COVERINGS OF ALGEBRAIC GROUPS AND LIE ALGEBRAS OF CLASSICAL TYPE

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Introduction. By a Lie algebra of classical type we shall mean a Lie algebra L over an algebraically closed field K of characteristic $p > 7$ which possesses a standard Cartan subalgebra, that is, an abelian Cartan subalgebra H such that L and H satisfy the axioms of Mills and Seligman [11].

Let G be the algebraic component of the identity in the automorphism group of L. In [7] Curtis constructs an irreducible projective representation of G from each of the irreducible restricted representations of L and a group G^* which is a covering group for G in the sense that

(i) there is a covering homomorphism mapping G^* onto G whose kernel is contained in the center of G^*;

(ii) each of the projective representations of G constructed can be lifted to an irreducible representation of G^*.

It is the purpose of this paper to investigate the structure of this covering group and to identify it. The main structure theorem presented here, after noting that G^* is an irreducible linear algebraic group, is that the decomposition of L as the direct sum of simple ideals induces a decomposition of G and G^* as the direct product of the corresponding groups for the simple ideals. The identification problem is thus reduced to the "simple case" and for Lie algebras of type A_n or C_n results have been obtained by Curtis [7]. In the last section we give a complete treatment for the orthogonal Lie algebra with respect to a quadratic form Q, that is, for types B_n and D_n. The principal result obtained is that the covering group is birationally isomorphic to the reduced Clifford group associated with Q.

These results are of some interest in view of recent work of Steinberg [13]. Working with the simple groups defined by Chevalley [5] Steinberg constructs a covering group in terms of generators and relations which is naturally isomorphic to the simply connected covering group of the simple Chevalley group. Combining Steinberg's results with ours it is not difficult to show that the two covering groups are birationally isomorphic.

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1. The group of invariant automorphisms. Let L be a Lie algebra of classical type and H a standard Cartan subalgebra of L. We assume that

(i) a basis for L is chosen such that the basis elements $e(a)$, $e(-a)$ spanning the one dimensional root spaces of the roots $a, -a$ (zero is not counted as a root) are normalized to give $a(h(a)) = 2$ where $h(a) = [e(-a), e(a)]$, and the basis elements spanning H are $h(a_1), \cdots, h(a_n)$ for some roots $a_1, \cdots, a_n$;

(ii) an ordering of the roots has been introduced which is compatible with the additive structure of the set of roots [6, pp. 95–96];

(iii) a p-power operation has been defined in L with respect to which L is a restricted Lie algebra.

Let S be the set whose elements are the automorphisms

$$s = s(a, k) = \exp(k \, ad \, e(a))$$

where k runs over K and a runs over the roots of L with respect to H. Let G be the subgroup of the group $A(L)$ of all automorphisms of L generated by the set S. G will be called the group of invariant automorphisms associated with H.

An immediate extension of Theorem 3 [7] and a theorem of Ono (Jour. Math. Soc. Japan 10 (1958), 307–313) made possible by the work of Block [1, Cor. 6.1, Th. 7.1] is

1.1. G is an irreducible algebraic group of linear transformations on L. If L possesses a nondegenerate trace form then G is the algebraic component of the identity in $A(L)$ and the Lie algebra of G is isomorphic to L.

2. The covering group of G. Let U be the U-algebra of L (see [10, p. 192]). Since the center of L is trivial every automorphism is a restricted map and can be extended to an automorphism of U. We use the same notation for an automorphism in G and the extended automorphism of U.

When M is a finite dimensional irreducible right U-module Curtis [7, pp. 317–319] has constructed an irreducible projective representation of G, F, acting in M such that

$$(x \cdot u) F(g) = (xF(g)) \cdot u^g \,, \qquad x \in M, u \in U, g \in G \,. \tag{1}$$

Moreover, if $x_+(x_-)$ is a maximal (minimal) vector in M then F is normalized so that for any $s \in S$

$$\begin{cases} x_+ F(s) = x_+, \; s = s(a, k), \; a > 0 \,, \\ x_- F(s) = x_-, \; s = s(a, k), \; a < 0 \,. \end{cases} \tag{2}$$

Immediate consequences of (1), (2), and the definitions which will be used are:

2.1. For any $s(a, k) \in S$, and $k, k' \in K$,

$$F(s(a, k + k')) = F(s(a, k))F(s(a, k')) .$$

2.2. (Curtis [7, II. 2.1]). For any $x \neq 0$ in M and any root a there exist vectors $x_2, x_3, \cdots, x_t$ in M, depending upon x and a, such that for all $k \in K$

$$xF(s(a, k)) = x + kx \cdot e(a) + k^2 x_2 + \cdots + k^t x_t .$$

Let $\mathscr{M}$ be a nonempty set of nonisomorphic, finite dimensional, irreducible right U-modules, and for each M in $\mathscr{M}$ let $G(M)$ be the group consisting of the linear transformations $F_M(s_1) \cdots F_M(s_r)$, where $r \geqq 1$, $s_i \in S$, and F_M is an irreducible projective representation of G acting in M which satisfies (1) and (2). Denote by $G(\mathscr{M})$ the sub-group of $\Pi\, G(M)\, (M \in \mathscr{M})$ generated by the elements $g(s_1, \cdots, s_r)$ for every nonempty ordered sequence $(s_1, \cdots, s_r)$ of elements from S where $g(s_1, \cdots, s_r)$ has as its M^{th} coordinate $F_M(s_1) \cdots F_M(s_r)$. When $\mathscr{M}$ contains a representative from each of the p^n, $n = \operatorname{rank} L$, isomorphism classes of irreducible U-modules $G(\mathscr{M}) = G^*$ is the covering group defined by Curtis, the covering homomorphism ω mapping $g(s_1, \cdots, s_r)$ onto $s_1 \cdots s_r$.

THEOREM 1. *For any nonempty set of nonisomorphic irreducible U-modules $\mathscr{M}$, $G(\mathscr{M})$ is an irreducible linear algebraic group on $\boldsymbol{M} = \times M$ $(M \in \mathscr{M})$. The center of $G(\mathscr{M})$ is finite and has order relatively prime to p.*

Proof. $\boldsymbol{M}$ is made into a right U-module by defining for any $u \in U$ and $\boldsymbol{m} = (m, m', \cdots) \in \boldsymbol{M}$

$$\boldsymbol{m} \cdot u = (m \cdot u, m' \cdot u, \cdots,) .$$

The action of $G(\mathscr{M})$ on $\boldsymbol{M}$ is given by

$$\boldsymbol{m}g(s(a, k)) = (mF_M(s(a, k)), m'F_{M'}(s(a, k)), \cdots) ,$$

and therefore since each F_M, $M \in \mathscr{M}$, satisfies (1) we have

$$(3) \qquad (\boldsymbol{m} \cdot u)g(s_1, \cdots, s_r) = (\boldsymbol{m}g(s_1, \cdots, s_r)) \cdot u^{s_1 \cdots s_r} ,$$

for any $\boldsymbol{m} \in \boldsymbol{M}$, $u \in U$, and $g(s_1, \cdots, s_r) \in G(\mathscr{M})$. By similar reasoning we obtain 2.1 and 2.2 for the elements $g(s(a, k))$ and the module $\boldsymbol{M}$. It follows from 2.1 and 2.2 that for each root a the mapping $k \rightarrow$

$g(s(a, k))$ is a polynomial representation of the additive group of K, hence Proposition 2, [2, p. 121] enables us to conclude that $\{g(s(a, k)): k \in K\}$ is an irreducible algebraic group. But these subgroups generate $G(\mathscr{M})$ therefore the first assertion follows from Corollary 3, [2, p. 123]. For any $M \in \mathscr{M}$ 2.1 implies that $G(M)$ is a subgroup of the unimodular group on M. F_M is an irreducible projective representation hence the center of $G(M)$ is finite and has order relatively prime to p. The second assertion follows from the fact that $G(\mathscr{M})$ is a subdirect product of the groups $G(M)$, $M \in \mathscr{M}$.

3. The structure of the covering group. Before discussing structure theorems for G^* some remarks are in order concerning the dependence of the groups $G(M)$ and $G(\mathscr{M})$ upon the method of construction. Two projective representations of G on an irreducible U-module M which satisfy (1) and (2) yield the same group $G(M)$, and if M' is an irreducible U-module, θ a U-module isomorphism of M onto M', then the mapping $g \to \theta^{-1}F_M(g)\theta$ is a projective representation of G on M' satisfying (1) and (2). This mapping induces naturally an isomorphism of $G(M)$ onto $G(M')$ which is birational (both the isomorphism and its inverse are rational maps). Thus we see that $G(\mathscr{M})$ is determined up to birational isomorphism by the isomorphism classes of the irreducible right U-modules contained in $\mathscr{M}$.

For any nonempty set of nonisomorphic irreducible U-modules $\mathscr{M}$ let $\boldsymbol{M}$ be defined and given the structure of right U-module as in Theorem 1. For any x in L the linear transformation on $\boldsymbol{M}$ representing x can be extended uniquely to a derivation $D(x)$ of the tensor algebra $T(\boldsymbol{M})$ over the vector space $\boldsymbol{M}$. The map $x \to D(x)$ is a restricted representation of L, hence $T(\boldsymbol{M})$ is a right U-module. Finally let $\mathscr{M}'$ be a complete set of nonisomorphic irreducible right U-modules containing $\mathscr{M}$. The covering group G^* is birationally isomorpic to $G(\mathscr{M}')$ and we shall denote by $\pi(\mathscr{M})$ the composite of this isomorphism with the projection of $G(\mathscr{M}')$ onto $G(\mathscr{M})$.

THEOREM 2. *Let $\mathscr{M}$ be a nonempty set of nonisomorphic irreducible right U-modules such that every irreducible right U-module is the homomorphic image of a finite dimensional U-submodule of $T(\boldsymbol{M})$. Then $\pi(\mathscr{M})$ is a birational isomorphism of G^* onto $G(\mathscr{M})$.*

Proof. Without loss of generality we may assume $G^* = G(\mathscr{M}')$, where $\mathscr{M}'$ is a complete set of nonisomorphic irreducible U-modules containing $\mathscr{M}$, and that $\pi(\mathscr{M})$ is the canonical projection of $G(\mathscr{M}')$ onto $G(\mathscr{M})$. Thus $\pi(\mathscr{M})$ is a polynomial representation of $G(\mathscr{M})$.

Equation (3), 2.1, and 2.2 hold for the U-module $\boldsymbol{M}$ and the elements $g(s)$, $s \in S$, hence the proof of Theorem 5 [7, p. 321] carries over to this situation. As in Curtis' proof one shows that for each $M \in \mathscr{M}'$ there exists a group homomorphism $\pi(M)$, $\pi(M): G(\mathscr{M}) \to G(M)$, sending $g(s_1, \cdots, s_r)$ onto $F_M(s_1) \cdots F_M(s_r)$. From the construction of $\pi(M)$ (see [7, p. 321–322]) and Proposition 2 [3, p. 11] it is easily deduced that $\pi(M)$ is a polynomial representation of $G(\mathscr{M})$. For each $M \in \mathscr{M}'$ $\pi(\mathscr{M}) \circ \pi(M)$ is the coordinate projection of $G(\mathscr{M}')$ onto $G(M)$ hence $\pi(\mathscr{M})$ is an isomorphism. $\pi(\mathscr{M})^{-1}$ is the homomorphism $\Delta \circ \psi$ where Δ is the canonical isomorphism of $G(\mathscr{M})$ with the diagonal of $G(\mathscr{M})^q$, $q = p^n =$ cardinality of $\mathscr{M}'$, and ψ the homomorphism $\pi(M) \times \pi(M') \times \cdots$, $\mathscr{M}' = \{M, M', \cdots\}$. Our proof is now complete since both Δ and ψ are polynomial representations.

REMARK. Let $\lambda_1, \cdots, \lambda_n$ be a basis for the space of integral linear functions on H [7, p. 313]; take M_i, $i \in [1, n]$ to be an irreducible right U-module with maximal weight λ_i, and set $\mathscr{M} = \{M_i \colon i \in [1, n]\}$. Then $\mathscr{M}$ satisfies the hypothesis of Theorem 2.

Let $L = L_1 + \cdots + L_t$ be the decomposition of L into the direct sum of simple ideals, then $H \cap L_i$ is a standard Cartan subalgebra of L_i and $H = H_1 + \cdots + H_t$ (direct). In terms of this decomposition we have

THEOREM 3. *Let L_i, $i \in [1, t]$, be the simple ideals in the decomposition of L and G_i the group of invariant automorphism of L_i determined by $H_i = L_i \cap H$. If G_i^* is the covering group of G_i, G^* the covering group of G, then G is birationally isomorphic to $G_1 \times \cdots \times G_t$, and G^* is birationally isomorphic to $G_1^* \times \cdots \times G_t^*$.*

Proof. For each $i \in [1, t]$ let T_i be the set of all automorphisms $s = s(a, k) \in S$ whose restriction to H_j, $j \neq i$ is the identity transformation. For any $s(a, k), s(a', k') \in S$, $s(a, k) = s(a', k')$ if and only if $k = k' = 0$, in which case $s(a, k)$ is the identity automorphism on L, or $k = k' \neq 0$ and $a = a'$. This fact together with the partitioning of the roots implied by the decomposition of L enables one to construct a birational isomorphism between the subgroup $G^{(i)}$ of G generated by T_i and the group G_i. As G is the direct product of the subgroups $G^{(i)}$, $i \in [1, t]$, G is birationally isomorphic to $G_1 \times \cdots \times G_t$. For future use we denote the composite of this isomorphism with the projection of $G_1 \times \cdots \times G_t$ onto G_i by ρ_i.

For each i let $\lambda_{i,j}$, $j \in [1, n_i]$, be a basis for the space of integral linear functions on H_i, and take $M_{i,j}$ to be an irreducible restricted L_i-module with maximal weight $\lambda_{i,j}$. The canonical projection of L onto L_i followed by the restricted representation of L_i defining the L_i-module structure in $M_{i,j}$ is an irreducible restricted representation of L. We *identify* $\lambda_{i,j}$ with its image under the canonical injection of H_i^* into H^* and note that as an irreducible restricted L-module $M_{i,j}$ has maximal weight $\lambda_{i,j}$.

Let $F_{i,j}$ be an irreducible projective representation of G_i acting in $M_{i,j}$ which satisfies (1) and (2) when $M_{i,j}$ is considered as an L_i-module. Setting $\bar{F}_{i,j} = \rho_i \circ F_{i,j}$ we obtain an irreducible projective representation of G acting in $M_{i,j}$ which satisfies (1) and (2) when $M_{i,j}$ is viewed as an L-module. The mapping $\psi_{i,j}$ sending $\bar{F}_{i,j}(s_1) \cdots \bar{F}_{i,j}(s_r)$ to $F_{i,j}(\rho_i(s_1)) \cdots F_{i,j}(\rho_i(s_r))$, $r \geqq 1$, $s_i \in S$, is a birational isomorphism of $G(M_{i,j})$ onto $G_i(M_{i,j})$. For each i let $\mathscr{M}_i = \{M_{i,j}: j \in [1, n_i]\}$. It follows that ψ_i, the restriction of $\psi_{i,1} \times \cdots \times \psi_{i,n_i}$ to $G(\mathscr{M}_i)$, is a birational isomorphism of $G(\mathscr{M}_i)$ onto $G_i(\mathscr{M}_i)$. If $\mathscr{M}$ is the union of the sets $\mathscr{M}_i$, π_i the canonical projection of $G(\mathscr{M})$ onto $G(\mathscr{M}_i)$ (along the coordinates in $\mathscr{M} - \mathscr{M}_i$) then $\delta \circ (\pi_1 \times \cdots \times \pi_t) \circ (\psi_1 \times \cdots \times \psi_t)$ is a birational isomorphism of $G(\mathscr{M})$ onto $G_1(\mathscr{M}_1) \times \cdots \times G_t(\mathscr{M}_t)$, where δ is the natural isomorphism of $G(\mathscr{M})$ with the diagonal of $G(\mathscr{M}) \times G(\mathscr{M}) \cdots \times G(\mathscr{M})$ (t factors). But $G_i(\mathscr{M}_i)$ is birationally isomorphic to G_i^* by our choice of $\mathscr{M}_i$ and the remark following Theorem 2. The elements $\lambda_{i,j}$, $j \in [1, n_i]$, $i \in [1, t]$, are basis elements for the space of integral linear functions on H hence G^* is birationally isomorphic to $G(\mathscr{M})$, and Theorem 3 is proved.

We are now in a position to establish the following:

THEOREM 4. *If ω is the covering homomorphism of G^* onto G then ω is a rational representation of G^* and the kernel of ω is the center of G^*.*

Proof. In view of Theorem 3, it is sufficient to consider the case where L is simple. In this situation the adjoint representation is irreducible, and if F_L is an irreducible projective representation of G acting in L which satisfies (1) and (2) then $F_L(s) = s$ for any $s \in S$, hence $G(L) = G$. Let $\mathscr{M}$ be a complete set of nonisomorphic irreducible U-modules which contains L. Let $\pi(L)$ be the canonical projection of $G(\mathscr{M})$ onto $G(L) = G$ and ψ the birational isomorphism of G^* onto $G(\mathscr{M})$. Then we have $\omega = \psi \circ \pi(L)$ and hence ω is rational.

From the definition of G^* it follows that ker ω is contained in the center of G^*. The enveloping algebra of G contains the Lie algebra of G. The latter however contains ad L, hence any element in the center of G commutes with the elements of ad L. Thus the center of G is $\{1\}$ and this establishes the second assertion.

THEOREM 5. *If L possesses a nondegenerate trace form then the Lie algebra of G^* is isomorphic to the Lie algebra of G. In particular the differential of the covering homomorphism is such an isomorphism.*

Proof. It is sufficient to prove the following.

(*) if $\mathscr{M}$ is any nonempty set of nonisomorphic irreducible U-modules and R the representation of L afforded by the U-module $\boldsymbol{M} = \times M(M \in \mathscr{M})$ then the Lie algebra of $G(\mathscr{M})$ contains $R(L)$.

Indeed, we have $G^* = G(\mathscr{M}_0)$ where $\mathscr{M}_0$ is a complete set of irreducible U-modules, hence the Lie algebra of G^* contains $R_0(L)$ an isomorphic copy of L. The hypothesis on L implies (see 1.1) that the Lie algebra of G is isomorphic to L, consequently the first assertion of the theorem follows from Theorem 4.

In view of Theorem 3 and Proposition 3 [2, p. 139; 3, p. 13] it is sufficient to prove the second assertion when L is simple. If $\mathscr{M}_0$ is a complete set of irreducible U-modules containing L then there is a birational isomorphism ψ mapping G^* onto $G(\mathscr{M}_0)$. The group $G(L)$ is G (see the proof of Theorem 4) consequently $d\omega = d\psi \circ d\pi$ where $\pi = \pi(L)$ is the projection of $G(\mathscr{M}_0)$ onto $G(L)$. $d\psi$ is an isomorphism (see [2, Prop. 5, p. 140]). Since the Lie algebra of $G(\mathscr{M}_0)$ is $R_0(L)$ we obtain that $d\pi$ is the mapping $R_0(l) \rightarrow \operatorname{ad} l$, $l \in L$, hence $d\omega$ is an isomorphism.

We prove (*) by direct computation using Proposition 4 [2, p. 132]. Let X be transcendental over K and set $\Omega = K(X)$. For any root a the maps

$$g_a\colon k \rightarrow g_a(k) = g(s(a, k)) \text{ and } s_a\colon k \rightarrow s_a(k) = s(a, k)$$

are polynomial representations of $\mathscr{K}$ the additive group of the field K. We denote by $g'_a(s'_a)$ the unique polynomial representation of $\mathscr{K}^\Omega$ which extends $g_a(s_a)$. $\mathscr{K}^\Omega$ is the additive group of the field Ω and for any $k' \in \Omega$, $s'_a(k')$ is the automorphism $s(a, k')$ of L^Ω. As $\boldsymbol{M}^\Omega$ is naturally a restricted L^Ω-module we obtain from (3)

$$(\boldsymbol{m}\cdot x)g'_a(k) = (\boldsymbol{m}g'_a(k))\cdot x^{s(a,k)},\ \boldsymbol{m} \in \boldsymbol{M}^\Omega,\ x \in L,\ k \in \Omega\ , \tag{4}$$

and

$$\begin{aligned} \boldsymbol{m}_+ g'_a(k) &= \boldsymbol{m}_+ \text{ if } a > 0,\ k \in \Omega\ , \\ \boldsymbol{m}_- g'_a(k) &= \boldsymbol{m}_- \text{ if } a < 0,\ k \in \Omega\ , \end{aligned} \tag{5}$$

where $\boldsymbol{m}_+(\boldsymbol{m}_-)$ is the image of a maximal (minimal) vector in M, $M \in \mathscr{M}$, under the canonical injection of M into $\boldsymbol{M}$.

Let D be the derivation with respect to X in Ω. If V is any finite dimensional vector space over K, D induces a unique endomorphism in V^Ω, also denoted by D, which commutes with every linear function on V. Furthermore, if V is an algebra over K then the induced endomorphism of V^Ω is a derivation of the algebra V^Ω. With these conventions in mind we shall prove that if Δ is a maximal simple system of roots of L with respect to H then

$$(**)\qquad \boldsymbol{m}(g'_a(X)\cdot D) = (\boldsymbol{m}g'_a(X))\cdot e(a),\ \boldsymbol{m} \in \boldsymbol{M}^{\Omega},\ \pm a \in \Delta\ .$$

For a given $a \in \Delta$ let N be the set of all $\boldsymbol{m}$ in $\boldsymbol{M}^{\Omega}$ for which (**) holds. Applying D to the first of the equations in (5) yield that N contains the image of a maximal vector in M under the canonical injection of M into $\boldsymbol{M}$ for each M in $\mathscr{M}$. If $\boldsymbol{m} \in N \cap \boldsymbol{M}$ and $x = e(-a)$ then upon applying D to the right side of (4) we obtain with the aid of Lemma 3 [2, p. 132]

$$\begin{aligned}[(\boldsymbol{m}g'(X))\cdot e(-a)^{s(a,X)}]D &= [(\boldsymbol{m}g'_a(X))\cdot(e(-a) + Xh(a) - X^2e(a))]D \\ &= (\boldsymbol{m}g'_a(X))\cdot(e(a)\,[e(-a) + Xh(a) - X^2e(a)] + h(a) - 2Xe(a)) \\ &= ((\boldsymbol{m}g'_a(X))\cdot e(-a)^{s(a,X)})\cdot e(a)\ .\end{aligned}$$

Thus $\boldsymbol{m}\cdot e(-a)$ belongs to $N \cap \boldsymbol{M}$ whenever $\boldsymbol{m}$ does. Similar calculations with $x = e(-b)$, $b \in \Delta$, $b \neq a$, give $\boldsymbol{m}\cdot e(-b) \in N \cap \boldsymbol{M}$ whenever $\boldsymbol{m} \in N \cap \boldsymbol{M}$. But each M in $\mathscr{M}$ is spanned over K by vector monomials [8, p. 853–855] consequently $N = \boldsymbol{M}^{\Omega}$. An analogous argument can be given for the case $-a \in \Delta$.

Now (**) and Proposition 4 [2, p. 132] imply that $e(a)^R$, $\pm a \in \Delta$, belongs to the Lie algebra of $G(\mathscr{M})$ which gives the desired conclusion.

4. The covering group of the orthogonal Lie algebra. Let V be an m-dimensional vector space over K, B a nondegenerate symmetric bilinear form on V, and L the set of all K-endomorphisms T of V such that $B(xT, y) + B(x, yT) = 0$. L is a Lie subalgebra of the associative algebra of all K-endomorphisms of V which is closed under the associative p^{th} power, hence a restricted Lie algebra over K. We refer to L as the orthogonal Lie algebra determined by B.

The nondegeneracy of B enables one to identify the space of K-endomorphisms of V with $V \otimes V$, $v \otimes w\colon x \to B(x, v)w$ for any $x, v, w \in V$; to show that the elements $T(v, w) = v \otimes w - w \otimes v$, $v, w \in V$, span L; and to show that the identity mapping is a restricted representation with nondegenerate trace form. When $m > 4$ L is simple [9, or the remark following 4.1.1], and is of classical type D_n for $m = 2n$, $n \geqq 4$, and of type B_n for $m = 2n + 1$, $n \geqq 3$.

THEOREM 6. *Let L be the orthogonal Lie algebra determined by a nondegenerate, symmetric, bilinear form B on an m-dimensional vector space V over K. The group G^* is birationally isomorphic to the reduced Clifford group associated with the orthagonal group of B.*

Proof. In 4.2 and 4.4 a set of modules $\mathscr{M}$ is determined which satisfies the hypothesis of Theorem 2. For m even V together with the spaces of half spinors constitute such a set while for m odd we obtain V and the space of spinors (we are following the terminology of [4]). The projective representation of the group G associated with the modules in $\mathscr{M}$ are computed in 4.3 and 4.5 utilizing the formula $y\cdot\exp(ad\, x) = \exp(-x)y\exp(x)$ which holds in any associative algebra at characteristic p provided $x^t = 0$, $t \leqq (p+1)/2$. In all cases we have $F_M(s(a, k)) = \exp(k\, e(a)^R)$ where R is the representation of L in M, $M \in \mathscr{M}$. Lemma 4.1.3 concerning generators of the reduced Clifford group Γ_0^+ enables us to show that $G(V)$ is the image of Γ_0^+ under the vector representation while if M is the space of spinors (respectively: one of the spaces of half-spinors) then $G(M)$ is the image of Γ_0^+ under the spin representation (respectively: one of the half spin representations). These facts lead easily to the conclusion of the theorem.

4.1. Preliminary results on the representations of L. For each integer $r \in [1, m]$ let $A(r)$ be the space of skew-symmetric tensors of rank r in the tensor algebra on V. Each K-endomorphism in L can be uniquely extended to a derivation of the tensor algebra, and it is well known that $A(r)$ is invariant under each such derivation. Thus $A(r)$ becomes a U-module, and as is the case when K is of characteristic zero

4.1.1. For each positive integer r such that $2r < m$ the space $A(r)$ is an irreducible U-module.

Proof. A basis $v(1), \cdots, v(m)$ for V determines a basis for $A(r)$ as follows: let $\Sigma(r)$ be the set of strictly increasing sequences of length r from the integers $1, \cdots, m$ then the basis for $A(r)$ consists of the elements $a(\tau)$, $\tau \in \Sigma(r)$, where if $\tau = \{i_1, \cdots, i_r\}$

$$a(\tau) = \Sigma\gamma\cdot v(i_1') \cdots v(i_r')$$

the summation extending over all permutations $P = \begin{pmatrix} i_1, & \cdots, & i_r \\ i_1', & \cdots, & i_r' \end{pmatrix}$ of the set τ, while γ equals the signature of P.

L has been identified with $A(2)$ hence the elements $T(v(i), v(j)) = a(\{i, j\})$, $i, j \in [1, m]$, $i < j$, form a basis for L, and choosing the $v(i)$'s to be an orthogonal basis gives

$$(6) \qquad a(\tau)\cdot T(v(i), v(j)) = \begin{cases} 0, \text{ if } \{i, j\} \subset \tau \text{ or } \{i, j\} \subset ([1, m] - \tau) \\ \pm b(\tau \cap \{i, j\})\, a(\tau_1), \text{ otherwise}, \end{cases}$$

where τ_1 is the sequence in $\Sigma(r)$ whose elements comprise the set $(\tau \cup \{i, j\}) - (\tau \cap \{i, j\})$, and $b(i) = B(v(i), v(i))$, $i \in [1, m]$.

Let $W \neq \{0\}$ be an irreducible U-submodule of $A(r)$, and for each

$w \neq 0$ set $l(w)$ equal to the number of $a(\tau)$, $\tau \in \Sigma(r)$, which appear with nonzero coefficient in the expansion of w. Let $w_0 \neq 0$ be an element of W with $l(w_0)$ minimal. Assume $l(w_0) > 1$ and suppose that $a(\tau_1)$, $a(\tau_2)$, $\tau_1 \neq \tau_2$, appear with nonzero coefficients in the expansion of w_0. Since $2r < m$ there exist $i, j \in [1, m]$ such that $i \neq j$, $i \in \tau_1 \cap ([1, m] - \tau_2)$, while $j \in ([1, m] - \tau_1) \cap ([1, m] - \tau_2)$. Then $w_1 = w_0 T(v(i), v(j)) \neq 0$ and $l(w_1) < l(w_0)$, consequently W contains an element $a(\tau)$ for some $\tau \in \Sigma(r)$. Repeated application of (6) yields the desired result.

REMARK. (i) The adjoint representation of L is equivalent to the representation of $A(2)$ just discussed, thus we obtain another proof of the simplicity of L.

(ii) Whereas 4.1.1 seems to be in the domain of common knowledge the above proof is included for its simplicity and characteristic free nature.

Let C be the Clifford algebra on V with respect to the quadratic form $B(x, x)$. Identifying V with its image under the canonical injection of V into C, we have

$$v^2 = B(v, v) \cdot 1, \; v \in V. \tag{7}$$

and

$$vw + wv = 2B(v, w) \cdot 1, \; v, w \in V. \tag{8}$$

We obtain additional irreducible U-modules from the spin representation of C_+ since

4.1.2 The subspace $[V, V]$ of C spanned by all elements $[v, w] = vw - wv$, $v, w \in V$, is a Lie algebra, and the linear mapping of L onto $[V, V]$ given by $T(v, w)^\varphi = 4^{-1}[v, w]$ is a restricted isomorphism. The enveloping algebra of $\varphi(L)$ in C is C_+ the subalgebra of even elements.

Proof. Except for the restrictedness of φ these results follow from the identity

$$w \cdot T(x, y) = [w, T(x, y)^\varphi], \; x, y, w \in V, \tag{9}$$

(see [10, Th. 7, p. 231]). For any x, y in V, $(T(x, y))^3 = kT(x, y)$ and $(T(x, y)^\varphi)^3 = 4^{-1}kT(x, y)^\varphi$. These relations imply that φ is restricted.

4.1.3. If $v(i), v(-i), (v(0), v(i), v(-i))$, $i \in [1, n]$, constitute a basis for V such that $B(v(i), v(j)) \neq 0$ if and only if $i + j = 0$ then the elements $\exp(k\, v(i)\, v(j))$, $|i| \neq |j|$, $i < j$, $k \in K$, generate the reduced Clifford group Γ_0^+.

Proof. For $|i| \neq |j|$, $i < j$, $B(v(i), v(j)) = 0$ and at least one of

$v(i)$, $v(j)$ is isotropic so $(v(i)v(j))^2 = 0$ and $\exp(k\, v(i)v(j))$ is well defined by the exponential power series. Equations (7)–(9) imply

$$\exp(-k\, v(i)v(j))x \exp(k\, v(i)v(j)) = x \cdot \exp(2k\, T(v(i), v(j))) \ . \tag{10}$$

As $\exp(k\, v(i)v(j))$ is an invertible element of C_+ which has norm one [4, p. 52] (10) tells us that it belongs to Γ_0^+ the reduced Clifford group. Let G' be the subgroup of Γ_0^+ generated by the elements $\exp(k\, v(i)v(j))$, $k \in K$, $|i| \neq |j|$, $i < j$, and ν the vector representation of the group Γ_0^+. For $j \neq 0$ $\exp(k\, T(v(i), v(j))) = W_{j,i,k}$ while for $j = 0$ $\exp(k\, T(v(i), v(0))) = V_{-i,k}$ where $W_{j,i,k}$ and $V_{-i,k}$ are the generators of the commutator subgroup of the orthogonal group given in [12, p. 397, 398]. For $0 < i < j$ set $g(i, j) = \exp(\sqrt{2}/2\, v(i)v(-j)) \exp(-\sqrt{2}/2\, v(i)v(-j))$ then $g(i, j)$ and $g(-i, j)$ belong to G' as does $(g(i, j)g(-i, j))^2 = -1$. Thus $\nu(G') = \nu(\Gamma_0^+)$ and G' contains $\{\pm 1\}$, the kernel of ν, implying that $G' = \Gamma_0^+$.

4.2. Assume $m = 2n$, $n \geqq 4$. Since V has maximal index with respect to B we may select a basis $w(i)$, $|i| \in [1, n]$, for V such that $B(w(i), w(j)) = 0$ if $i + j \neq 0$, $B(w(-i), w(i)) = 1$. With respect to this basis for V, L has a basis consisting of the transformations $T(i, j) = T(w(i), w(j))$, $|i|$, $|j| \in [1, n]$, $i < j$. The subspace H spanned by the $T(-i, i)$, $i \in [1, n]$, is a standard Cartan subalgebra, and the root elements of L with regard to H are $e(a) = T(i, j)$, $i < j$, $|i| \neq |j|$. The root a belonging to $T(i, j)$ is the linear function $\mu_i + \mu_j$ where for any i, $w(i) \cdot h = \mu_i(h)w(i)$, $h \in H$. A maximal simple system of roots Δ is given by

$$a_i = \mu_i + \mu_{-(i+1)},\ i \in [1, n-1],\ a_n = \mu_n + \mu_{n-1}\ ,$$

and canonical generators for L in terms of Δ are

$$\begin{aligned} e(a_i) &= T(-i-1, i),\ i \in [1, n-1],\ e(a_n) = T(n, n-1)\ ; \\ h(a_i) &= T(-i, i) - T(-i, -1, i+1),\ i \in [1, n-1]\ , \\ h(a_n) &= T(-n, n) + T(-n+1, n-1)\ ; \\ e(-a_i) &= T(-i, i+1),\ i \in [1, n-1],\ e(a_n) = T(-n+1, -n)\ . \end{aligned}$$

If λ_i, $i \in [1, n]$, is the basis for H^* dual to the basis $h(a_i)$, $i \in [1, n]$, for H, then the λ_i span the space of integral linear functions on H. The irreducible U-module $A(r)$, $r \in [1, n-2]$, has maximal weight λ_r since the basis element $a(\{1, \cdots, r\})$ is a maximal vector. The irreducible U-modules with maximal weight λ_{n-1} and λ_n are obtained from the spin representation of C_+ as follows: in C set $f = w(1) \cdots w(n)$ and let I be the right ideal of C generated by f. I has a basis consisting of the elements $f = f(\emptyset)$, $f(\tau) = f \cdot w(-i_1) \cdots w(-i_s)$, where $\tau = \{i_1, \cdots, i_s\}$

belongs to the set Σ of all strictly increasing sequences of integers from $[1, n]$. I is a minimal right ideal of C and can be taken as the space of spinors (see [4, p. 55]). The restriction to C_+ of the representation of C afforded by I is the spin representation of C_+, and the subspace $I_+(I_-)$ spanned by the elements $f(\tau)$, length of τ even (odd), is an irreducible C_+ submodule of I ([4, p. 45)] hence an irreducible U-module. The mapping φ sends

$$T(i, j) \to -2^{-1}w(j)w(i),\ i < j,\ |i| \neq |j|\ ;$$
$$T(-i, j) \to 2^{-1}(1 - w(i)w(-i)),\ i \in [1, n]\ .$$

Hence (7) and (8) imply that $f(\{n\})\ (f(\emptyset) = f)$ is a maximal vector of weight $\lambda_{n-1}(\lambda_n)$ in $I_-(I_+)$. Since the tensor algebra on $V \times I_- \times I_+$ contains a maximal vector of weight λ_k, $k \in [1, n]$, $\mathscr{M} = \{V, I_-, I_+\}$ satisfies the hypothesis of Theorem 2.

4.3. The root elements of L are represented on the modules in $\mathscr{M}$ by nilpotent linear transformations of degree 2 consequently the formula $y \exp(ad\ x) = \exp(-x)y \exp(x)$ can be applied when y and x are linear transformations representing the root elements of L on these spaces. Computing the value of $F_M(s(a, k))$ on "vector monomials" with the aid of this formula and (1) we obtain that for each M in $\mathscr{M}$ $F_M(s(a, k)) = \exp(k\, e(a)^R)$, $s(a, k) \in G$, R the representation of L afforded by M. Since the root elements of L are the transformations $T(i, j)$, $i < j$, $|i| \neq |j|$, 4.1.3 together with the above result implies that $F_V(s(a, k)) = \nu(\exp(k\, e(a)^\varphi))$, $F_{I\pm}(s(a, k)) = \rho_\pm(\exp(k\, e(a)^\varphi))$ where ν is the vector representation of Γ_0^+ and $\rho_\pm$ are the half-spin representations.

Let η be the canonical isomorphism of Γ_0^+ with the diagonal of $\Gamma_0^+ \times \Gamma_0^+ \times \Gamma_0^+$. Then $\psi = \eta \circ (\nu \times \rho_- \times \rho_+)$ is a homomorphism of Γ_0^+ onto $G(\mathscr{M})$ whose kernel, the intersection of the kernels of ν, ρ_-, and ρ_+, is $\{1\}$ [4, III. 6.1].

The structure of C [4, II. 2.1] enables one to view the elements of Γ_0^+ as linear automorphisms on the 2^n-dimensional vector space I. Γ_0^+ is an irreducible linear algebraic group being generated by the one dimensional irreducible subgroups $\{\exp(k\, e(a)^\varphi : k \in K\}$. The subspace $I_+(I_-)$ is an invariant subspace under the identity representation of Γ_0^+ and the induced representation of Γ_0^+ is $\rho_+(\rho_-)$. Thus ψ^{-1} is easily seen to be a rational representation of $G(\mathscr{M})$. The rationality of ψ follows immediately from Proposition 2 [3, p. 11] and Proposition 6 [2, p. 141].

4.4. Assume $m = 2n + 1$, $n \geqq 3$. Select a basis $w(i)$, $|i| \in [0, n]$, for V such that

$$B(w(i), w(j)) = 0 \text{ if } i + j \neq 0,\ B(w(-i), w(i)) = 1,\ i \in [1, n]\ ,$$

and

$$B(w(0), w(0)) = 2 .$$

With respect to this basis for V, L has basis $T(i, j) = T(w(i), w(j))$, $|i|, |j| \in [0, n]$, $i < j$. The subspace of L spanned by the elements $T(-i, i)$ $i \in [1, n]$, is a standard Cartan subalgebra H, and the root elements of L with respect to H are the transformations $e(a) = T(i, j)$, $|i|, |j| \in [0, n]$, $|i| \neq |j|$, $i < j$. The root a belonging to $T(i, j)$ is the linear function $\mu_i + \mu_j$ where for any i, μ_i is defined by $w(i)\cdot h = \mu_i(h)w(i)$, $h \in H$. A maximal simple system of roots is given by $a_i = \mu_i + \mu_{-(i+1)}$ $i \in [1, n-1]$, $a_n = \mu_n + \mu_0 = \mu_n$, and generators of L are

$$e(a_i) = T(-(i+1), i),\ i \in [1, n-1],\ e(a_n) = T(0, n) ;$$
$$h(a_i) = T(-i, i) - T(-(i+1), i+1),\ i \in [1, n-1],\ h(a_n) = 2T(-n, n) ;$$
$$e(-a_i) = T(-i, i+1),\ i \in [1, n-1],\ e(a_n) = T(-n, 0) .$$

If λ_i, $i \in [1, n]$, is the basis in H^* dual to $h(a_i)$, $i \in [1, n]$, then $A(r)$, $r \in [1, n-1]$, has maximal weight λ_r since $a(\{1, \cdots, r\})$ is a maximal vector with respect to the chosen basis for V.

Let V' be a $2n$-dimensional vector space over K with basis $v(i)$, $|i| \in [1, n]$, and B' a symmetric bilinear form on V' defined by $B'(v(i), v(j)) = 0$ if $i + j \neq 0$, $B(v(-i), v(i)) = -2$, $i \in [1, n]$. Let C' be the Clifford algebra on V' with respect to $Q'(x) = B(x, x)$. C' is isomorphic to C_+ [4, II. 2.6], the isomorphism sends $w(0)w(i)$ to $v(i)$, $|i| \in [1, n]$. C' is represented faithfully on the minimum right ideal $I' = fC'$, $f = v(1) \cdots v(n)$, and the isomorphism of C_+ onto C' following φ maps

$$T(i, j) \to 4^{-1}v(j)v(i),\ |i|, |j| \in [1, n],\ |i| \neq |j| ;$$
$$T(-i, 0) \to -2^{-1}v(-i),\ i \in [1, n] ;$$
$$T(-i, i) \to 2^{-1} + 4^{-1}v(i)v(-i),\ i \in [1, n] .$$

Thus f is a maximal vector of weight λ_n in I' and $\mathscr{M} = \{V, I'\}$ satisfies the hypothesis of Theorem 2.

4.5. As in 4.3 we have the reduced Clifford group Γ_0^+ generated by the elements $\exp(k\, e(a)^\varphi)$, and

$$F_V(s(a, k)) = \nu'(\exp(k\, e(a)^\varphi) ,$$
$$F_{I'}(s(a, k)) = \rho(\exp(k\, e(a)^\varphi)) ,$$

where ν' is the restriction of the vector representation to Γ_0^+, and ρ is the composite of the isomorphism of C_+ onto C' with the representation of C' on I'. ρ induces a group monomorphism ρ' on Γ_0^+, and so $\psi = \eta' \circ (\nu' \times \rho')$, η' the canonical injection of Γ_0^+ onto the diagonal of

$\Gamma_0^+ \times \Gamma_0^+$, is the required isomorphism. The birationality of ψ follows exactly as in 4.3.

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