ON CONTINUITY OF MULTIPLICATION IN A COMPLEMENTED ALGEBRA

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The present study was originally motivated by reading a paper of M. Rajagopalan [6]. The author and Sr. K. A. Bellcourt were able to obtain the same result as M. Rajagopalan under much weaker hypothesis. Latter the author realized that in the case of an H^*-algebra and a two-sided H^*-algebra the condition "$\|xy\| \leqq M\|x\|\cdot\|y\|$" is a consequence of the other axioms in the definition. The same is true about right H^*-algebra if we assume continuity of involution (It was pointed out to the author that P. J. Laufer established this result in 1958. The author arrived at it independently of Laufer).

The present paper deals with the question whether the same is true about complemented algebras. It turns out that we have to assume topological semi-simplicity in some sense and continuity of the mapping $x \rightarrow xa$ (Sr. Bellcourt should be credited with the idea of assuming topological semi-simplicity). Below we have a new characterization of complemented algebras. Lemma 1 may be of interest by itself.

2. LEMMA 1. *Let A be an* (associative) *algebra whose underlying vector space is a Banach space* (in other words A would be a Banach algebra if we would assume that $\|xy\| \leqq M\|x\|\cdot\|y\|$). *Suppose that the mapping R_a: $x \rightarrow xa$ is continuous for each $a \in A$; suppose also that the mapping L_a: $x \rightarrow ax$ is continuous for each a in some dense subset B of A. Then A is a Banach algebra.*

Proof. Let $a \in A$ and let a_n be a sequence of members of B such that $a_n \rightarrow a$. Then the sequence $\|a_n\|$ is bounded; also the sequence $\|a_n x\|$ is bounded for each x in A ($\|a_n x\| \leqq \|R_x\|\cdot\|a_n\|$). From Theorem 5, page 80, of [3] we may conclude that there exists a positive number M such that $\|L_{a_n}\| \leqq M$ for each n.

Now let $x_m, x \in A$ be such that $x_m \rightarrow x$. Then $\|ax - a_n x_m\| = \|ax - a_n x + a_n x - a_n x_m\| \leqq \|a - a_n\|\cdot\|R_x\| + M\|x - x_m\|$.

From this we may draw two conclusions. First of all $B = A$ (note that the above inequality implies that $ax_m \rightarrow ax$: $\|ax - ax_m\| \leqq \|ax - a_n x_m\| + \|a_n - a\|\cdot\|R_{x_m}\|$). Secondly, combining this fact with the above we see that the mapping $<x, y> \rightarrow xy$ is continuous. This conclusion can be obtained also using a result of [2].

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3. DEFINITION. Let A be a complex (associative) algebra whose underlying vector space is a Hilbert space. Then A will be called a right almost complemented algebra (r. a. c. algebra) if it has the following properties:

(i) In each ideal $I \neq 0$ there exists an element a such that there is no sequence x_n with property that $a + x_n - ax_n \rightarrow 0$ (we may say that a is not topologically quasi-regular) (compare with [11]). We will refer to this property as a topological semi-simplicity.

(ii) The mapping R_a: $x \rightarrow xa$ is continuous for each $a \in A$.

(iii) The orthogonal complement R^p of a right ideal R is a right ideal.

Note that (ii) implies that the closure of a right ideal is also a right ideal.

Let A be a fixed r. a. c. algebra.

LEMMA 2. *Every nonzero left ideal L in A contains a left projection* (*as in* [9] *and* [8] *it is understood that a projection is a nonzero element*).

Proof. The proof is a modification of the last part of the proof of Theorem 3.2 of [11]. (The first part of the proof is not valid since the mapping $x \rightarrow ax$ does not have to be continuous. However property (i) of the above definition is stronger than topological semi-simplicity of [11]). We take an element a in L which is not topologically right quasi-regular, consider $R = \{ax - x \mid x \in A\}$ and project a upon R^p.

LEMMA 3. *If e is a left projection in A then the mapping L_e: $x \rightarrow ex$ is continuous and $\| L_e \| \geqq 1$.*

Proof. Note that $\| x \|^2 = \| ex \|^2 + \| x - ex \|^2$ for each $x \in A$.

LEMMA 4. *There exists a positive number r such that $r \leqq \| e \|$ for each left projection e in A.*

Proof. If the lemma is not true then we can find a sequence e_n of left projections such that $\| e_n \| \rightarrow 0$. For each n let λ_n be a positive number such that $\| \lambda_n e_n \| = 1$. Then $\lambda_n \rightarrow \infty$ and $\| \lambda_n e_n x \| \leqq \| R \|$ for each $x \in A$. By Theorem 5 of [3] there exists a positive number M such that $\lambda_n \cdot \| L_{e_n} \| = \| L_{\lambda_n e_n} \| \leqq M$ for each n. It means that $\| L_{e_n} \| < 1$ for n large enough.

COROLLARY. *Each left projection in A is a finite sum of orthogonal primitive* (minimal) *projections.*

The following lemma is a generalization of Mazur-Gelfand theorem. It was established by the author and Sr. K. A. Bellcourt in the latter part of 1961 (presented to the society on February 22, 1962 [4]).

LEMMA 5. *Let A be a complex* (associative) *algebra whose underlying vector space is a Banach space. Assume further that the mapping R_a: $x \to xa$ is continuous for each $a \in A$ and that A is a division algebra. Then A is isomorphic to the field of complex numbers.*

Proof. The general idea of the proof is the same as in the proof of Theorem 22F in [5].

For each $a \in A$ let us write $|a|$ instead of $\|R_a\|$ ($|a|$ is the norm of the operator $x \to xa$). Let $x \in A$, $x \neq 0$; for each complex λ let y_λ be the inverse of $e - \lambda x$. If λ_0 is fixed then $y_\lambda = \sum_{k=0}^{\infty}(\lambda - \lambda_0)^k(y_{\lambda_0}x)^k y_{\lambda_0}$ provided λ_0 is close enough to λ_0 (note that that the series converges if $|\lambda - \lambda_0|\ \|y_{\lambda_0}x\| < 1$ since $\|(y_{\lambda_0}x)^k y_{\lambda_0}\| \leqq \|y_{\lambda_0}x\|\ (|y_{\lambda_0}x|)^{k-1}|y_{\lambda_0}|)$ for each k). If f is a bounded linear functional on A then the function

$$\varphi(\lambda) = f(y_\lambda) = \sum_{k=0}^{\infty}(\lambda - \lambda_0)^k f((y_{\lambda_0}x)^k y_{\lambda_0})$$

is analytic for each λ such that $e \neq \lambda x$. Now consider

$$u_\lambda = -\sum_{n=0}^{\infty}\lambda^{-n}(x^{-1})^{n+1}$$

in the region $|x^{-1}| < |\lambda|$. By direct inspection we verify that $\lim_{\lambda\to\infty}u_\lambda = -x^{-1}$ and $y_\lambda = (1/\lambda)u_\lambda$. It follows that

$$\lim_{\lambda\to\infty}\varphi(\lambda) = \lim_{\lambda\to\infty}\frac{1}{\lambda}f(u_\lambda) = 0 .$$

Invoking Liouville's theorem we conclude that $x = \lambda e$ (or rather $x = (1/\lambda)e$) for some complex λ.

LEMMA 6. *Let A be an algebra whose underlying vector space is a Hilbert space. If a A has a left adjoint* [8, page 52] *then the mapping L_a: $x \to ax$ is continuous.*

Proof. Let x_n, x and u in A be such that $x_n \to x$, $ax_n \to u$. Then $(u, z) = \lim_n\ (ax_n, z) = \lim_n\ (x_n, a^l z) = (ax, z)$ for each $z \in A$. It simply means that $u = ax$. The lemma now follows from closed graph theorem.

In the following lemmas A will again denote a fixed r.a.c. algebra.

LEMMA 7. *If e is a primitive left projection in A then eA and*

Ae are minimal ideals and eAe is isomorphic to the complex field.

Proof. The first part of the proof is almost identical with the proof of Lemma 3.3 of [11]. We may conclude that ue is not topologically right quasi-regular (if eu is not) since the mapping $x \to ex$ is continuous. The last part of the lemma follows from Lemma 5.

LEMMA 8. *If e is a primitive left projection in A then every element in eA has a left adjoint.*

Proof. If $a \in eA$ and $ae \neq 0$ consider the left ideal $Aa = Aea$. By Lemma 2 it contains a left projection $f, f = bea$ for some $b \in A$. Then $ef = ebea = \lambda a$ for some $\lambda \neq 0$ (if $\lambda = 0$ then $0 = fe = beae$ and so $be = 0$). Thus $a^l = \bar{\lambda}^{-1} fe$. If $ae = 0$ then we consider $a' = a + e$ instead of a...

REMARK. We could not use the first part of the proof of Theorem 1 of [9] since we do not know whether every idempotent in A is not topogically right quasi-regular.

Now we state our main result.

THEOREM 1. *Every r.a.c. algebra is a complemented algebra.*

Proof. As in the proof of Theorem 2 of [9] we show that there exists a dense subset B of A such that every element in B has a left adjoint in A. The theorem now follows Lemmas 1 and 6 above.

THEOREM 2. *Let A be an algebra whose underlying space is a Hilbert space. Suppose that every element x in A has a right adjoint x^r in A. Suppose that A has at least one of the following properties:*

(i) *The set of elements of A having a left adjoint is dense in A.*

(ii) *The mapping $x \to x^r$ is continuous.*

(iii) *A satisfies condition* (i) *of the definition of r.a.c. algebra.*

Then A is a two-sided H^*-algebra.

Proof. This theorem follows from Lemmas 1, 6 and Theorem 1. (Note that an orthogonal complement of a right ideal in A is again a right ideal).

BIBLIOGRAPHY

1. W. Ambrose, *Structure theorems for a special class of Banach Algebras,* Trans. Amer. Math. Soc., **57** (1945), 364-386.

2. R. Arens, *Linear topological division algebras*, Bull. Amer. Math. Soc., **53** (1947), 623-630.

3. S. Banach, *Théorie des opérations linéaires*, Warszawa, 1932.

4. Sister K. A. Bellcourt and P. P. Saworotnow, *Structure theorems for certain inner product algebras*, Notices Amer. Math. Soc., **9** (1962), 29.

5. L. H. Loomis, *An Introduction to Abstract Harmonic Analysis*, New York, 1953.

6. M. Rajagopalan, *Classification of algebras*, J. Indian Math. Soc., **22** (1958), 109-116.

7. ———, *H-algebras*, J. Indian Math. Soc., (N. S.) **25** (1961), 1-25.

8. P. P. Saworotnow, *On a generalization of the notion of H^*-algebra*, Proc. of Amer. Math. Soc., **8** (1957), 49-55.

9. *On the imbedding of a right complemented algebra into Ambrose's H^*-algebra*, Proc. Amer. Math. Soc., **8** (1957) 56-62.

10. M. F. Smiley, *Right H^*-algebras*, Proc. Amer. Math. Soc., **4** (1953), 1-4.

11. J. F. Smith and P. P. Saworotnow, *On some classes of scalar-product algebras*, Pacific J. of Math., **11** (1961), 739-750.

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