H} g\alpha \leqq \varphi A + \varepsilon .$$

(iii) Suppose for every $\alpha \in \mathscr{B}$, $g\alpha = \lim_i \mu_i \alpha$. Then for $\alpha_0, \cdots, \alpha_n$ in $\mathscr{B}$ we have

$$\begin{aligned} \sum_{k=0}^{n} g\alpha_k &= \lim_i \sum_{k=0}^{n} \mu_i \alpha_k \\ &= \lim_i \mu_i\Big(\bigcup_{k=0}^{n} \alpha_k\Big) \\ &= g\Big(\bigcup_{k=0}^{n} \alpha_k\Big) . \end{aligned}$$

Hence for any compact C,

$$\varphi C = \inf \{g\alpha ; \alpha \in \mathscr{B}, C \subset \alpha\} .$$

Proof of 3.2

(i) Clearly, for any compact C, $\varphi C < \infty$ and, for any open α,

$$\varphi^*\alpha = \sup\{\varphi C\,;\ C \text{ compact},\ C \subset \alpha\} \leqq \varphi\alpha\ .$$

Thus, for any $A \subset X$, using 3.1 (ii) we have

$$\begin{aligned}\varphi^* A &= \inf\{\varphi^*\alpha\,;\ \alpha \text{ open},\ A \subset \alpha\} \\ &\leqq \inf\{\varphi\alpha\,;\ \alpha \text{ open},\ A \subset \alpha\} \\ &= \varphi A\ .\end{aligned}$$

(ii) For any compact C and open $\alpha \supset C$, we have $\varphi C \leqq \varphi^*\alpha$, hence $\varphi C \leqq \varphi^* C$. By (i) then $\varphi^* C = \varphi C$.

(iii) To see that φ^* is a Radon measure, we now only need to check that open sets are φ^*-measurable. Let α be open, $T \subset X$ and $\varepsilon > 0$. Let T' be open, $T \subset T'$ and $\varphi^* T' < \varphi^* T + \varepsilon$. Note that if C is compact, β is open and $C \subset \beta$ then, by regularity, $\bar{C} \subset \beta$. Thus, since $T' \cap \alpha$ is open, there exists a closed, compact $C_1 \subset T' \cap \alpha$ with $\varphi^*(T' \cap \alpha) \leqq \varphi C_1 + \varepsilon$. Also, since $T' - C_1$ is open, there exists a closed compact $C_2 \subset T' - C_1$ with $\varphi^*(T' - C_1) \leqq \varphi C_2 + \varepsilon$. Then

$$\begin{aligned}\varphi^*(T \cap \alpha) + \varphi^*(T - \alpha) &\leqq \varphi^*(T' \cap \alpha) + \varphi^*(T' - C_1) \\ &\leqq \varphi C_1 + \varphi C_2 + 2\varepsilon \\ &= \varphi(C_1 \cup C_2) + 2\varepsilon \quad \text{(by 3.1 (i))} \\ &\leqq \varphi^* T' + 2\varepsilon \\ &\leqq \varphi^* T + 3\varepsilon\ .\end{aligned}$$

Proof of 3.3. We need only show that $\varphi^* A = \varphi A$ for open A. Given such A, by assumption, $A = \bigcup_{n\in\omega} C_n$ where the C_n are compact and $C_n \subset C_{n+1}$. Because of regularity, we may assume that the C_n are closed compact. We shall show that $\varphi A = \lim_n \varphi C_n$. To this end, let $\varepsilon > 0$ and define α_n and C'_n by recursion as follows: let $C' = C_0$ and, for any $n \in \omega$, let α_n be open, $C'_n \subset \alpha_n$, $\varphi\alpha_n \leqq \varphi C'_n + \varepsilon/2^{n+1}$ and

$$C'_{n+1} = C_{n+1} - \bigcup_{j=0}^{n} \alpha_j\ .$$

Then the C'_n are closed compact, mutually disjoint and $A \subset \bigcup_{n\in\omega}\alpha_n$. Thus,

$$\begin{aligned}\varphi A &\leqq \sum_{n\in\omega}\varphi\alpha_n \leqq \sum_{n\in\omega}\varphi C'_n + \varepsilon \\ &= \lim_N \sum_{n=0}^{N}\varphi C'_n + \varepsilon = \lim_N \varphi\Big(\bigcup_{n=0}^{N} C'_n\Big) + \varepsilon \\ &\leqq \lim_N \varphi C_N + \varepsilon\ .\end{aligned}$$

4. Weak* convergence. Let X be a locally compact, Hausdorff space, $\mathscr{M}$ be the family of Radon measures on X, μ be a net in $\mathscr{M}$. It is well known that $\mathscr{M}$ can be identified with the set of positive linear functionals on $C_0(X)$ so that the weak* or vague limit of the μ_i is defined by

4.1. DEFINITION. $(W^*)\text{-}\lim_i \mu_i = \nu$ if and only if $\nu \in \mathscr{M}$ and, for every $f \in C_0(X)$,

$$\lim_i \int f d\mu_i = \int f d\nu .$$

On the other hand, for any base $\mathscr{B}$ for the topology of X, let

4.2. DEFINITION. $\mathscr{B}\text{-}\underline{\mathrm{Lim}}_i\, \mu_i$ be the measure φ^* defined in § 3. If $\mathscr{B}$ is the family of all open sets then we simply write $\underline{\mathrm{Lim}}_i\, \mu_i$ instead of $\mathscr{B}\text{-}\underline{\mathrm{Lim}}_i\, \mu_i$.

We then have the following:

4.3. THEOREM. *$(W^*)\text{-}\lim_i \mu_i$ exists if and only if there exists a base $\mathscr{B}$ for the topology of X, closed under finite unions and intersections, such that, for every $\alpha \in \mathscr{B}$, $\lim_i \mu_i \alpha < \infty$, in which case,*

$$(W^*)\text{-}\lim_i \mu_i = \mathscr{B}\text{-}\underline{\mathrm{Lim}}_i\, \mu_i = \underline{\mathrm{Lim}}_i\, \mu_i .$$

The proof of this theorem is given in Lemmas A, B, C, D, E below. A restricted version of Lemma B was proved by Wulfsohn [4].

LEMMA A. *Let $\nu \in \mathscr{M}$ and*

$$\mathscr{B} = \{\alpha : \alpha \textit{ is open, } \bar{\alpha} \textit{ is compact and } \nu \text{ (boundary } \alpha) = 0\} .$$

Then $\mathscr{B}$ is a base for the topology of X and is closed under finite unions and intersections.

Proof. Let A be open and $a \in A$. Then there exists $f \in C_0(X)$ such that: $0 \leqq f(x) \leqq 1$ for $x \in X$, $f(a) = 1$ and $f(x) = 0$ for $x \notin A$. Since $\int f d\nu < \infty$, there exists $0 < t < 1$ such that $\nu(f^{-1}\{t\}) = 0$. Let $\alpha = \{x : f(x) > t\}$. Then α is open, $a \in \alpha \subset A$ and boundary $\alpha = f^{-1}\{t\}$ so that $\alpha \in \mathscr{B}$. Thus, $\mathscr{B}$ is a base. It is closed to finite unions and intersections since boundary $(\alpha \cup \beta) \cup$ boundary $(\alpha \cap \beta) \subset$ boundary $\alpha \cup$ boundary β for any open α, β.

LEMMA B. *$(W^*)\text{-}\lim_i \mu_i = \nu$ if and only if $\nu \in \mathscr{M}$ and $\lim_i \mu_i \alpha = \nu\alpha$ for every open α with $\bar{\alpha}$ compact and ν* (boundary α) $= 0$.

Proof. Let $(W^*)\text{-}\lim_i \mu_i = \nu$, α be open, $\bar{\alpha}$ compact, ν (boundary

$\alpha) = 0$. For any compact $C \subset \alpha$, let $f \in C_0(X)$, $0 \leqq f(x) \leqq 1$ for all $x \in X$, $f(x) = 1$ for $x \in C$, $f(x) = 0$ for $x \notin \alpha$. Then

$$\nu C \leqq \int f d\nu = \lim_i \int f d\mu_i \leqq \varliminf_i \mu_i \alpha .$$

Hence

$$\nu\alpha \leqq \varliminf_i \mu_i \alpha .$$

Now, since ν (boundary $\alpha) = 0$, given $\varepsilon > 0$, let β be open, $\bar{\alpha} \subset \beta$ and $\nu\beta \leqq \nu\bar{\alpha} + \varepsilon = \nu\alpha + \varepsilon$. Let $f \in C_0(x)$, $0 \leqq f(x) \leqq 1$ for $x \in X$, $f(x) = 1$ for $x \in \bar{\alpha}$, $f(x) = 0$ for $x \notin \beta$. Then

$$\varlimsup_i \mu_i \alpha \leqq \lim_i \int f d\mu_i = \int f d\nu \leqq \nu\beta \leqq \nu\alpha + \varepsilon .$$

Thus,

$$\nu\alpha = \lim_i \mu_i \alpha .$$

Conversely, suppose $\nu \in \mathscr{M}$ and $\lim_i \mu_i \alpha = \nu\alpha$ for every open α with $\bar{\alpha}$ compact and ν (boundary $\alpha) = 0$. Let $f \in C_0(X)$, $\varepsilon > 0$. Then there exist $t_k \neq 0$ for $k = 0, \cdots, n$ such that $t_k < t_{k+1}$, $t_0 \leqq f(x) \leqq t_n$ for $x \in X$, $\nu(f^{-1}\{t_k\}) = 0$ and

$$\sum_{k=0}^{n-1} t_{k+1} \nu\alpha_k - \varepsilon \leqq \int f d\nu \leqq \sum_{k=0}^{n-1} t_k \nu\alpha_k + \varepsilon$$

where

$$\alpha_k = \{x : t_k < f(x) < t_{k+1}\}$$

so that α_k is open, $\bar{\alpha}_k$ is compact and ν (boundary $\alpha_k) = 0$. Then $\lim_i \mu_i \alpha_k = \nu\alpha_k$ and

$$\begin{aligned}\int f d\nu &\leqq \lim_i \sum_{k=1}^{n-1} t_k \mu_i \alpha_k + \varepsilon \\ &\leqq \varliminf_i \int f d\mu_i + \varepsilon .\end{aligned}$$

Now, let β_k be open, $\bar{\beta}_k$ be compact, ν (boundary $\beta_k) = 0$, $\bar{\alpha}_k \subset \beta_k$ and $\nu\beta_k \leqq \nu\alpha_k + \varepsilon/(n\,|\,t_{k+1}\,|)$. Then $\lim_i \mu_i \beta_k = \nu\beta_k$ and

$$\begin{aligned}\varlimsup_i \int f d\mu_i &\leqq \lim_i \sum_{k=0}^{n-1} t_{k+1} \mu_i \beta_k \\ &= \sum_{k=0}^{n-1} t_{k+1} \nu\beta_k \\ &\leqq \sum_{k=0}^{n-1} t_{k+1} \nu\alpha_k + \varepsilon \\ &\leqq \int f d\nu + 2\varepsilon .\end{aligned}$$

LEMMA C. *If* $(W^*)\text{-lim}_i \mu_i = \nu$ *and*

$$\mathscr{B} = \{\alpha : \alpha \textit{ is open, } \bar{\alpha} \textit{ is compact, } \nu \text{ (boundary } \alpha) = 0\}$$

then

$$\nu = \mathscr{B}\text{-}\underline{\text{Lim}}_i\, \mu_i .$$

Proof. Let $g\alpha = \underline{\text{lim}}_i\, \mu_i\alpha$ for any $\alpha \in \mathscr{B}$, φ be the measure generated by g and $\mathscr{B}$ (see § 3). Then, in view of Lemma B and 3.1 (iii), for any compact $C \subset X$,

$$\varphi C = \inf \{g\beta ; \beta \in \mathscr{B} ; C \subset \beta\} .$$

Now, for any open $\alpha \supset C$ there exists, by Lemma A, $\beta \in \mathscr{B}$ with $C \subset \beta \subset \alpha$. Therefore, using Lemma B, and the outer regularity of ν, we have

$$\begin{aligned} \nu C &= \inf \{\nu\alpha ;\ \alpha \text{ open, } C \subset \alpha\} \\ &= \inf \{\nu\beta ;\ \beta \in \mathscr{B},\ C \subset \beta\} \\ &= \inf \{g\beta ;\ \beta \in \mathscr{B},\ C \subset \beta\} \\ &= \varphi C . \end{aligned}$$

Hence, for any $A \subset X$,

$$\begin{aligned} \nu A &= \inf_{\substack{\alpha \text{ open} \\ A \subset \alpha}} \ \sup_{\substack{C \text{ compact} \\ C \subset \alpha}} \nu C \\ &= \inf_{\substack{\alpha \text{ open} \\ A \subset \alpha}} \ \sup_{\substack{C \text{ compact} \\ C \subset \alpha}} \varphi C = \mathscr{B}\text{-}\underline{\text{Lim}}_i\, \mu_i A . \end{aligned}$$

LEMMA D. *Let* $\mathscr{B}$ *be a base for the topology of* X, *closed under finite unions and intersections, such that for any* $\alpha \in \mathscr{B}$, $\lim_i \mu_i\alpha < \infty$. *Then*

$$\mathscr{B}\text{-}\underline{\text{Lim}}_i\, \mu_i = (W^*)\text{-}\lim_i \mu_i .$$

Proof. For $\alpha \in \mathscr{B}$, let $g\alpha = \underline{\text{lim}}_i\, \mu_i\alpha = \lim_i \mu_i\alpha$, φ be the measure generated by g and $\mathscr{B}$ and $\varphi^* = \mathscr{B}\text{-}\underline{\text{Lim}}_i\, \mu_i$ (see § 3). Then, by Theorem 3.2, $\varphi^* \in \mathscr{M}$. Let α be open, $\bar{\alpha}$ compact, $\varphi^*(\text{boundary } \alpha) = 0$. By 3.2 (ii), we have

$$\varphi^*\alpha = \varphi^*\bar{\alpha} = \varphi\bar{\alpha}$$

and by 3.1 (iii),

$$\varphi\bar{\alpha} = \inf \{g\beta ;\ \beta \in \mathscr{B},\ \bar{\alpha} \subset \beta\} .$$

Given $\varepsilon > 0$, let $\beta \in \mathscr{B}$, $\bar{\alpha} \subset \beta$ and $g\beta \leqq \varphi^*\alpha + \varepsilon$. Then

$$\overline{\lim_i}\, \mu_i\alpha \leqq \lim_i \mu_i\beta = g\beta \leqq \varphi^*\alpha + \varepsilon .$$

On the other hand, let C be compact, $C \subset \alpha$ and $\varphi^*\alpha < \varphi^*C + \varepsilon = \varphi C + \varepsilon$. Then there exists $\gamma \in \mathscr{B}$ such that $C \subset \gamma \subset \alpha$ and therefore

$$\varphi C \leqq g\gamma = \lim_i \mu_i\gamma \leqq \underline{\lim_i}\, \mu_i\alpha .$$

Thus,

$$\overline{\lim_i}\, \mu_i\alpha \leqq \varphi^*\alpha \leqq \underline{\lim_i}\, \mu_i\alpha$$

so that $\lim_i \mu_i\alpha = \varphi^*\alpha$. By Lemma B then $\varphi^* = (W^*)\text{-}\lim_i \mu_i$.

LEMMA E. *Let $\mathscr{B}$ be a base for the topology of X, closed under finite unions and for every $\alpha \in \mathscr{B}$, $\lim_i \mu_i\alpha < \infty$. Then*

$$\mathscr{B}\text{-}\underline{\mathrm{Lim}}_i\, \mu_i = \underline{\mathrm{Lim}}_i\, \mu_i .$$

Proof. For any open α, let $g\alpha = \underline{\lim}_i \mu_i\alpha$, φ_1 be the measure generated by g and $\mathscr{B}$ and φ_2 be the measure generated by g and the family of all open sets. We have to show that for any compact C, $\varphi_1 C = \varphi_2 C$. Now, clearly $\varphi_2 C \leqq \varphi_1 C$. Suppose $\varphi_2 C < \infty$ and $\varepsilon > 0$. Let α_i be open for $i = 0, \cdots, n$, $C \subset \bigcup_{i=0}^n \alpha_i$ and

$$\sum_{i=1}^n g\alpha_i \leqq \varphi_2 C + \varepsilon .$$

For each $x \in C$ there exists $\beta \in \mathscr{B}$ such that $x \in \beta \subset \alpha_i$ for some $i = 0, \cdots, n$. Since C is compact, there is a finite family $H \subset \mathscr{B}$ which covers C and is a refinement of $\{\alpha_0, \cdots, \alpha_n\}$. For each i, let β_i be the union of all those elements in H which are contained in α_i. Then $\beta_i \in \mathscr{B}$, $\beta_i \subset \alpha_i$ and $C \subset \bigcup_{i=0} \beta_i$. Thus,

$$\varphi_1 C \leqq \sum_{i=0}^n g\beta_i \leqq \sum_{i=0}^n g\alpha_i \leqq \varphi_2 C + \varepsilon .$$

5. **Remarks.** Let $\mathscr{B}$, g, φ be as in § 3. The following example shows that φ need not be a Radon measure.

Let X be the set of all ordinals up to and including the first uncountable ordinal Ω. Then, in the order-topology, X is compact Hausdorff. For each $i < \Omega$, let μ_i be the point mass at i, that is, $\mu_i\alpha = 1$ if $i \in \alpha$ and $\mu_i\alpha = 0$ if $i \notin \alpha$. Let

$$\mathscr{B} = \{\alpha ;\ \alpha \text{ is open and } \Omega \notin (\bar{\alpha} - \alpha)\}.$$

For any $\alpha \in \mathscr{B}$, if $\Omega \notin \alpha$ then α is countable and hence $g\alpha = \underline{\lim}_i \mu_i\alpha = 0$; if $\Omega \in \alpha$ then $g\alpha = 1$. Let $A = X - \{\Omega\}$. Then A is open and, being uncountable, for any countable family $H \subset \mathscr{B}$ which covers A there exists $\alpha \in H$ with $g\alpha = 1$. Thus, $\varphi A = 1$. On the other hand, if C is compact $C \subset A$ then C is countable and hence $\varphi C = 0$. Thus,

$$\varphi A \neq \sup \{\varphi C ; \ C \text{ compact, } C \subset A\} .$$

Note, however, that if, instead of taking $\mathscr{B}$ as above, we let $\mathscr{B}$ be the family of all open sets in X then there exist uncountable, disjoint $\alpha, \beta \in \mathscr{B}$ with $A = \alpha \cup \beta$. Then $g\alpha = g\beta = 0$ so that $\varphi A = 0$. In this case, φ is the point mass at Ω and $\varphi = \varphi^*$.

We are unable to determine if this holds true in general for compact or locally compact Hausdorff spaces, i.e. if $\varphi = \varphi^*$ whenever $\mathscr{B}$ is the family of all open sets in X.

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