ON ABSTRACT AFFINE NEAR-RINGS

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1. Introduction. We shall limit ourselves to near-rings for which addition is commutative. They will be known as abelian near-rings. We assume that the distributive law $(b + c)a = ba + ca$ holds, but the law $a(b + c) = ab + ac$ does not necessarily hold. (This is consistent with the usual convention that the product AB of two operators A and B stands for B followed by A, e.g., consider the near-ring of all mappings of a group into itself.) Our aim is to generalize the results of [1] and [2] to a class of near-rings which we call abstract affine near-rings.

2. Abelian near-rings. We first define two subsets $L(R)$ and $C(R)$ of a near-ring R. (When convenient, we call these sets L and C. $L(R)$ is the set of all elements $a \in R$ which satisfy $a(b + c) = ab + ac$ for all b and c in R. $C(R)$ is the set of all elements $a \in R$ which satisfy $ab = a$ for all b in R. Note that, in general, $0 \cdot a = 0$ and $(-a)b = -(ab)$.

PROPOSITION 1. *L is a subring of B.*

Proof. If $a, b \in L$, then

$$\begin{aligned}(a + b)(x + y) &= a(x + y) + b(x + y) = ax + ay + bx + by \\ &= (ax + bx) + (ay + by) = (a + b)x + (a + b)y\,,\end{aligned}$$

hence $a + b \in L$. Since $0 \cdot a = 0$ for all $a, 0 \in L$. Also if $a \in L$, then

$$\begin{aligned}(-a)(x + y) &= -[a(x + y)] = -[ax + ay] = (-ax) + (-ay) \\ &= (-a)x + (-a)y\,,\end{aligned}$$

hence $-a \in L$. Furthermore if $a, b \in L$, then $ab(x + y) = a(bx + by) = abx + aby$, hence $ab \in L$. This completes the proof. Note that if R contains an identity e, then $e \in L$.

DEFINITION. An r-ideal is a subgroup closed under multiplication on the left and right by arbitrary elements of R. An ideal I is a subgroup closed under right multiplication by elements of R and which furthermore satisfies $y(x + a) - yx \in I$ for all $a \in I$, $x \in R$, $y \in R$.

PROPOSITION 2. *C is an r-ideal of R.*

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Proof. If $a, b \in C$, then $(a+b)x = ax + bx = a + b$, hence $a + b \in C$. $0 \cdot x = 0$, hence $0 \in C$. If $a \in C$, then $(-a)x = -(ax) = -a$, hence $-a \in C$. If $a \in C$, then $(ax)y = a(xy) = a = ax$ and $(xa)y = x(ay) = xa$. This proves the result.

PROPOSITION 3. $L \cap C = 0$.

Proof. Let $a \in L \cap C$. Let x be arbitary in R. Then $a = a(x + x) = ax + ax = a + a$. Thus $a = 0$.

3. Abstract affine near-rings.

DEFINITION. An abstract affine near-ring R is an abelian near-ring R which satisfies $R = C + L$. C can be regarded as a module over L. If $r \in L$ and $a \in C$ define $r \circ a = ra$. The axioms for a module are clearly satisfied. Also if $l_1 l_2 \in L$, and $c_1, c_2 \in C$, then

$$\begin{aligned}(l_1 + c_1)(l_2 + c_2) &= l_1(l_2 + c_2) + c_1(l_2 + c_2) \\ &= l_1 l_2 + l_1 c_2 + c_1 = l_1 l_2 + l_1 \circ c_2 + c_1\end{aligned}$$

Thus multiplication can be expressed in terms of the ring and module operations. Conversely, let M be any left R module. We make the group direct sum $R \oplus M$ into a near-ring as follows. Let $r_1, r_2 \in R$ and $m_1, m_2 \in M$. Define $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + m_1)$.

PROPOSITION 4. With this definition for multiplication $R \oplus M$ is an abstract affine near-ring with $L(R \oplus M) = R, 0)$ and $C(R \oplus M) = (0, M)$.

Proof.

$$\begin{aligned}[(r_1, m_1)(r_2, m_2)](r_3, m_3) &= (r_1 r_2, r_1 m_2 + m_1)(r_3, m_3) \\ &= (r_1 r_2 r_3, r_1 r_2 m_3 + r_1 m_2 + m_1) \\ (r_1, m_1)[(r_2, m_2)(r_3, m_3)] &= (r_1, m_1)(r_2 r_3, r_2 m_3 + m_2) \\ &= r_1 r_2 r_3, r_1 r_2 m_3 + r_1 m_2 + m_1) \,.\end{aligned}$$

This verifies the associative law.

$$\begin{aligned}[(r_1, m_1) + (r_2, m_2)](r_3, m_3) &= (r_1 + r_2, m_1 + m_2)(r_3, m_3) \\ &= [(r_1 + r_2)r_3, (r_1 + r_2)m_3 + m_1 + m_2] \\ (r_1, m_1)(r_3, m_3) + (r_2, m_2)(r_3, m_3) &= (r_1 r_3, r_1 m_3 + m_1) + (r_2 r_3, r_2 m_3 + m_2) \\ &= (r_1 r_3 + r_2 r_3, r_1 m_3 + r_2 m_3 + m_1 + m_2).\end{aligned}$$

This verifies the distributive law. Hence $R \oplus M$ is an abelian near-ring. Furthermore,

$$\begin{aligned}(r_1, 0)[(r_2, m_2) + (r_3, m_3)] &= (r_1, 0)(r_2 + r_3, m_2 + m_3)\\ &= [r_1(r_2 + r_3), r_1(m_2 + m_3)] .\\ (r_1, 0)(r_2, m_2) + (r_1, 0)(r_3, m_3) &= (r_1r_2, r_1m_2) + (r_1r_3, r_1m_3)\\ &= (r_1r_2 + r_1r_3, r_1m_2 + r_1m_3) .\end{aligned}$$

Hence $(r_1, 0) \in L$, $(0, m_1)(r_2, m_2) = (0r_2, 0m_2 + m_1) = (0, m_1)$. Hence $(0, m_1) \in C$. Since $L \cap C = 0$. This completes the proof.

We are now ready to discuss the connection with [1] and [2]. Embed M in a module M_1 so that R is faithful, i.e., $rm = 0$ for all $m \in M_1$ implies $r = 0$. This can always be done. If the element $(r, m) \in R \oplus M$ is identified with the map of M_1 into itself defined by $x \rightarrow rx + m$ for all $x \in M_1$, we obtain an isomorphism of the abstract affine near-ring and a near-ring of maps of M_1 into M_1. (It is easily verified that the operations are preserved.) Furthermore each map is the sum of an endomorphism and a constant map. Thus the near-ring considered in [2] corresponds to the special case where M is a vector space and R is the ring of all linear transformations.

4. The Ideals in $R \oplus M$. Henceforth we write $r + m$ for (r, m). We now classify the ideals and r-ideals of $R \oplus M$. Let J be an ideal or an r-ideal and let $r + m \in J$. Then $(r + m)0 \in J$, i.e., $m \in J$. Thus $r \in J$. This shows that $J = R_1 \oplus M_1$ where R_1 and M_1 are subgroups of R and M respectively. If $r_1 \in R_1$ and $r \in R$, then r_1r and rr_1 are in J, hence in R_1. Thus R_1 is an ideal in R. (Note that an ideal is closed under left multiplication by elements of $L(R \oplus M)$.) If $m_1 \in M_1$ and $r \in R$, then $rm_1 \in J$. Hence $rm_1 \in M_1$. Thus M_1. is a submodule of M.

At this point we consider the ideals and r-ideals separately. Let J be an r-ideal. Let $m \in M$. Since $0 \in J, m = m \cdot 0 \in J$. Hence $M_1 = M$. Thus all r-ideals have the form $R_1 \oplus M$ where R_1 is an ideal in R. Conversely, let J be any set of the form $R_1 \oplus M$ where R_1 is an ideal of R. Clearly, J is a subgroup. Let $r_1 \in R_1$, $r \in R$, $m_1 \in M_1$ and $m \in M$. Then $(r_1 + m_1)(r + m) = r_1r + r_1m + m_1 \in R_1 \oplus M$ and $(r + m)(r_1 + m_1) = rr_1 + rm_1 + m \in R_1 \oplus M$. Thus J is an r-ideal.

Now let J be an ideal. Let $r_1 \in R_1$ and $m \in M$. Then $r_1m \in J$. Hence $R_1M \subset M_1$. (Note that left multiplication by elements of M give no new information since $m(y + x) - mx = 0$ for all $m \in M$ and $x, y \in R$.) Conversely, let J be of the form $R_1 \oplus M_1$ where R_1 is an ideal of R and M_1 is a submodule of M containing R_1M. Again J is a subgroup. Let $r_1 \in R_1$, $m_1 \in M_1$, $r \in R$ and $m \in M$. Then

$$(r_1 + m_1)(r + m) = r_1r + r_1m + m_1 \in R_1 + R_1M + M_1 \subset R_1 \oplus M_1 = J.$$

On the left it suffices to check with r and m separately. For r we may use left multiplication. Thus $r(r_1 + m_1) = rr_1 + rm_1 \in R_1 \oplus M_1$. For m the result is automatically 0 since $mx = my$ for all $x, y \in R$. Thus J is an ideal.

We have proved the following theorem.

THEOREM. *The r-ideals of $R \oplus M$ are exactly the sets of the form $R_1 \oplus M$ where R_1 is an ideal of R. The ideals of $R \oplus M$ are exactly the sets of the form $R_1 \oplus M_1$ where R_1 is an ideal of R and M_1 is a submodule of M containing R_1M. Thus every r-ideal is an ideal.*

In the special case considered in [2], M is a simple R module and $R_1M = M$ for all ideals $R \neq 0$. Thus the result there that classifies all ideals other than (0) as those sets which have the form $R_1 \oplus M$ where R_1 is an ideal of R follows from our theorem.

BIBLIOGRAPHY

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