A GENERALIZATION OF POWER-ASSOCIATIVITY

J. MARSHALL OSBORN

Probably the most promising new identity to arise in a recent study of identities on commutative algebras [3] is

$$2((x^2\cdot x)x)x + (x^2\cdot x)x^2 = 3(x^2\cdot x^2)x \; . \tag{2}$$

This identity generalizes not only the power-associative identity, $x^2\cdot x^2 = (x^2\cdot x)x$, but also the generalization of the Jordan identity considered in [4]. In the present paper, we study the structure of commutative rings of characteristic relatively prime to 2, 3, 5, or 7 satisfying (1). This restriction on the characteristic will be assumed throughout the paper without further mention.

There are two obvious ways in which the structure theory of the class of rings studied here is noticeably weaker than the structure theory of power-associative rings. First of all, given a ring A satisfying (1) containing an idempotent e, there can exist elements of A which are annihilated by the operator $(2R_e - I)^2$ but not by $(2R_e - I)$. Secondly, defining the additive subgroups $A_\lambda = A_e(\lambda) = \{x \mid x \in A, \ xe = \lambda x\}$ for $\lambda = 0$, $1/2$, and 1, the relations $A_1A_0 = 0$ and $A_{1/2}A_{1/2} \subset A_1 + A_0$ are not valid in general. Despite these impediments, we see in § 1 that A may be decomposed simultaneously with respect to a set of mutually orthogonal idempotents in much the usual fashion. In § 2 we prove that, if A is simple of degree ≥ 3 satisfying the condition that $x(2R_e - I)^2 = 0$ if and only if $x(2R_e - I) = 0$ for all x in A, then A is a Jordan ring.

1. We begin our investigation by partially linearizing (1) to obtain

$$\begin{aligned} 4((yx\cdot x)x)x + 2(yx^2\cdot x)x + 2yx^3\cdot x + 2y(x^3\cdot x) + 2(yx\cdot x)x^2 \\ + yx^2\cdot x^2 + 2yx\cdot x^3 = 12(yx\cdot x^2)x + 3y(x^2\cdot x^2) \; . \end{aligned} \tag{2}$$

Then, setting $x = e$ in (2) immediately yields

$$4yR_e{}^4 - 8yR_e{}^3 + 5yR_e{}^3 - yR_e = 0, \quad \text{or}$$

$$y[(R_e - I)(2R_e - I)^2R_e] = 0 \; . \tag{3}$$

Defining $B_{1/2} = B_e(1/2) = \{x \mid x \in A, \ \ x(2R_e - I)^2 = 0\}$, it follows from (3) that A may be decomposed into the additive direct sum

$$A = A_1 + B_{1/2} + A_0 \; . \tag{4}$$

Received August 28, 1953. The research for this paper was supported by National Science Foundation Grant G-19052.

Another additive subgroup of A which will be useful is $C_{1/2} = C_e(1/2) = \{x \mid \exists y \varepsilon B_{1/2} \in y(2R_e - I) = x\}$. It is easy to see that $C_{1/2} \subset A_{1/2} \subset B_{1/2}$.

THEOREM 1 *Let A be a ring satisfying* (1) *with an idempotent e, and let $A_1, A_0, A_{1/2}, B_{1/2}, C_{1/2}$ be defined as above. Then A_1 and A_0 are subrings, $A_1A_0 \subset A_{1/2}$, $A_1B_{1/2} \subset B_{1/2} + A_0$, $A_0B_{1/2} \subset B_{1/2} + A_1$, $A_1C_{1/2} \subset A_{1/2}$, $A_0C_{1/2} \subset A_{1/2}$, $A_{1/2}A_{1/2} \subset A_1 + A_{1/2} + A_0$, $A_{1/2}C_{1/2} \subset A_1 + C_{1/2} + A_0$,* and *$C_{1/2}C_{1/2} \subset C_{1/2}$.*

To establish this theorem, we first partially linearize (2) and assume that the new variable of degree 3 is idempotent. This gives

$$\begin{aligned}
4(yx)R_e^2 &+ 4(ye\cdot x)R_e^2 + 4(yR_e^2\cdot x)R_e + 4yR_e^3\cdot x + 4(y\cdot xe)R_e^2 \\
&+ 2(ye\cdot x)R_e + 2yR_e^2\cdot x + 4(y\cdot xR_e^2)R_e + 2(y\cdot xe)R_e + 2ye\cdot x \\
&+ 4y(xR_e^3) + 2y(xR_e^2) + 2y\cdot xe + 2(yx)R_e^2 + 2(ye\cdot x)R_e \\
&+ 4(yR_e^2)(xe) + 2(y\cdot xe)R_e + 2(ye)(xe) + 2(yx)R_e \\
&+ 4(yx)(xR_e^2) + 2(ye)(xe) \\
&= 12(yx)R_e^2 + 24(ye\cdot xe)R_e + 12yR_e^2\cdot x + 12y(xR_e^2)\ ,
\end{aligned}$$

which simplifies to

$$\begin{aligned}
&(yx)[2R_e^3 - 5R_e^2 + R_e] + (ye\cdot x + y\cdot xe)[2R_e^2 + 2R_e + I] \\
(5)\qquad &+ (yR_e^2\cdot x + y\cdot xR_e^2)[2R_e - 5I] + 2(yR_e^3\cdot x + y\cdot xR_e^3) \\
&+ (ye\cdot xe)[-12R_e + 2I] + 2(yR_e^2\cdot xe + ye\cdot xR_e^2) = 0\ .
\end{aligned}$$

First, letting $x,\ y \in A_1$, this reduces to $(yx)[2R_e^3 - R_e^2 - 3R_e + 2I] = 0$. Since 1 is a root of this operator but 0 and 1/2 are not for any characteristic, we have $yx \varepsilon A_1$, or $A_1A_1 \subset A_1$. Similarly, if $x, y \varepsilon A_0$, (5) reduces to $(yx)[2R_e^3 - 5R_e^2 + R_e] = 0$, which gives $A_0A_0 \subset A_0$. And, choosing $y \in A_1$, $x \in A_0$ in (5) yields $(yx)[2R_e^3 - 3R_e^2 + 5R_e - 2I] = 0$, or $A_1A_0 \subset A_{1/2}$.

Suppose next that $y \in A_1$ and $x \in B_{1/2}$. Letting $w = x(R_e - 1/2I)$, we have $xe = 1/2x + w$, $we = 1/2w$, $xe\cdot e = 1/4x + w$, $(xe\cdot e)e = 1/8x + 3/4w$, and (5) becomes $(yx)[(2R_e^3 - 5R_e^2 + R_e) + (3R_e^2 + 3R_e + 3/2I) + (5/2R_e - 25/4I) + 9/4I + (-6R_e + I) + 3/2I] + (yw)[2R_e^2 + 2R_e + I + 2R_e - 5I + 3/2I - 12R_e + 2I + 4I] = 0$, or

$$(6)\qquad (yx)\left[2R_e^3 - 2R_e^2 + \frac{1}{2}R_e\right] + (yw)\left[2R_e^2 - 8R_e + \frac{7}{2}I\right] = 0\ .$$

Taking $w = 0$ in (6), we see that $A_1A_{1/2} \subset B_{1/2} + A_0$. But then $(yw) \varepsilon B_{1/2} + A_0$ in general and the component of (6) in A_1 is $\frac{1}{2}(yx)_1 = 0$, giving $A_1B_{1/2} \subset B_{1/2} + A_0$. This shows that the first term in (6) is zero, which implies that $(yw)[(2R_e - I)(R_e - 7/2I)] = 0$, or $A_1C_{1/2} \subset A_{1/2}$. Similarly, letting $y \in A_0$, $x \in B_{1/2}$ in (5) yields

$$(yx)\left[2R_e^3 - 4R_e^2 + \frac{5}{2}R_e - \frac{1}{2}I\right] + (yw)\left[2R_e^2 + 4R_e - \frac{5}{2}I\right] = 0\ ,$$

from which one gets $A_0B_{1/2} \subset B_{1/2} + A_1$ and $A_0C_{1/2} \subset A_{1/2}$.

Finally, let $x, y \in B_{1/2}$, $x(R_e - 1/2I) = w$, $y(R_e - 1/2I) = z$ in (5) to get

$$\begin{aligned}(yx)[(2R_e^3 &- 5R_e^2 + R_e) + (2R_e^2 + 2R_e + I) + \left(R_e - \frac{5}{2}I\right) + \frac{1}{2}I \\ &+ \left(-3R_e + \frac{1}{2}I\right) + \frac{1}{2}I\Big] + (yw + zx)[(2R_e^2 + 2R_e + I) \\ &+ (2R_e - 5I) + \frac{3}{2}I + (-6R_e + I) + \frac{3}{2}I\Big] \\ &+ (zw)[(-12R_e + 2I) + 4I] = 0\ ,\end{aligned}$$

or

$$\begin{aligned}(7) \qquad (yx)[2R_e^3 - 3R_e^2 + R_e] &+ (yw + zx)[2R_e^2 - 2R_e] \\ &+ (zw)[-12R_e + 6I = 0\ .\end{aligned}$$

Taking $w = z = 0$ in (7), we obtain first the relation $A_{1/2}A_{1/2} \subset A_1 + A_{1/2} + A_0$. If only z is zero, then the component of (7) in $B_{1/2}$ is $\{(yx)_{1/2}[2R_e - I] + 2(yw)_{1/2}\}(R_e - I)R_e = 0$, showing that $A_{1/2}C_{1/2} \subset A_1 + C_{1/2} + A_0$. If neither w nor z is zero, we may apply the operator $(2R_e - I)^2$ to (7) to get $(zw)(2R_e - I)^3 = 0$, or $C_{1/2}C_{1/2} \subset B_{1/2}$. But since $C_{1/2}C_{1/2} \subset A_{1/2}C_{1/2} \subset A_1 + C_{1/2} + A_0$, we have $C_{1/2}C_{1/2} \subset C_{1/2}$ to finish the proof of Theorem 1.

By constructing examples, it is not difficult to show that the relations given in Theorem 1 cannot be improved. To illustrate this proceedure, we shall show that the relation $A_1A_0 \subset A_{1/2}$ cannot be improved. Consider the commutative algebra spanned by the four elements e, a_1, $a_{1/2}$, a_0 over any field F, and let multiplication be defined by $e^2 = e$, $a_1a_0 = a_{1/2}$, $ea_i = ia_i$ ($i = 0, 1/2, 1$), where all other products of basis elements are assumed to be zero. To show that this algebra satisfies (1), it is sufficient to show that the complete linearization of of (1) is satisfied for all ways of replacing the variables by basis elements. If either four or five of these variables are replaced by e, the equation is satisfied by (3). If exactly three of the variables are replaced by e and the other two variables by a_1 and a_0 respectively, then the equation reduces to $(a_1a_0)[2R_e^3 - 3R_e^2 + 5R_e - 2I] = 0$ as in the proof of Theorem 1, and hence is satisfied. If any other combination of basis elements is substituted into the linearized form of (1), it is clear that every term will vanish, and the identity will be trivially satisfiəd.

Suppose now that a ring A satisfying (1) contains two orthogonal idempotents u and v. Although the elements of of $A_v(1)$ are not in general orthogonal to the elements of $A_u(1)$, we can prove that v is orthogonal to $A_u(1)$.

LEMMA 1 *If u and v are orthogonal idempotents, then* $A_u(1) \subset A_v(0)$.

For the proof of this lemma we linearize (2) so that two of the x's in each term become u's and the other two become v's. This gives

$$\begin{aligned} 4((yu\cdot u)v)v &+ 4((yu\cdot v)u)v + 4((yu\cdot v)v)u + 4((yv\cdot u)u)v \\ &+ 4((yv\cdot u)v)u + 4((yv\cdot v)u)u + 2(yu\cdot v)v + 2(yv\cdot u)u \\ &+ 2(yu\cdot u)v + 2(yv\cdot v)u + yu\cdot v + yv\cdot u \\ &= 12(yu\cdot v)u + 12(yv\cdot u)v \ . \end{aligned}$$

Taking $y \in A_u(1)$ and using the relation $yv\cdot u = 1/2yv$ which follows from Theorem 1, this becomes $(yv\cdot v)[4R_u^2 + 8R_u + 3I] = 2yv$, or

$$(yv\cdot v)](2R_u + I)(2R_u + 3I)] = 2yv \ . \tag{9}$$

Since $yv \in A_u(1/2)$, we see from (9) that $(yv\cdot v) \in A_u(1/2)$ also. But then (9) reduces to $8yv\cdot v = 2yv$, or $(yv)[4R_v - I] = 0$. Thus, $yv = 0$ and $A_u(1) \subset A_v(0)$ as desired.

We are now ready to consider how the decomposition of A with respect to the idempotent $u + v$ is related to the decompositions with respect to u and v separately. We shall prove.

THEOREM 2 *Let u and v be orthogonal idempotents in a ring A satisfying* (1). *Then $R_uR_v = R_vR_u$ and*

$$\begin{aligned} A_{u+v}(1) &= A_u(1) + B_u\left(\frac{1}{2}\right) \cap B_v\left(\frac{1}{2}\right) + A_v(1) \ , \\ B_{u+v}\left(\frac{1}{2}\right) &= B_u\left(\frac{1}{2}\right) \cap A_v(0) + A_u(0) \cap B_v\left(\frac{1}{2}\right) , \\ A_{u+v}\left(\frac{1}{2}\right) &= A_u\left(\frac{1}{2}\right) \cap A_v(0) + A_u(0) \cap A_v\left(\frac{1}{2}\right) , \\ C_{u+v}\left(\frac{1}{2}\right) &- C_u\left(\frac{1}{2}\right) \cap A_v(0) + A_u(0) \cap C_v\left(\frac{1}{2}\right) , \\ A_{u+v}(0) &= A_u(0) \cap A_v(0) \ . \end{aligned}$$

For the proof of Theorem 2 we shall need

LEMMA 2 *If u and v are orthogonal idempotents and if* $y \in B_n(1/2) \cap B_v(1/2)$, *then* $yv \in B_u(1/2) \cap B_v(1/2)$, $yu\cdot v = yv\cdot u = 1/4y$, *and* $y \in A_{u+v}(1)$. *Hence,* $A_u(1/2) \cap B_v(1/2) = A_u(1/2) \cap A_v(1/2)$.

By Theorem 1, we have $yv \in B_u(1/2) + A_u(1)$ and hence $(yv)(2R_u - I)^2 \in A_u(1) \subset A_v(0)$. On the other hand, $yv \in B_v(1/2)$, giving $(yv)(2R_u - I)^2 \in B_v(1/2) + A_v(1)$. Thus, $(yv)(2R_u - I)^2 = 0$, or $(yv) \in B_u(1/2)$, to give the first assertion of the lemma.

From Theorem 1 we also get the relation $y(2R_u - I)R_v(2R_u - I) = 0$, or $4(yu\cdot v)u = 2yu\cdot v + 2yv\cdot u - yv$. Using this relation and $4yu\cdot u = 4yu - y$, equation (8) with $y \in B_u(1/2) \cap B_v(1/2)$ becomes

$$\begin{aligned}4(yu\cdot v)v &- yv\cdot v + 2(yu\cdot v)v + 2(yv\cdot u)v - yv\cdot v + 4(yu\cdot v)u\\ &- yu\cdot u + 4(yv\cdot u)v - yv\cdot v + 2(yv\cdot u)v + 2(yv\cdot v)u - yv\cdot v\\ &+ 4(yv\cdot v)u - yv\cdot v + 2(yu\cdot v)v + 2(yv\cdot u)u + 2(yu\cdot u)v\\ &+ 4(yv\cdot v)u - yv\cdot v + 2(yu\cdot v)v + 2(yv\cdot u)u + 2(yu\cdot u)v\\ &+ 2(yv\cdot v)u + yu\cdot v + yv\cdot u - 12(yu\cdot v)u - 12(yv\cdot u)v = 0\ ,\end{aligned}$$

or

$$\begin{aligned}8(yu\cdot v)v &- 4(yv\cdot u)v + 8(yv\cdot v)u + 2(yv\cdot u)u - 8(yu\cdot v)u\\ &+ 2(yu\cdot u)v + yu\cdot v + yv\cdot u - 5yv\cdot v - yu\cdot u = 0\ .\end{aligned}$$

Reducing this equation again given

$$\begin{aligned}8yu\cdot v &- 2yu - 2yu\cdot v - 2yv\cdot u + yu + 8yv\cdot u - 2yu + 2yv\cdot u\\ &- \frac{1}{2}yv - 4yu\cdot v - 4yv\cdot u + 2yv + 2yu\cdot v - \frac{1}{2}yv + yu\cdot v\\ &+ yv\cdot u - 5yv + \frac{5}{4}y - yu + \frac{1}{4}y = 0\ ,\end{aligned}$$

or $5yu\cdot v + 5yv\cdot u - 4yu - 4yv + 3/2y = 0$, which may be put in the form

$$y\Big[\Big(R_u - \frac{1}{2}I\Big)\Big(5R_v - \frac{3}{2}I\Big) + \Big(R_v - \frac{1}{2}I\Big)\Big(5R_u - \frac{3}{2}I\Big)\Big] = 0\ .$$

If $y \in A_u(1/2) \cap B_v(1/2)$, then (10) reduces to $y(R_v - 1/2I)(5R_u - 3/2I) = 0$, or, $y \in A_v(1/2)$. Thus $A_u(1/2) \cap B_v(1/2) = A_u(1/2) \cap A_v(1/2)$. But then $y \in B_u(1/2) \cap B_v(1/2)$ implies that $y(R_u - 1/2I) \in A_u(1/2) \cap A_v(1/2)$ and $y(R_u - 1/2I)(5R_v - 3/2I) = y(R_u - 1/2I)$. Using this relation, (10) reduces to $y[R_u - 1/2I + R_v - 1/2I] = 0$, or $y \in A_{u+v}(1)$. Since $y(R_u - 1/2I)R_v = 1/2y(R_u - 1/2I)$, we also have $yu\cdot v = 1/2yv + 1/2yu - 1/4y = 1/2y(u+v) - 1/4y = 1/4y$. And finally, $yv\cdot u = 1/4y$ by symmetry.

Returning to the proof of the theorem, let y be an arbitrary element of $A_{u+v}(1)$ and let $y = y_1 + y_{1/2} + y_0$ be its decomposition with respect to u. Then the equation $y(u+v) = y$ gives $y_1 + y_{1/2}(u+v) + y_0v = y_1 + y_{1/2} + y_0$, which breaks into the two equations $y_{1/2}(u+v) = y_{1/2}$

and $y_0v = y_0$ since $y_{1/2}(u+v) \in B_u(1/2) + A_u(1)$ and $y_0v \in A_u(0)$. Thus, $y_0 \in A_v(1)$ and $y_{1/2}(2R_v - I) = - y_{1/2}(2R_u - I) \in A_u(1/2) \cap A_{u+v}(1)$, leading to $y_{1/2}(2R_v - I)^2 = y_{1/2}(2R_v - I)[2(R_u + R_v) - I - 2R_u] = y_{1/2}(2R_v - I)(I - 2R_u) = 0$ and $y_{1/2} \in B_v(1/2)$. We have shown that $A_{u+v}(1)$ is contained in $A_u(1) + B_u(1/2) \cap B_v(1/2) + A_v(1)$. Conversely, $A_u(1)$ and $A_v(1)$ are clearly in $A_{u+v}(1)$, while $B_u(1/2) \cap B_v(1/2)$ is in by Lemma.

Next, suppose that $y \in B_{u+v}(1/2)$ and let $y = y_1 + y_{1/2} + y_0$ again be the decomposition of y with respect to u. Then,

$$0 = (y_1 + y_{1/2} + y_0)\Big[(R_u + R_v)^2 - (R_u + R_v) + \frac{1}{4}I\Big] = \frac{1}{4}y_1 + y_{1/2}\Big[R_u^2 - R_u + \frac{1}{4}I + R_uR_v + R_vR_u + R_v^2 - R_v\Big] + y_0\Big[R_v^2 - R_v + \frac{1}{4}I\Big],$$

and breaking this equation into components gives $1/4y_1 + y_{1/2}[R_uR_v + R_vR_u + R_v^2 - R_v] = 0$, and $y_0[R_v^2 - R_v + 1/4I] = 0$ or $y_0 \in B_v(1/2)$. Letting $y_{1/2} = w_1 + w_{1/2} + w_0$ be the decomposition of $y_{1/2}$ with respect to v, the former equation becomes $1/4y_1 + w_{1/2}[R_uR_v + R_vR_u + R_v^2 - R_v] = 0$. But $1/4y_1$ is the only term in the last equation with a component in $A_v(0)$, so that $y_1 = 0$ and $y_{1/2} \in B_{u+v}(1/2)$. By symmetry, $w_1 = 0$ and $w_0 \in B_u(1/2)$, giving $w_{1/2} = (y_{1/2} - w_0) \in B_u(1/2) \cap B_v(1/2)$. Then Lemma

$$0 = w_{1/2}[R_uR_v + R_vR_u + R_v^2 - R_v] = w_{1/2}\Big[\frac{1}{4}I + \frac{1}{4}I - \frac{1}{4}I\Big]\frac{1}{4}w_{1/2},$$

showing that $y_{1/2} = w_0 \in A_v(0)$. This proves that $B_{u+v}(1/2)$ is contained in $B_u(1/2) \cap A_v(0) + A_u(0) \cap B_v(1/2)$, and the converse is immediate.

If $y \in A_{u+v}(1/2)$, the argument above shows that $y = y_{1/2} + y_0$ where $y_{1/2} \in B_u(1/2) \cap A_v(0)$ and $y_0 \in A_u(0) \cap B_v(1/2)$. Then, $0 = (y_{1/2} + y_0)[R_u + R_v - 1/2I] = y_{1/2}(R_u - 1/2I) + y_0(R_v - 1/2I)$, and breaking into components gives $y_{1/2} \in A_u(1/2)$ and $y_0 \in A_v(1/2)$. Hence $A_{u+v}(1/2)$ is contained in $A_u(1/2) \cap A_v(0) + A_u(0) \cap A_v(1/2)$, and the converse is obvious. If $z \in C_{u+v}(1/2)$, then there exists an element $y \in B_{u+v}(1/2)$ such that $z = y[R_u + R_v - 1/2I]$. But then $z = (y_{1/2} + y_0)[R_u + R_v - 1/2I] = y_{1/2}(R_u - 1/2I) + y_0(R_v - 1/2I) \in C_u(1/2) \cap A_v(0) + A_u(0) \cap C_v(1/2)$, and the converse is again obvious.

Finally, let $y \in A_{u+v}(0)$ and let $y = y_1 + y_{1/2} + y_0$ be the decomposition of y with respect to u. Then $0 = y(u+v) = y_1 + y_{1/2}(u+v) + y_0v = 0$, giving $y_0 \in A_v(0)$ and $y_1 + y_{1/2}(u+v) = 0$. If $y_{1/2} = w_1 + w_{1/2} + w_0$ is the decomposition of $y_{1/2}$ with respect to v, the latter equation gives $y_1 + w_1 + w_{1/2}(u+v) + w_0u = 0$, and the component of this equation in $A_v(0)$ is $y_1 + w_0u = 0$. But then $w_0 \in A_u(1) + A_u(0)$, so that

$0 = y_{1/2}(4R_u^2 - 4R_u + I) = w_1 + w_{1/2}(4R_u^2 - 4R_u + I) + w_0$. The component of the last equation in $A_v(0)$ is $w_0 = 0$, implying that $y_1 = 0$ and that $y_{1/2} \in A_{u+v}(0)$. By symmetry, we also have $w_1 = 0$, so that $y_{1/2} = w_{1/2} \in B_u(1/2) \cap B_v(1/2) \subset A_{u+v}(1)$. Thus, $y_{1/2} = 0$, and $A_{u+v}(0) \subset A_u(0) \cap A_v(0)$. The The converse of this inclusion is trivial.

The relation $R_uR_v = R_vR_u$ was shown to hold on elements of $B_u(1/2) \cap B_v(1/2)$ in Lemma 2, and it is easy to check that it also holds for elements of each of the other additive subgroups into which we have decomposed A.

Now that we have established Theorem 2, it is an easy matter to decompose A simultaneously with respect to any number of mutually orthogonal idempotents.

THEOREM 3 *Let $e_1, e_2, \cdots, e_n$ be a set of orthogonal idempotents in a ring A satisfying* (1) *whose sum is the unity element of A, and define $A_i = A_{e_i}(1)$, $A_{ij} = A_{e_i}(1/2) \cap A_{e_j}(1/2)$, $B_{ij} = B_{ei}(1/2) \cap B_{e_j}(1/2)$, and $C_{ij} = C_{e_i}(1/2) \cap C_{e_j}(1/2)$ for $1 \leq i, j \leq n$ and $i \neq j$. Then A is the additive direct sum of the A_i's and the B_{ij}'s, and $A_iA_i \subset A_i$, $A_iA_j \subset A_{ij}$, $A_iB_{ij} \subset B_{ij} + A_j$, $A_iC_{ij} \subset A_{ij}$, $B_{ij}B_{ij} \subset A_i + B_{ij} + A_j$, $A_{ij}A_{ij} \subset A_i + A_{ij} + A_j$, $A_{ij}C_{ij} \subset A_i + C_{ij} + A_j$, $C_{ij}C_{ij} \subset C_{iz}$, $B_{ik}B_{jk} \subset B_{ik}$, $A_{ij}A_{jk} \subset A_{ik}$, $B_{ij}C_{jk} \subset C_{ik}$,* and *$C_{ij}C_{jk} = A_iB_{jk} = B_{ij}B_{kl} = 0$* for *$1 \leq i, j, k, l \leq n$* and *$i, j, k, l$* distinct.

The first eight inclusion relations listed in this theorem follow immediately from Theorem 1. To show $B_{ij}B_{jk} \subset B_{ik}$, we let $u = e_i + e_j$ and $w = e_i + e_j + e_k$ and observe that $B_{ij}B_{jk} \subset B_u(1/2) + A_u(0)$ and $B_{ij}B_{jk} \subset A_w(1)$, leading to $B_{ij}B_{jk} \subset B_{ik} + B_{jk} + A_k$. But, by symmetry, we also have $B_{ij}B_{jk} \subset B_{ik} + B_{ij} + A_k$. But, by symmetry, we also have $B_{ij}B_{jk} \subset B_{ik} + B_{ij} + A_i$, giving $B_{ij}B_{jk} \subset B_{ik}$. This same calculation also shows that $C_{ij}C_{jk} \subset B_{ik}$. However, $C_{ij}C_{jk} \in C_{e_j}(1/2) \cap A_w(1) = C_{ij} + C_{jk}$, giving $C_{ij}C_{jk} = 0$. Looking at the product A_iB_{jk} with respeot to the three idempotents e_i, e_j, e_k, we get that this product is contained respectively in $A_{ij} + A_{ik}$, $B_{jk} + B_{ij} + A_j$, and $B_{jk} + B_{ik} + A_k$. Since the mutual intersection of three is zero, $A_iB_{jk} = 0$. Observing that $B_{ij}B_{kl} \subset A_u(1)B_{kl}$ for $u = e_i + e_j$, we also have $B_{ij}B_{kl} = 0$.

For the two remaining inclusion relations given in Theorem 3, we must make a little longer calculation. Linearing (2) completely and setting two of the variables equal to e_i, and the other three equal to e_j, x, y respectively where $x \in B_{ij}$ and $y \in B_{jk}$, we get

$$\begin{aligned}
&4((ye_j\cdot x)e_i)e_i + 4((y\cdot xe_j)e_i)e_i + 4(y(xe_i\cdot e_j))e_i + 4(y(xe_j\cdot e_i))e_i \\
&\qquad 4y((xe_i\cdot e_i)e_j) + 4y((xe_i\cdot e_j)e_i + 4y((xe_j\cdot e_i)e_i + 2y(xe_i\cdot e_j) \\
&\qquad 2(ye_j\cdot x)e_i + 2(y\cdot xe_j)e_i + 4(ye_j)(xe_i\cdot e_i) + 2(ye_j)(xe_i) \\
&\qquad = 24(ye_j\cdot xe_i)e_i + 12y(xe_j\cdot e_i)\,.
\end{aligned}$$

Using the relation $xe_i \cdot e_j = xe_j \cdot e_i = 1/4x$ from Lemma 2, this reduces to

$$(ye_j \cdot x + xe_j \cdot y)[4R_{e_i}^2 + 2R_{e_i}] + 3xe_i \cdot y + (yx)\left[2R_{e_i} - \frac{5}{2}I\right]$$
$$+ 4(ye_j)(xe_i \cdot e_i) + 2(ye_j)(xe_i) - 24(ye_j \cdot xe)e_i = 0 .$$

Letting $xe_j = 1/2x + w$ and $ye_j = 1/2y + z$, and noting that $xe_i = x - xe_j = 1/2x - w$ and that $zw = 0$, our equation becomes

$$(yx)[4R_{e_i}^2 - 2R_{e_i}] + (zx)[4R_{e_i}^2 - 10R_{e_i} + 2I]$$
$$+ (yw)[4R_{e_i}^2 + 14R_{e_i} - 6I] = 0 .$$

Since yx, zx, and yw are all in B_{ik}, we may replace $4R_{e_i}^2$ by $4R_{e_i} - I$ here, giving

$$(yx)[2R_{e_i} - I] + (zx)[- 6R_{e_i} + I[+ (yw)[18R_{e_i} - 7I] = 0 , \tag{11}$$

Applying the operator $(2R_{e_i} - I)$ to (11), we get

$$(zx)[- 12R_{e_i}^2 + 8R_{e_i} - I] + (yw)[36R_{e_i}^2 - 32R_{e_i} + 7I] = 0 ,$$

which reduces to $(zx)[- 4R_{e_i} + 2I] + (yw)[4R_{e_i} - 2I] = 0$, or $(yw - zx) \in A_{ik}$. On the other hand, we may set $e = e_i$ in (7) to obtain

$$(yx)[2R_{e_i}{}^3 - 3R_{e_i}^2 + R_{e_i}] + (yw + zx)[2R_{e_i}^2 - 2R_{e_i}] = 0 ,$$

which simplifies to $(yx)[1/2R_{e_i} - 1/4I] + (yw + zx)[- 1/2I] = 0$, or $(yw + zx) \in C_{ik}$. Thus, yw and zx are both in A_{ik}, and (11) reduces to $(yx)[2R_{e_i} - I] - 2zx + 2yw = 0$, or $(yw - zx) \in C_{ik}$. We finally have $zx \in C_{ik}$, giving the relation $B_{ij}C_{jk} \subset C_{ik}$. The remaining relation — $A_{ij}A_{ik} \subset A_{ik}$ — may be derived by taking $z = w = 0$ in (11).

2. This section will be devoted to the proof of

THEOREM 4. *Let A be a simple ring satisfying* (1) *and containing two orthogonal idempotents u and v such that $u + v$ is not the unity element of A and such that $B_u(1/2) = A_u(1/2)$ and $B_v(1/2) = A_v(1/2)$. Then A is a Jordan ring.*

If A doesn't contain a unity element, then we may adjoin one and the resulting ring will still satisfy the same identity [3, Theorem 1]. It is therefore sufficient to prove the theorem for a ring R which contains a unity element and which is either simple or is the result of adjoining a unity element to a simple ring. In the latter case, every ideal of the augmented ring contains the original ring [1, Lem. 2, p. 506], and in either case, the idempotents $e_1 = u$, $e_2 = v$, and $e_3 = 1 - u - v$ are mutually orthogonal idempotents of R which add to the unity element. Adopting the terminology of Theorem 3, we see from the last sentence of Lemma 2 that the remaining hypotheses of Theorem 4 are equivalent to the relations $B_{ij} = A_{ij}$ for $1 \leq i, j \leq 3$ and $i \neq j$.

We must next deduce more information from our identity about the products of elements from different components of R. Linearizing (1) completely, replacing two of the variables by the idempotent $e = e_i$ $(i = 1, 2, 3)$, and assuming that the other three variables satisfy $xe = \lambda x$, $ye = \mu y$, and $ze = \nu z$, we obtain

$$\begin{aligned}
&[yz\cdot x + yx\cdot z + xz\cdot y]R_e^2 + [(yz\cdot e)x + (yx\cdot e)z + (xz\cdot e)y]R_e \\
&\quad + (\lambda + \mu + \nu - 3)[(yz\cdot e)x + (yx\cdot e)z + (xz\cdot e)y] \\
&\quad + [(yz)R_e^2\cdot x + (yx)R_e^2\cdot z + (xz)[R_e^2\cdot y] \\
&\quad + \Big[\Big(\mu + \nu - 6\lambda + \frac{1}{2}\Big)(yz\cdot x) + \Big(\lambda + \mu - 6\nu + \frac{1}{2}\Big)(yx\cdot z) \\
&\quad + \Big(\lambda + \nu - 6\mu + \frac{1}{2}\Big)(xz\cdot y)]R_e \\
&\quad + \Big(\lambda^2 + \mu^2 + \nu^2 + \lambda\mu + \lambda\nu - 6\mu\nu + \frac{1}{2}\lambda \\
&\quad + \frac{1}{2}\mu + \frac{1}{2}\nu\Big)yz\cdot x + (\lambda^2 + m^2 + \nu^2 \\
&\quad + \lambda\mu + \mu\nu - 6\lambda\mu + \frac{1}{2}\lambda + \frac{1}{2}\mu + \frac{1}{2}\nu\Big)yx\cdot z \\
&\quad + \Big(\lambda^2 + \mu^2 + \nu^2 + \lambda\mu + \mu\nu - 6\lambda\nu + \frac{1}{2}\lambda + \frac{1}{2}\mu \\
&\quad + \frac{1}{2}\nu\Big)xz\cdot y = 0\,.
\end{aligned} \tag{12}$$

We first set $\lambda = \mu = 0$, $\nu = 1$ in this equation to get

$$\begin{aligned}
&(yz\cdot x + xz\cdot y)\Big[R_e^2 + \frac{3}{2}R_e + \frac{3}{2}I\Big] + [(yz\cdot e)x + (xz\cdot e)y][R_e - 2I] \\
&\quad + (yz)R_e^2\cdot x + (xz)R_e^2\cdot y + (yx\cdot z)\Big[R_e^2 - \frac{11}{2}R_e + \frac{3}{2}I\Big] = 0\,,
\end{aligned}$$

which reduces to

$$(yz\cdot x + xz\cdot y)\Big[R_e^2 + 2R_e + \frac{3}{4}I\Big] + (yx\cdot z)\Big[R_e^2 - \frac{11}{2}R_e + \frac{3}{2}I\Big] = 0\,.$$

Separating this equation into components and using the convention that the subscript 1, 1/2, or 0 indicates the component in $A_e(1)$, $A_e(1/2)$, or $A_e(0)$ respectively, the last equation yields

$$2[yz\cdot x + xz\cdot y]_{1/2} = yx\cdot z\,. \tag{13}$$

Next, setting $\lambda = \mu = 0$, $\nu = 1/2$ in (12) gives

$$(yz\cdot x + xz\cdot y)\left[R_e^2 + R_e + \frac{1}{2}I\right] + (yx\cdot z)\left[R_e^2 - \frac{5}{2}R_e + \frac{1}{2}I\right]$$
$$+ [(yz\cdot e)x + (xz\cdot e)y][R_e - 5/2I] + [(yz)R_e^2\cdot x + (xz)R_e^2\cdot y] = 0 ,$$

which becomes

$$[(yz)_1\cdot x + (xz)_1\cdot y][R_e^2 + 2R_e - I] + [(yz)_{1/2}\cdot x + (xz)_{1/2}\cdot y][R_e^2$$
$$+ 3/2R_e - 1/2I] + (yx\cdot z)[R_e^2 - 5/2R_e + 1/2I] = 0 .$$

This separates into the two equations

$$2[(yz)_{1/2}\cdot x + (xz)_{1/2}\cdot y]_1 = [yx\cdot z]_1 , \tag{14}$$

$$[(yz)_1\cdot x + (xz)_1\cdot y] + 2[(yz)_{1/2}\cdot x + (xz)_{1/2}\cdot y]_{1/2} = 2[yx\cdot z]_{1/2} . \tag{15}$$

The equations that we have just derived may be put in operator form by defining for each $x \in A_e(0)$ the mappings $S_x\colon A_e(1) \to A_e(1/2)$, $T_x\colon A_e(1/2) \to A_e(1)$, and $U_x\colon A_e(1/2) \to A_e(1/2)$ by the equations $(z_1)S_x = zx$, $(z_{1/2})T_x = (zx)_1$, and $(z_{1/2})U_x = (zx)_{1/2}$ respectively. In this notation, equations (13) – (15) become

$$\frac{1}{2}S_{yx} = S_yU_x + S_xU_y , \tag{16}$$

$$\frac{1}{2}T_{yx} = U_yT_x + U_xT_y , \tag{17}$$

$$U_{yx} = U_yU_x + U_xU_y + \frac{1}{2}(T_yS_x + T_xS_y) . \tag{18}$$

We shall make use of these relations to prove.

LEMMA 3 *In the ring R, $A_{ij}A_{ij} \subset A_i + A_j$ for $1 \leq i, j \leq 3$ and $i \neq j$.*

Choosing e to be that one of e_1, e_2, e_3 which is neither e_i nor e_j, we see that $A_e(0) = A_i + A_{ij} + A_j$. Consider the subalgebra of $A_e(0)$ defined by $D = \{x \mid x \in A_0,\ S_x = T_x = 0\}$. By (18), the mapping $x \to U_x$ defines a homomorphism of D into the Jordan ring of all endomorphisms of $A_e(1/2)$ with kernel $C = \{x \mid x \in A_0,\ S_x = T_x = U_x = 0\}$. If $x \in C$ and $y \in A_0$, then $S_{yx} = T_{yx} = U_{yx} = 0$ by (16), (17), and (18), so that C is an ideal of A_0. Furthermore, $CA_e(1) = CA_e(1/2) = 0$ by the definition of S, T, and U, showing that C is an ideal of R. But R contains no nonzero ideals lying within $A_e(0)$, implying that $C = 0$ and that D is a Jordan ring.

Since e_i and e_j are contained in D, we have $D = D_i + D_{ij} + D_j$, where $D_i \subset A_i$, $D_{ij} \subset A_{ij}$, and $D_j \subset A_j$. The fact that D is Jordan implies that $D_{ij}D_{ij} \subset D_i + D_j \subset A_i + A_j$. But since $A_{ij}A_e(1) = 0$ and $A_{ij}A_e(1/2) \subset A_e(1/2)$, we see that $A_{ij} \subset D$, giving $A_{ij} \subset D_{ij}$ and $A_{ij}A_{ij} \subset D_{ij}D_{ij} \subset A_i + A_j$.

In order to prove our next lemma, we need to compute two more special cases of (12). Using Lemma 2, we may now assume that $A_e(1/2A_e(1/2) \subset A_e(1) + A_e(0)$. First, taking $\lambda = 0$, $\mu = 1/2$, $\nu = 1$ and saving just the component in $A_e(0)$ gives

$$-\frac{3}{2}\Big[\frac{1}{2}yx\cdot z + \frac{1}{2}xz\cdot y\Big]_0 + \Big[\frac{1}{4}yx\cdot z + \frac{1}{4}xz\cdot y]_0$$
$$+\Big[-yz\cdot x + \frac{5}{2}yx\cdot z + \frac{5}{2}xz\cdot y\Big]_0 = 0\ ,$$

or

$$2[y_{1/2}x_0\cdot z_1 + x_0z_1\cdot y_{1/2}]_0 = (y_{1/2}z_1)_0\cdot x_0\ . \tag{19}$$

Secondly, setting $\lambda = 0$, $\mu = \nu = 1/2$ in (12) and keeping just the component in $A_e(0)$, we get

$$-2\Big[\frac{1}{2}(yx)_{1/2}\cdot z + (yx)_1\cdot z + \frac{1}{2}(xz)_{1/2}\cdot y + (xz)_1\cdot y]_0$$
$$+\Big[\frac{1}{4}(yx)_{1/2}\cdot z + (yx)_1\cdot z + \frac{1}{4}(xz)_{1/2}\cdot y + (xz)_1\cdot y\Big]_0$$
$$+\Big[-\frac{1}{2}yz\cdot x + \frac{5}{4}yx\cdot z + \frac{5}{4}xz\cdot y\Big]_0 = 0\ ,$$

which simplifies to

$$\begin{aligned} 2[(y_{1/2}x_0)_{1/2}\cdot z_{1/2} + (z_{1/2}x_0)_{1/2}\cdot y_{1/2}]_0 \\ + [(y_{1/2}x_0)_1\cdot z_{1/2} + (z_{1/2}x_0)_1 y_{1/2}]_0 = 2(y_{1/2}z_{1/2})_0\cdot x_0\ . \end{aligned} \tag{20}$$

LEMMA 4. *Let G_0 be the additive subgroup of $A_0 = A_e(0)$ generated by all elements of the form $(y_{1/2}z_1)_0$ and $(y_{1/2}z_{1/2})_0$. Then either $G_0 = A_0$ or we may adjoin e_3 to G_0 to obtain A_0.*

If x_0 is any element of A_0, we see from (19) that $(y_{1/2}z_1)_0\cdot x_0$ is in G_0 and from (20) that $(y_{1/2}z_{1/2})_0\cdot x_0$ is in G_0. Thus, G_0 is an ideal of A_0. Defining the ideal G_1 of A_1 analogously, we now consider $G = G_1 + A_{1/2} + G_0$. But $GA_1 = G_1A_1 + (A_{1/2}A_1)_{1/2} + (A_{1/2}A_1)_0 + A_0A_1 \subset G_1 + A_{1/2} + G_0 + A_{1/2} = G$, and similarly $GA_{1/2} \subset G$ and $GA_0 \subset G$. Thus, G is an ideal of R and is nonzero since $A_{1/2} \subset G$. It follows from the definition of R that either $G = R$ or we may adjoin e_3 to obtain R. In either case the lemma holds.

We shall assume hereafter that i, j, k form a permutation of 1, 2, 3, and we shall indicate in which part of the decomposition of R an element lies by attaching the appropriate subscripts. Then, taking $e = e_j$ in Lemma 4 yields the following.

COROLLARY. *A_{ik} is generated by the elements of the form $(y_{ij}z_{jk})$ and A_i is generated by e_i and by the elements of the form $(y_{ij}z_j)_i$* and $(y_{ij}z_{ij})_i$.

LEMMA 5. *The following relations hold in R: $x_{ik}(y_{ij}z_j)_i \in A_k$,* $x_k(y_{ij}z_{ij}) = 0$, and $x_{ik}(y_{ij}z_{ij})_i \in A_{ik}$.

To establish this lemma, we observe first that $x_{ik}(y_{ij}z_j)_i \in A_{ik}A_i \subset A_{ik} + A_k$. On the other hand, using (19) with $e = e_j$ gives $x_{ik}(y_{ij}z_j)_i = 2[y_{ij}x_{ik} \cdot z_j + x_{ik}z_j \cdot y_{ij}] = 2y_{ij}x_{ik} \cdot z_j \in A_{jk}A_j \subset A_{jk} + A_k$, and combining the two relations gives $x_{ik}(y_{ij}z_j)_i \in A_k$. Secondly, taking $e = e_k$ in (13) gives $x_k(y_{ij}z_{ij}) = 2[x_k y_{ij} \cdot z_{ij} + x_k z_{ij} \cdot y_{ij}] = 0$. And finally, setting $e = e_j$ in (20) yields $x_{ik}(y_{ij}z_{ij})_i = y_{ij}x_{ik} \cdot z_{ij} + z_{ij}x_{ik} \cdot y_{ij} \in A_{ik}$.

LEMMA 6. *If x_i is an element of A_i such that $x_iA_{ik} \subset A_{ik}$, then $x_iA_{ij} \subset A_{ij}$. Similarly, $x_iA_{ik} \subset A_k$ implies $x_iA_{ij} = 0$.*

Suppose that $x_iA_{ik} \subset A_{ik}$. Then (20) gives

$$x_i(y_{ik}z_{jk}) = (x_iy_{ik})z_{jk} \in A_{ik}z_{jk} \subset A_{ij}$$

to show that $x_iA_{ij} \subset A_{ij}$. On the other hand, if $x_iA_{ik} \subset A_k$, then (20) yields $(x_i(y_{ik}z_{jk}) = (x_iy_{ik})z_{jk} \in A_kz_{jk} \subset A_{jk} + A_k$. However, we also have $x_i(y_{ik}z_{jk}) \in x_iA_{ij} \subset A_{ij} + A_j$, and thus $x_iA_{ij} = 0$.

We are now in a position to prove.

LEMMA 7. *In the ring R, $A_iA_{ij} \subset A_{ij}$ and $A_iA_j = 0$. Hence $A_i + A_{ij} + A_j$ is a Jordan ring.*

By the corollary to Lemma 4, A_i is generated by e_i and elements of the form $(y_{ij}z_j)_i$ and $(y_{ij}z_{ij})_i$. Then $(y_{ij}z_j)_iA_{ik} \subset A_k$ by Lemma 5 and so $(y_{ij}z_j)_iA_{ij} = 0$ by Lemma 6. On the other hand, Lemma 5 also gives $(y_{ij}z_{ij})_iA_{ik} \subset A_{ik}$, which implies $(y_{ij}z_{ij})_iA_{ij} \subset A_{ij}$ by Lemma 6. Hence, $A_iA_{ij} \subset A_{ij}$, $(y_{ij}z_j)_i = 0$, and A_i is generated by e_i and elements of the form $(y_{ij}z_{ij})_i$. But then $A_iA_k = 0$ by the second relation of Lemma 5. The relations which we have just established show that the Jordan ring $D = \{x \mid x \in A_{e_k}(0),\ xA_e(1) = 0,\ xA_{e_k}(1/2) \subset A_{e_k}(1/2)\}$ used in the proof of Lemma 3 is all of $A_{e_k}(0) = A_i + A_{ij} + A_j$.

Now that Lemma 7 has been proved, equation (15) yields the three special cases

$$
\begin{aligned}
&z_{ij}x_i \cdot y_{ik} = z_{ij} \cdot x_i y_{ik} \,, \quad z_{ik}y_{ij} \cdot x_i = z_{ik} \cdot y_{ij}x_i \,, \\
&z_{ij}x_{ik} \cdot y_{ik} + z_{ij}y_{ik} \cdot x_{ik} = z_{ij} \cdot x_{ik}y_{ik} \,,
\end{aligned} \tag{21}
$$

while (20) yields

$$
[y_{jk}x_{ij} \cdot z_{ik}]_i = [y_{jk}z_{ik} \cdot x_{ij}]_i \,. \tag{22}
$$

Since $A_i + A_{ij} + A_j$ is a Jordan ring, we also have

$$
\begin{aligned}
&z_{ij} \cdot x_i y_i = z_{ij}x_i \cdot y_i + z_{ij}y_i \cdot x_i \,, \ z_{ij}x_i \cdot y_j = z_{ij}y_j \cdot x_i \,, \\
&y_{ij}z_{ij} \cdot x_i = [y_{ij}x_i \cdot z_{ij} + z_{ij}x_i \cdot y_{ij}]_i \,, \\
&[y_{ij}x_i \cdot z_{ij}]_j = [z_{ij}x_i \cdot y_{ij}]_j \,.
\end{aligned} \tag{23}
$$

Theorem 4 may now be established by verifying that the linearized Jordan identity is satisfied for all possible ways of choosing the arguments in the various components of R. These calculations all proceed easily using Theorem 3, Lemma 3, Lemma 7, and equations (21)–(23). However, this computation may be avoided by appealing to [2, Theorem 5], which states that a certain set of hypotheses implied by the properties that we have established for R implies power-associativity. It should be remarked that Mrs. Losey's theorem is stated only for simple algebras in which the decomposition is well behaved with respect to any idempotent in the algebra. However, her proof actually establishes the theorem for simple rings containing a unity element or with unity element adjoined in which properties about the decomposition with respect to an idempotent are only assumed for three particular idempotents which add to the unity element.

References

1. A. A. Albert, *A theory of power-associative commutative algebras*, Trans. Amer. Math. Soc., **69** (1950), 503–527.

2. N. Losey, *Simple commutative non-associative algebras satisfying a polynomial identity of degree five*, Ph. D. thesis, University of Wisconsin, 1963.

3. J. M. Osborn, *Identities on non-associative algebras*, to appear in Can. J. Math.

4. ———, *An identity of degree four*, submitted to Proc. Amer. Math. Soc.

University of Wisconsin