OPEN IDEALS IN $C(X)$

LOWELL A. HINRICHS

A group topology on the ring $C(X)$, of all real-valued, continuous functions on X, is said to have the *ideal closure property*, (I.C.P), in case the closure of any ideal is simply the intersection of all maximal ideals containing it. In this paper we consider which ideals of $C(X)$ can be open with respect to such a topology.

In § 2, a characterization of such ideals is given and it is shown that the family $\mathscr{S}$, of all such ideals, is itself a fundamental system of neighborhoods of zero with respect to a ring topology having I.C.P. In § 3 we consider the two extremes, where $\mathscr{S}$ is the family of all ideals and where $\mathscr{S}$ consists only of finite intersections of maximal ideals. The former class is characterized as the class of p-space (spaces for which every prime ideal of $C(X)$ is maximal) and the latter as the class of pseudo-compact spaces (spaces for which every $f \in C(x)$ is bounded). In the final section it is shown that if P is a countable discrete subset of the Stone-Cech compactification of X, then $\bigcap\{M^p; p \in P\} \in \mathscr{S}$ if and only if P is C-embedded in $X \cup P$.

1. The notation and terminology will be that of [1]. Many of the arguments will depend upon theorems and exercises of [1]. In order to avoid lengthy restatements of these results, when such results are used, we will simply give a reference to the appropriate statement. To simplify the reader's task all references will be given to [1]. The original source of these results can be determined by consulting the notes at the end of this book.

Throughout the paper X will denote a topological space and $C(X)$ the ring of all real-valued continuous functions on X. The term "topology", unless explicitly stated to the contrary, will always mean Hausdorff topology.

Although it is assumed that the reader is familiar with the material in the first few chapters of [1], we will recall some of the basic definitions and results which will be used throughout. For $f \in C(X)$ we set $Z(f) = \overleftarrow{f}(0)$ and for an ideal I we set $Z[I] = \{Z(f); f \in I\}$. An ideal I is called a z-ideal if $Z(g) \in Z[I]$ implies $g \in I$. It will be recalled that there is a one-to-one correspondence between the maximal ideals of $C(X)$ and the points of the Stone-Cech compactification, βX, of X. Explicitly this correspondence, $p \rightarrow M^p$, is given by the Gelfand-

Received November 21, 1963. This research was supported jointly by the National Science Foundation and the United States Air Force under grants NSF 81-6199 and AF AFOSR 62-162.

Kolmogorff theorem, [1; p. 120], which states that for each $p \in \beta X$

$$M^p = \{f \in C(X); p \in cl_{\beta X} Z(f)\} .$$

A second class of ideals which will be of importance for our purposes is the class of ideals of the form

$$O^p = \{f \in C(X); p \in \text{int. } cl_{\beta X} Z(f)\} .$$

We will frequently make use of the fact that every $f \in C(X)$ has a unique continuous extension to a function f^* of βX into the one-point compactification of the real number system. In particular it should be noted that if $X \subseteqq T \subseteqq \beta X$ and $f \in C(T)$ then $(f \mid X)^*(t) = f(t)$ and that if $f, g \in C(X)$, $q \in \beta X$ with $f^*(q)$ and $g^*(q)$ real numbers then $(f + g)^*(q) = f^*(q) + g^*(q)$ and $(fg)^* = f^*(q) \cdot g^*(q)$. Finally, we recall that there exists a largest subspace υX of βX such that for every $f \in C(X)$, $(f^* \mid \upsilon X) \in C(\upsilon X)$.

If A is a subset of βX, the symbol $\bar{A}$ will always refer to the closure of A in βX and A^0 will always denote the interior of A in βX. The symbol N will be reserved throughout to denote the set of nonnegative integers.

2. If I is an ideal of $C(X)$ we will set $\theta(I) = \{p \in \beta X; I \subseteqq M^p\}$ and if P is a subset of βX we will set $J(P) = \cap \{M^p; p \in P\}$. It is apparent that a topology $\mathscr{T}$ on $C(X)$ will have I.C.P. if and only if for every ideal I of $C(X)$, $cl_{\mathscr{T}} I = J \circ \theta(I)$. We begin our investigation with a straightforward observation regarding the restrictions placed on open ideals by this condition.

2.1. LEMMA. *Suppose that an ideal I is open with respect to some group topology on $C(X)$ having* I.C.P. *Then $I = J \circ \theta(I)$ and hence I is a z-ideal. Moreover, for every $p \in \beta X$ and every $f \in M^p$, $[f + I] \cap O^p \neq \phi$.*

Proof. Suppose I is an ideal of $C(X)$ and $\mathscr{T}$ is a topology, having I.C.P., for which I is open. Since I is open and $cl_{\mathscr{T}} I = J \circ \theta(I)$, for any $f \in J \circ \theta(I)$ we must have $[f + I] \cap I \neq \phi$. Let $g, h \in I$ such that $f + h = g$. Thus $f = g - h \in I$ and hence $I = J \circ \theta(I)$. That I is a z-ideal now follows from the fact that every maximal ideal is a z-ideal and an intersection of z-ideals is a z-ideal. The final asssertion follows from the fact that, for every $p \in \beta X$, M^p is the unique maximal ideal containing O^p, [1; 7.13, page 106], and hence $cl_{\mathscr{T}} O^p = M^p$.

In the last lemma we saw that, if $\mathscr{T}$ has I.C.P., then for every $p \in \beta X$, $cl_{\mathscr{T}} O^p = M^p$. The following theorem shows that under certain conditions the converse holds. In particular we have:

2.2. THEOREM. *Suppose $\mathscr{T}$ is a group topology for $C(X)$ possessing a basis for the neighborhood system of zero consisting of ideals. Then $\mathscr{T}$ has* I.C.P. *if and only if for every $p \in \beta X$, $cl_{\mathscr{T}} O^p = M^p$.*

Proof. The necessity is obvious, in view of the fact that, for each $p \in \beta X$, M^p is theunique maximal ideal containing O^p. On the other hand if the condition is statisfied then every maximal ideal is closed and hence $cl_{\mathscr{T}} I \subseteqq J \circ \theta(I)$ for every ideal I. To complete the proof it is sufficient to show that, if I is an ideal, $f \in J \circ \theta(I)$ and V a $\mathscr{T}$-open ideal, then $[f + V] \cap I \neq \phi$.

We begin by considering the case where f is a nonnegative function. For each $p \in \theta(I)$, there exists $h_p \in V$ such that $f - h_p \in O^p$; i.e., $\overline{Z(f - h_p)}^0$ is a neighborhood of p. Let us suppose that, for each $p \in \theta(I)$, we have chosen such a function $h_p \in V$. Then $\{\overline{Z(f - h_p)}^0; p \in \theta(I)\}$ is an open cover of $\theta(I)$ and $\theta(I)$ is compact [1; 7.0, p. 112]. Thus there exists a finite subcover, say $\{\overline{Z(f - h_i)}^0;\ i = 1, \cdots, n\}$. Without loss of generality we may assume that each h_i is nonnegative; for since V is a z-ideal and f is nonnegative we have $|h_i| \in V$ and $Z(f - h_i) \subseteqq Z(f - |h_i|)$. Thus the functions $f^{1/n}, h^{1/n}, \cdots, h_n^{1/n}$ are well-defined continuous functions Moreover $Z(h_i^{1/n}) = Z(h_i)$ and hence $h_i^{1/n} \in V$ for $i = 1, \cdots, n$. Now, $(f^{1/n} - h_1^{1/n})(f^{1/n} - h_2^{1/n}) \cdots (f^{1/n} - h_n^{1/n}) \in f + V$; i.e., there exists $h \in V$ such that $f + h = (f^{1n} - h_1^{1/n}) \cdots (f^{1/n} - h_n^{1//n})$. In addition,

$$Z(f + h) = Z(f^{1/n} - h_1^{1/n}) \cdots (f^{1/n} - h_n^{1/n}))$$
$$= \bigcap_{i=1}^{n} Z(f^{1/n} - h_i^{1/n}) = \bigcup_{i=1}^{n} Z(f - h_i)$$

and hence

$$\overline{Z(f + h)}^0 = \left(\bigcup_{i=1}^{n} \overline{Z(f - h_i)}\right)^0 \supseteqq \bigcup_{i=1}^{n} \overline{Z(f - h_i)}^0 \supseteqq \theta(I)\ .$$

By [1; 7.0, p. 112] we have $f + h \in I$.

Now if f is arbitrary in $J \circ \theta(I)$ we decompose f, in the usual manner, into the difference of two positive functions. Explicitly, let $f = f^+ - f^-$ where $f^+ = \frac{1}{2}(|f| + f)$ and $f^- = \frac{1}{2}(|f| - f)$. Since $Z(f^+) \supseteqq Z(f)$ and $Z(f^-) \supseteqq Z(f)$, we have $f^+, f^- \in J \circ \theta(I)$. Using the first part of the proof we choose functions $g_1, g_2 \in V$ and $k_1, k_2 \in I$ such that $f^+ + g_1 = k_1$ and $f^- + g_2 = k_2$. Thus $f + (g_1 - g_2) = (f + g_1) - (f^- + g_2) = k_1 - k_2 \in I$. Since V is an ideal $g_1 - g_2 \in V$ and hence $[f + V] \cap I \neq \phi$.

We now turn our attention to whether or not there exists any topologies satisfying the conditions of the above theorem. The next lemma will lay the basis for such an example

2.3. LEMMA. *Suppose* I_1 *and* I_2 *are ideals of* $C(X)$ *such that for every* $p \in \beta X$ *and every* $f \in M^p$, $[f + I_j] \cap O^p \neq \phi$, $j = 1, 2$. *Then* $[f + (I_1 \cap I_2)] \cap O^p \neq \phi$.

Proof. As in the last proof, we begin by supposing that f is nonnegative. Then $f^{1/2} \in M^p$ and hence there exists $g_1 \in I_1$ and $g_2 \in I_2$ such that $f^{1/2} + g_1 \in O^p$ and $f^{1/2} + g_2 \in O^p$. But $Z(f - g_1 g_2) \supseteqq Z(f^{1/2} + g_1) \cap Z(f^{1/2} + g_2) \in Z[O^p]$ and hence $f - g_1 g_2 \in O^p$. Since $-g_1 g_2 \in I_1 \cap I_2$ we have $[f + (I_1 \cap I_2)] \cap O^p \neq \phi$. If f is arbitrary in M^p, we express f as $f^+ - f^-$ and proceed as in the proof of the last theorem.

2.4. THEOREM. *The family of all maximal ideals of* $C(X)$ *is a subbase for the neighborhood system of zero, with respect to a ring topology on* $C(X)$ *having* I.C.P. *Moreover, an ideal is open with respect to this topology if and only if it is a finite intersection of maximal ideals.*

Proof. It is easily seen that any nonvoid family of ideals of $C(X)$ is a subbase for the neighborhood system of zero, with respect to a (possibly not Hausdorff) ring topology on $C(X)$. That the topology described in the theorem is Hausdorff follows from the fact $C(X)$ is semi-simple; i.e., the singleton zero is closed.

In view of Theorem 2.2 and Lemma 2.3, to see that this topology has I.C.P. it is sufficient to see that, if $p, q \in \beta X$ and $f \in M^q$, then $[f + M^p] \cap O^q \neq \phi$. If $p = q$, then $-f \in M^p = M^q$ and hence $0 = f - f \in [f + M^p] \cap O^q$. If $p \neq q$ then there exists a $g \in C(\beta X)$ that vanishes on a neighborhood of p and takes on only the value 1 on a neighborhood of q. It follows that $gf \in M^p$ and $f - fg \in O^q$.

To prove the concluding statement suppose that I is an ideal, open with respect to this topology. Then there exists a finite subset P of βX such that $J(P) \subseteqq I$ and hence $\theta(I) \subseteqq \theta \circ J(P)$. But $\theta \circ J(P) = P$, for if $q \notin P$ then $\{q\}$ and P are completely separated and hence there exists $g \in C(\beta X)$ such that g vanishes on a neighborhood of P and $g(q) = 1$. Thus $P \subseteqq \overline{Z(g)}$ and $q \notin \overline{Z(g)}$. It follows that $q \notin \theta \circ J(P) = \cap \{\overline{Z(h)}; h \in J(P)\}$, [1; 7.0 p. 112]. Thus $\theta(I) \subseteqq \theta \circ J(P) = P$ and as such $\theta(I)$ is finite. The proof is completed by noting that from Lemma 2.2 we have $I = J \circ \theta(I) = \cap \{M^p; p \in \theta(I)\}$.

In the remainder of the paper, the topology defined by taking the family of maximal ideals as a subbase for the neighborhood system of zero will be referred to as the *maximal ideal topology.*

2.5. THEOREM. *There exists a largest element in the lattice of all ring topologies on* $C(X)$ *which have* I.C.P. *and a fundamental system of neighborhoods of zero consisting of ideals. The family*

$\mathscr{S}$ of all ideals which are open with respect to this topology contains every ideal which is open with respect to some group topology having I.C.P. *Moreover, the family $\mathscr{S}$ can be characterized as the collection of all ideals I satisfying the following condition.*

(∗) *For every $p \in \beta X$ and every $f \in M^p$, $[f + I] \cap O^p \neq \phi$.*

Proof. Let $\mathscr{S}$ be the family of all ideals satisfying (∗). From Lemma 2.3 it follows that $\mathscr{S}$ is closed under finite intersection. As was noted earlier, any family of ideals closed under finite intersection is a basis for the neighborhood system of zero with respect to a (possibly not Hausdorff) ring topology on $C(X)$. In particular $\mathscr{S}$ is such a basis. Moreover since $\mathscr{S}$ contains all maximal ideals, the topology determined by $\mathscr{S}$ is finer than the maximal ideal topology. Since the latter is Hausdorff so is that determined by $\mathscr{S}$. From Theorem 2.2 it follows that $\mathscr{S}$ has I.C.P. Finally from Lemma 2.1 we know, if an ideal I is open with respect to some group topology having I.C.P., then I satisfies (∗) and hence is a member of $\mathscr{S}$.

3. In the last section we saw that for an arbitrary space X there exists at least one topology on $C(X)$ having I.C.P. and possessing a fundamental system of neighborhoods of zero, consisting of ideals; namely, the maximal ideal topology. From Theorem 2.5 we know that the topology determined by $\mathscr{S}$ is another and that it is at least as fine as the maximal ideal topology. We now ask, for which spaces, if any, these topologies coincide.

That they do not invariably coincide is easily seen by considering an infinite p-space. Recall that X is a *p-space* in case every prime ideal of $C(X)$ is maximal. It is shown in [1; 4J, p. 63] that a p-space can be characterized as a space X for which every ideal of $C(X)$ is an intersection of maximal ideals. It follows that, for a p-space, $\mathscr{S}$ is simply the family of all ideals and the topology determined by $\mathscr{S}$ is the discrete topology. From Theorem 2.4 we know that an ideal is open with respect to the maximal ideal topology only if it is a finite intersection of maximal ideals. Hence the two topologies do not agree on any infinite p-space. Indeed, since every ideal of $\mathscr{S}$ is an intersection of maximal ideals, a p-spaces can be characterized as a space for which $\mathscr{S}$ is the family of all ideals or equivalently as a space for which the topology determined by $\mathscr{S}$ is the discrete topology.

There is nonetheless an extensive class of spaces for which the two topologies agree. In this section we shall characterize this class as the class of pseudo-compact spaces; i.e., those spaces for which every $f \in C(X)$ is bounded.

3.1. LEMMA. *If P is a countable discrete subset of βX and $J(P) \in \mathscr{S}$, then all of the accumulation points of P are in $\beta X - \upsilon X$.*

Proof. Suppose $P = \{p_i; i \in N\}$ is a countable discrete subset of $\beta X, J(P) \in \mathscr{S}$, and $q \in \upsilon X$ is an accumulation point of P. For each $n \in N$, set $P_n = \{p_i; i \leqq n\}$. Then, for each $n \in N, P_n$ and $\{q\}$ are disjoint subsets of βX and hence there exists $f_n \in C(\beta X)$ such that $0 \leqq f_n \leqq 1$, $f_n[P_n] = 1$ and f_n vanishes on a neighborhood of q(i.e., $(f_n \mid X) \in M^q$). Let such an f_n be chosen for each $n \in N$ and set $f = \sum_{n+1}^{\infty} f_n 2^{-n}$. It is easily seen that $f(p) \neq 0$, for any $p \in P$, and that $Z(f) \supseteqq \bigcap_{n=1}^{\infty} Z(f_n)$. Since $q \in \upsilon X, Z[M^p]$ is closed under countable intersections, [1; 8.4, p. 117] and hence $Z(f \mid X) \in Z[M^q]$. Since $J(P) \in \mathscr{S}$ it satisfies (*) and hence $[(f \mid X) + J(P)] \cap O^q \neq \phi$. Choose $h \in J(P)$ and $g \in O^q$ such that $(f \mid X) + h = g$. Since $\overline{Z(g)}$ is a neighborhood of q, there exists $p \in P$ such that $p \in \overline{Z(g)}$. Moreover $h \in J(P)$ so $p \in \overline{Z(h)}$ and hence $p \in \overline{Z(g - h)} = \overline{Z(f \mid X)}$. But the latter is impossible since $(f \mid X)^*(p) - f(p) \neq 0$.

3.2. LEMMA. *If P is a countable discrete subset of βX which is C-embedded in $X \cup P$, then $J(P) \in \mathscr{S}$.*

Proof. Suppose that P is as described in the lemma, $q \in \beta X$ and $f \in M^q$. First we will show that $A = P \cap (\beta X - \overline{Z(f)})$ and $Z(f)$ are completely separated. To this end we will construct a function $g \in C(X \cup P)$ such that $Z(f) = Z(g) \cap X$ and $g(a) \neq 0$ for any $a \in A$. If A is finite the existence of such a function is obvious. Thus suppose that A is infinite, say $A = \{a_n; n \in N\}$. By a standard argument it can be shown that for each $n \in N$ there exists a unit $u_n \in C(X)$ such that $0 \leqq f^2 \cdot u_n \leqq 1$ and $(f^2 \cdot u_n)^*(a_n) = 1$. For each n we choose such a unit u_n and set $g = \sum_{n=1}^{\infty} f^2 \cdot u_n \cdot 2^{-n}$. Clearly we have $Z(g) = Z(f)$ and $g^*(a_n) \neq 0$ for any $n \in N$. Since P is C-embedded in $X \cup P$, the function $(1/g^*)$ has a continuous extension to $X \cup P$, say h. Since $(gh)^*$ must take on both the value 0 and 1 at any point of $\bar{A} \cap \overline{Z(g)}$, we have $\bar{A} \cap \overline{Z(f)} = \bar{A} \cap \overline{Z(g)} = \phi$. Thus by the usual argument there exists $k \in C(X)$ such that $A \subseteqq \overline{Z(k)}$ and $\overline{Z(f)} \subseteqq \overline{Z(k-1)}^0$. Thus $P \subseteqq A \cup \overline{Z(f)} \subseteqq \overline{Z(f \cdot k)}$ and $q \in \overline{Z(f - fk)}^0$. Hence we have $f - fk \in [f + J(P)] \cap O^q$. This completes the proof.

3.3. THEOREM. *A necessary and sufficient condition that the family $\mathscr{S}$ consists precisely of all finite intersections of maximal ideals is that the underlying space be pseudo-compact.*

Proof. Suppose that X is pseudo-compact. Since $C(X) \cong C(\beta X)$

we need only consider $\mathscr{S}$ with respect to the compact space βX. If Q is any infinite subset of βX then Q contains a countably infinite discrete subset P. But P must have an accumulation point in $\beta X = \upsilon(\beta X)$. By Lemma 3.1, $J(P) \notin \mathscr{S}$. But clearly if $J(P)$ does not satisfy (*) and $J(Q) \subseteq J(P)$, then $J(Q)$ does not satisfy (*) and hence is not a member of $\mathscr{S}$.

Conversely if X is not pseudo-compact then X contains a countable discrete C-embedded subset P, [1; 1.20, p. 20]. By Lemma 3.2, $J(P) \in \mathscr{S}$ and hence $\mathscr{S}$ contains ideals other than finite intersections of maximal ideals.

The problem of determining the ideals of $\mathscr{S}$ is, of course, equivalent to determing those subsets Q of βX for which $J(Q) \in \mathscr{S}$. If we set $\theta(\mathscr{S}) = \{Q \subseteq \beta X; J(Q) \in \mathscr{S}\}$, we can rephrase the results of this section as follows. The family $\theta(\mathscr{S})$ consists of all subsets of βX if and only if X is a p-space and $\theta(\mathscr{S})$ consists of all finite subsets of βX if and only if X is pseudo-compact. Between these two extremes considerable variation in $\theta(\mathscr{S})$ is to be expected. In the next section we examine this question in more detail. However we note in passing, the following.

COROLLARY. *If $\theta(\mathscr{S})$ contains a set which is not finite then $\theta(\mathscr{S})$ contains a set of cardinality 2^c, where c is the power of the continuum.*

Proof. If $\theta(\mathscr{S})$ contains a set which is not finite then the underlying space is not pseudo-compact and hence contains a C-embedded copy of N with $\bar{N} = \beta N$. But $J(\bar{N}) = J(N)$ and $J(N) \in \mathscr{S}$ so $J(\bar{N}) \in \mathscr{S}$. The corollary follows since card βN is 2^c, [1; 9.3, p. 131].

4. The last section left unanswered the problem of determining $\mathscr{S}$ for arbitrary spaces. At the end of the last section it was noted that this is equivalent to determining those subsets Q of βX for which $J(Q) \in \mathscr{S}$. We are not able to give a complete characterization of these sets. However a beginning on the problem can be made by recalling that every infinite subset Q of βX contains a countably infinite discrete subset P and that if $J(Q)$ is in $\mathscr{S}$ then so is $J(P)$. From Lemma 3.2, we know that, if P is a countable discrete subset of βX which is C-embedded in $X \cup P$, then $J(P) \in \mathscr{S}$. The major result of this section is that this condition is also necessary. The major step in this proof is the establishment of a necessary and sufficient condition that a countable discrete subset of βX be C-embedded in $X \cup P$.

As our starting point we take the well known fact that, if $P \subseteq \beta X$ and $f \in C(X \cup P)$ such that $(f \mid P)$ is a homeomorphism of P onto a closed subset of the reals, then P is C-embedded in X. Our first lemma is a slight variation of this result.

4.1. Lemma. *Let P be a countable discrete subset of βX. If there exists $f \in C(X \cap P)$ such that $f[P]$ is closed and discrete and for each $r \in f[P]$, $P \cap \overleftarrow{f(r)}$ is finite, then P is C-embedded in $X \cup P$.*

Proof. Let us assume that f is nonnegative. There is no loss of generality in doing this since, if f satisfies the hypothesis so does f^2. Since a closed disrete subset of the positive real line is well ordered in the usual order, we may assume $f[P] = \{r_n; n \in N\}$ where $r_n < r_{n+1}$ for every $n \in N$. Using the axiom of choice we may suppose that for each $n \in N$ we have ordered $P \cap \overleftarrow{f(r_n)}$, say $P \cap \overleftarrow{f(r_n)} = \{p(n, 1), \cdots, p(n, t_n)\}$. We define a function g on P as follows. For each $n \in N$ and $1 \leqq j \leqq t_n$ we set

$$g(p(n, j)) = \left(\frac{j-1}{n \cdot t_n}\right) \cdot (\min. \{r_{n+1} - r_n, 1\}) .$$

Since $g(p(n, j)) \to 0$ as $n \to \infty$, we may extend g continuously to $\bar{P}$ by defining g to be 0 at all of the accumulation points (in βX) of P. Since every compact subset of βX is C^*-embedded in βX we may extend g continuously to all of βX. Let h be such an extension and let $k = f + (h \mid X \cup P)$. Then $k \in C(X \cup P)$ and $(k \mid P)$ is a homeomorphism of P onto a closed subset of the reals. This is sufficient to insure that P is C-embedded in $X \cup P$, [1; 1.19, p. 20].

4.2. Lemma. *Let P be a countable discrete subset of βX. If there exists $f \in C(X \cup P)$ such that $f^*[\bar{P} - P] = \infty$, then P is C-embedded in $X \cup P$.*

Proof. Clearly we may assume that f is nonnegative; for, if f satisfies the hypothesis, so does f^2. Since $f^*[\bar{P} - P] = \infty$, for each $n \in N$, $\overleftarrow{f}[n, n+1]$ is finite. Thus $f[P]$ is closed and discrete and, for each $r \in f[P]$, $\overleftarrow{f(r)} \cap P$ is finite. The lemma now follows from Lemma 4.1.

4.3. Theorem. *For a countable discrete subset P of βX, the following statements are equivalent.*

1. *P is C-embedded in $X \cap P$*
2. *$J(P) \in \mathscr{S}$*
3. *There exists a countable family $\{U_n\}$, of neighborhoods of $\bar{P} - P$, such that $(\cap \{U_n\}) \cap X = \phi$.*

Proof. The implication (1) implies (2) is the content of Lemma 3.2. We will show that (2) implies (3) by showing that the denial of (3) implies the denial of (2). Thus suppose that (3) is false; i.e., that, for any countable family $\{U_n\}$ of neighborhoods of $\bar{P}-P$, $(\cap\{U_n\})\cap X\neq\phi$. Let $P=\{p_n;\, n\in N\}$ and set $P_n=\{p_k;\, 0\leqq k\leqq n\}$, for each $n\in N$. Then, for each $n\in N$, P_n and $\bar{P}-P$ are disjoint closed subsets of βX. Hence there exists a closed neighborhood U_n, of $\bar{P}-P$, disjoint from P_n. Since βX is normal, there exists $f_n\in C(\beta X)$ such that $0\leqq f_n\leqq 1$, $f_n[P_n]=1$, and $f_n[U_n]=0$. Then, as is well known, the function $f=\Sigma 2^{-n}f_n$ is in $C(\beta X)$ and $Z(f)\supseteqq \cap Z(f_n)\supseteqq \cap U_n$. Moreover, $\overline{[(\cap U_n)\cap X]}\cap(\bar{P}-P)\neq\phi$; for, if it were, we could find a neighborhood W, of $\bar{P}-P$, disjoint from $(\cap W_n)\cap X$ and hence $\{W_n=W\cap U_n\}$ would satisfy condition (3), contrary to our assumption. Since $\overline{Z(f\,|\,X)}\supseteqq\overline{(\cap U_n)\cap X}$, it follows that $\overline{Z(f\,|\,X)}\cap(\bar{P}-P)\neq\phi$. Let $q\in\overline{Z(f\,|\,X)}\cap(\bar{P}-P)$. Then $(f\,|\,X)\in M^q$. If $J(P)\in\mathscr{S}$, it follows from Lemma 2.1 and Theorem 2.5 that $[(f\,|\,X)+J(P)]\cap Oq\neq\phi$. Thus suppose $g\in O^q$ and $h\in J(P)$, with $(f\,|\,X)=g-h$. Since $g\in O^q$, there exists a neighborhood V of q on which g^* vanishes. Since $h\in J(P)$, h^* vanishes on P and hence $(f\,|\,X)^*=(g-h)^*=g^*-h^*$ vanishes on $V\cap P$. But from the construction of f it is clear that $(f\,|\,X)^*=f$ does not vanish anywhere on P. Hence $J(P)\notin\mathscr{S}$. It remains to show that (3) implies (2). It is easily seen that (3) implies that there exists a countable family $\{V_n\}$ of open neighborhoods of $\bar{P}-P$ satisfying, (i) $\bar{V}_{n+1}\subseteqq V_n$ and (ii) $[\cap V_n]\cap X=\phi$. We will show that the existence of such a family $\{V_n\}$ implies the existence of a function $f\in C(X\cup P)$, with $f^*[\bar{P}-P]=\infty$.

For each $n\in N$, choose $f_n\in C(\beta X)$ such that $0\leqq f_n\leqq 1$, $f_n[\bar{V}_{n+1}]=1$ and $f_n[\beta X-V_n]=0$ and set $f=\Sigma f_n$. To see $f\in C(X\cup P)$, suppose $x\in X\cup P$. Then, by (ii), there exists $n\in N$ such that $x\notin\bar{V}_{n+1}$. From (i) it follows that, for $m>n$, $\beta X-V_m\supseteqq\beta X-V_n$ and hence $f_m[\beta X-\bar{V}_n]=0$. Thus, on the neighborhood $(\beta X-\bar{V}_n)\cap(X\cap P)$, f is a finite sum of continuous real-valued functions and, hence, is itself a continuous real-valued function. Thus $f\in C(X\cup P)$. Finally, since $\bar{P}-P\subseteqq\cap V_n$, $f^*[\bar{P}-P]=\infty$. The implication now follows from Lemma 4.2.

Bibliography

1. L. Gillman and M. Jerison, *Rings of Continuous Functions*, D. Van Nostrand, 1960.