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CHEBYSHEV APPROXIMATION TO ZERO

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In this paper we shall be concerned with the questions of existence, uniqueness and constructability of those polynomials in $k+1$ variables $(x_1, x_2, \cdots, x_k, y)$ of degree not greater than n_s in x_s and m in y which best approximate zero on $I_1 \times I_2 \times \cdots \times I_{k+1}, I_s = [-1, 1]$, in the Chebyshev sense.

It is a classic result that among all monic polynomials of degree not greater than n there is a unique polynomial whose maximum over the interval $[-1, 1]$ is less than the maximum over $[-1, 1]$ of any other polynomial of the same type and moreover it is given by $\widetilde{T}_n(x) = 2^{1-n} \cos [n \text{ are } \cos x]$, the normalized Chebyshev polynomial.

Our method of attack will be to prove a generalization of an inequality for monic polynomials in one variable concerning the lower bound of the maximum viz. $\max_{-1 \leqq x \leqq 1} |P_n(x)| \geqq 2^{1-n}$ where $P_n(x)$ is a monic polynomial of degree not greater than n. The theorem will show that the only hope for uniqueness is to normalize our class of polynomials. This is done in a very natural way viz. by considering only polynomials, if they exist, of the form:

$$(0.1) \qquad P(x_1, x_2, \cdots, x_k, y) = A_m(x_1, \cdots, x_k)y^m + A_{m-1}(\cdots)y^{m-1} + \cdots + A_0(\cdots)$$

for which $A_m(x_1, x_2, \cdots, x_k)$ is the best polynomial approximation to zero on $I_1 \times I_2 \times \cdots \times I_k$. Thus if $k = 1$, we consider only polynomials of the form:

$$(0.2) \qquad P(x_1, y) = \widetilde{T}_n(x_1)y^m + A_{m-1}(x_1)y^{m-1} + \cdots + A_0(x_1) .$$

We find in the case of (0.2) that there is a unique best polynomial approximation and it is given by $\widetilde{T}_n(x_1)\widetilde{T}_m(y)$. Thus we can consider the question of existence, uniqueness and constructability of a polynomial of the form:

$$(0.3) \qquad P(x_1, x_2, y) = \widetilde{T}_{n_1}(x_1)\widetilde{T}_{n_2}(x_2)y^m + A_{m-1}(x_1, x_2)y^{m-1} + \cdots + A_0(x_1, x_2)$$

that best approximates zero. We find in this case there is a unique best polynomial approximation and it is given by $\widetilde{T}_{n_1}(x_1)\widetilde{T}_{n_2}(x_2)\widetilde{T}_m(y)$. Continuing in this way we shall show that the question is meaning-

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ful in general and that there is a unique best polynomial approximation to zero of the form (0.1) given by $\tilde{T}_{n_1}(x_1)\tilde{T}_{n_2}(x_2) \cdots \tilde{T}_{n_k}(x_k)\tilde{T}_m(y)$.

The uniqueness and constructability are the most surprising results, since as Buck [1] has shown, $F(x, y) = xy$ has amongst those polynomials of the form

$$p(x, y) = a_0 + a_1(x + y) + a_2(x^2 + y^2)$$

infinitely many polynomials of best approximation which are given by:

$$\alpha f_1 + \beta f_2\,, \quad \alpha \geqq 0\,, \quad \beta \geqq 0\,, \quad \alpha + \beta = 1$$

where

$$f_1(x, y) = \frac{1}{2}(x^2 + y^2) - \frac{1}{4}\,,$$

$$f_2(x, y) = x + y - \frac{1}{2}(x^2 + y^2) - \frac{1}{4}\,.$$

We shall finally normalize the polynomials in a different way and show by construction, the existence of a polynomial, of best approximation in this class. However in this case the question of uniqueness remains open.

1. NOTATION. Let $n_1, n_2, \cdots, n_k$ be positive fixed integers. Let σ be the finite set of vectors $\{(x_{1j_1}, x_{2j_2}, \cdots, x_{kj_k})\}$, where $j_1, j_2, \cdots, j_k$ are integers with $0 \leqq j_1 \leqq n_1, 0 \leqq j_2 \leqq n_2, \cdots, 0 \leqq j_k \leqq n_k$; and where also $-1 \leqq x_{1j_1} \leqq 1, -1 \leqq x_{2j_2} \leqq 1, \cdots, -1 \leqq x_{kj_k} \leqq 1$ and no two of the x_{1j_1} are the same, no two of the x_{2j_2} are the same, $\cdots$, no two of the x_{kj_k} are the same. Let $Q(x, y) = Q(x_1, x_2, \cdots, x_k, y)$ be any polynomial in $x_1, x_2, \cdots, x_k$ and y of degree $\leqq n_1 + n_2 + \cdots + n_k + m - 1$ where Q is of degree $\leqq n_s$ in x_s, $s = 1, 2, \cdots, k$ and of degree $\leqq m$ in y. Let π be the set of all such polynomials. Thus if $Q(x, y)$ is in π

$$Q(x, y) = p_m(x)y^m + p_{m-1}(x)y^{m-1} + \cdots + p_0(x)$$

where $p_m(x)$ is a polynomial in $x_1, x_2, \cdots, x_k$ of

$$\text{degree} \leqq n_1 + n_2 + \cdots + n_k - 1$$

and $p_s(x), 0 \leqq s \leqq m - 1$, are polynomials of degree $\leqq n_1 + n_2 + \cdots + n_k$ in $x_1, x_2, \cdots, x_k$. Let

$$A[p_m; \pi, \sigma] = \min_{x \text{ in } \sigma} | x_1^{n_1}x_1^{n_2} \cdots x_k^{n_k} - p_m(x_1, x_2, \cdots, x_k) |$$

which does not depend on the particular Q, but only on the class π and the leading coefficient polynomial of y.

THEOREM 1. *If $Q(x, y)$ is any polynomial in π and if σ is any set of the type described above then*

$$\max_{\substack{-1\leqq x_s\leqq 1\\ -1\leqq y\leqq 1}} |x_1^{n_1}x_2^{n_2}\cdots x_k^{n_k}y^m - Q(x_1, x_2, \cdots, x_k, y)| \geqq A[p_m; \pi, \sigma]2^{1-m} .$$

Proof. Assume not. Then there exists a $Q^*(x, y)$ in π and a set σ of the type described such that:

$$\max_{\substack{-1\leqq x_s\leqq 1\\ -1\leqq y\leqq 1}} |x_1^{n_1}x_2^{n_2}\cdots x_k^{n_k}y^m - Q^*(x, y)| < A[p_m; \pi, \sigma]2^{1-m}$$

consider the polynomial:

$$\begin{aligned} P(x, y) = x_1^{n_1}x_2^{n_2}&\cdots x_k^{n_k}y^m \\ &- Q^*(x, y) - [x_1^{n_1}x_2^{n_2}\cdots x_k^{n_k} - p_m(x)]\tilde{T}_m(y) \end{aligned}$$

where $p_m(x)$ is the coefficient of y^m in $Q^*(x, y)$ and where

$$\tilde{T}_m(y) = 2^{1-m}T_m(y) = 2^{1-m}\cos[m \operatorname{arc}\cos y] . \tag{1}$$

Then $P(x, y)$ is a polynomial of degree $\leqq m-1$ in y and thus can be written:

$$P(x, y) = q_{m-1}(x)y^{m-1} + q_{m-2}(x)y^{m-2} + \cdots + q_0(x)$$

where $q_s(x)$, $0 \leqq s \leqq m-1$, are polynomials in $x_1, x_2, \cdots, x_k$ of degree $\leqq n_1 + n_2 + \cdots + n_k$.

Let $(x_{1j_1}, x_{2j_2}, \cdots, x_{kj_k})$ belong to σ and y_r be one of the points

$$y_r = \cos\frac{r\pi}{m} , \quad 0 \leqq r \leqq m , \quad r = \text{integer} .$$

Then $\tilde{T}_m(y_r) = (-1)^r 2^{1-m}$ and we can show that the sign of

$$P[x_{1j_1}, x_{2j_2}, \cdots, x_{kj_k}, y_r]$$

is the same as the sign of $-[x_{1j_1}^{n_1}\cdots x_{kj_k}^{n_k} - p_m(y_{1j_1}, \cdots, x_{kj_k})]$. $\tilde{T}_m(y_r)$. To see this note that:

$$\begin{aligned} |\tilde{T}_m(y_r)|\,&|x_{1j_1}^{n_1}\cdots x_{kj_k}^{n_k} - p_m(x_{1j_1}, \cdots, x_{kj_k})| \\ &= |x_{1j_1}^{n_1}\cdots x_{kj_k}^{n_k} - p_m(x_{1j_1}, \cdots, x_{1j_k})|\, 2^{1-m} \\ &\geqq A[p_m; \pi, \sigma]2^{1-m} . \end{aligned}$$

But by the assumption

$$\max_{\substack{-1\leqq x_s\leqq 1\\ -1\leqq y\leqq 1}} |x_1^{n_1}\cdots x_k^{n_k}y^m - Q^*(x, y)| < A[p_m; \pi, \sigma]2^{1-m}$$

and thus a fortiori

$$| x_{1j_1}^{n_1} \cdots x_{kj_k}^{n_k} y_r^m - Q^*(x_{1j_1}, \cdots, x_{kj_k}, y_r) | < A[p_m; \pi, \sigma] 2^{1-m} .$$

If we fix x in σ then $P(x, y)$ is a polynomial of the one variable y and of degree $\leqq m-1$. And as y takes on the values $y_r = \cos(\pi r/m)$, $P(x, y)$ changes sign $m+1$ times. Thus $P(x, y)$ has m zeros, which means $q_{m-1}(x) = 0, q_{m-2}(x) = 0, \cdots, q_0(x) = 0$ since $P(x, y)$ is only of degree $\leqq m-1$.

Since x was an arbitrary point of σ, then

$$q_s[x_{1j_1}, x_{2j_2}, \cdots, x_{kj_k}] = 0 , \quad 0 \leqq s \leqq m-1$$

where $0 \leqq j_1 \leqq n_1, 0 \leqq j_2 \leqq n_2, \cdots, 0 \leqq j_k \leqq n_k$. But $q_s(x)$ is a polynomial of degree $\leqq n_1$ in x_1, of degree $\leqq n_2$ in $x_2, \cdots$, of degree $\leqq n_k$ in x_k and thus

$$q_s[x_1, x_2, \cdots, x_k] \equiv 0 , \quad 0 \leqq s \leqq m-1 .$$

From which we see $P(x, y) \equiv 0$ and thus:

$$x_1^{n_1} \cdots x_k^{n_k} y^m - Q^*(x, y) \equiv [x_1^{n_1} \cdots x_k^{n_k} - p_m(x)] \widetilde{T}_m(y) .$$

But clearly:

$$\max_{\substack{-1 \leqq x_s \leqq 1 \\ -1 \leqq y \leqq 1}} | x_1^{n_1} \cdots x_k^{n_k} - p_m(x) | \, | \widetilde{T}(y) | \geqq A[p_m; \pi, \sigma] 2^{1-m}$$

which is a contradiction and thus the theorem is proved.

Let us now consider the subset of polynomials π_0 of π for which $Q(x, y)$ belongs to π and $p_m(x) = 0$. Then by the above theorem, a lower bound for the maximum is

$$A[0; \pi, \sigma] = \min_{x \text{ in } \sigma} | x_1^{n_1} \cdots x_k^{n_k} | < 1$$

which clearly depends on the set σ. We shall next show that for this subset π_0, we get a lower bound for the maximum that is independent of σ and moreover the lower bound is larger than $A[0; \pi, \sigma]$ for all σ, namely it is unity. In the third theorem we shall show that unity is the best possible lower bound i.e. there is a polynomial in π_0 for which the maximum is 2^{1-m}.

THEOREM 2. *Let $Q(x, y)$ be any polynomial in π_0, then*

$$\max_{\substack{-1 \leqq x_s \leqq 1 \\ -1 \leqq y \leqq 1}} | x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} y^m - Q(x_1, x_2, \cdots, x_k, y | \geqq 2^{1-m} .$$

Proof. By contradiction. Assume there exists a $Q(x_1, \cdots, x_k, y)$ in π_0 such that:

$$\max_{\substack{-1 \leqq x_s \leqq 1 \\ -1 \leqq y \leqq 1}} | x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} y^m - Q(x_1, \cdots, x_k, y) | < 2^{1-m} .$$

Then there exist δ_s's, $1 \leqq s \leqq k, 1 > \delta_s > 0$ such that:

$$\max_{\substack{-1 \leqq x_s \leqq 1 \\ -1 \leqq y \leqq 1}} |x_1^{n_1} \cdots x_k^{n_k} y^m - Q(x_1, \cdots, x_k, y)| < 2^{1-m} \prod_{s=1}^{k} \delta_s^{n_s} .$$

Let $\widetilde{T}_m(y)$ be given by (1) and consider the polynomial

$$P(x_1, \cdots, x_k, y) \equiv x_1^{n_1} \cdots x_k^{n_k} y^m - Q(x_1, \cdots, x_k, y) - x_1^{n_1} \cdots x_k^{n_k} \widetilde{T}_m(y) .$$

$P(x_1, \cdots, x_k, y)$ is a polynomial of degree $\leqq m-1$ in y and of degree $\leqq n_s$ in x_s $1 \leqq s \leqq k$.

Let $\sigma^* = \{(x_{1j_1}, x_{2j_2}, \cdots, x_{kj_k})\}$ where $j_1, \cdots, j_k$ are integers with

$$0 \leqq j_1 \leqq n_1 + 1, 0 \leqq j_2 \leqq n_2 + 1, \cdots, 0 \leqq j_k \leqq n_k + 1;$$
$$\delta_1 < x_{1j_1} \leqq 1, \delta_2 < x_{2j_2} \leqq 1, \cdots, \delta_k < x_{kj_k} \leqq 1$$

and the x_{1j_1} are distinct, $\cdots$, the x_{kj_k} are distinct.

Note that for x in σ^*, the sign of $P(x_{1j_1}, \cdots, x_{kj_k}, y)$ is the same as the sign of $-x_{1j_1}^{n_1} \cdots x_{kn_k}^{n_k} \widetilde{T}_m(y_r)$ for $y_r = \cos(r\pi/m)$, $r = 0, 1, \cdots, m$. This follows from the fact that:

$$|x_{1j_1}^{n_1} \cdots x_{kj_k}^{n_k} y_r^m - Q(x_1, \cdots, x_k, y_r)| < 2^{1-m} \prod_{s=1}^{k} \delta_s^{n_s}$$

and the fact that:

$$|x_{1j_1}^{n_1} \cdots x_{kj_k}^{n_k} \widetilde{T}_m(y_r)| = 2^{1-m} \prod_{s=1}^{k} x_{sj_s}^{n_s} > 2^{1-m} \prod_{s=1}^{k} \delta_s^{n_s} .$$

Thus we conclude that $P(x_{1j_1}, \cdots, x_{kj_k}, y)$ has $m+1$ sign changes for $(x_{1j_1}, \cdots, x_{kj_k})$ in σ^*. Let us write

$$P(x, y) = p_{m-1}(x)y^{m-1} + p_{m-2}(x)y^{m-2} + \cdots + p_0(x)$$

where $p_s(x)$, $0 \leqq s \leqq m-1$, are polynomials of degree $\leqq n_s$ in x_s, $0 \leqq s \leqq k$. For each x in σ^*, $P(x, y)$ has $m+1$ sign changes and thus $p_{m-1}(x) = 0$, $p_{m-2}(x) = 0, \cdots, p_0(x) = 0$ for each x in σ^*. If for $(x_{1j_1}, x_{2j_2}, \cdots, x_{kj_k})$ in σ^*, we fix all but the first component, we get $n_1 + 2$ values in σ^* for which $p_s(x) = 0, 0 \leqq s \leqq m-1$, but these $p_s(x)$ are of degree $\leqq n_1$ in x_1 and thus $p_s(x_1, x_{2j_2}, x_{3j_3}, \cdots, x_{kj_k}) = 0$ for all real x_1. Continuing in this way, we see that $p_s(x_1, x_2, \cdots, x_k) \equiv 0$ for all $(x_1, x_2, \cdots, x_k)$, x_s real. Thus:

$$P(x_1, x_2, \cdots, x_k, y) \equiv 0$$

for all real x_s and real y. Thus

$$x_1^{n_1} \cdots x_k^{n_k} \widetilde{T}_m(y) \equiv x_1^{n_1} \cdots x_k^{n_k} y^m - Q(x_1, \cdots, x_k, y) .$$

But

$$\max_{\substack{-1\leqq x_s\leqq 1\\ -1\leqq y\leqq 1}} |x_1^{n_1}\cdots x_k^{n_k}\tilde{T}_m(y)| = 2^{1-m}$$

which gives a contradiction and the theorem is proved.

2. Normalization of competing polynomials and construction of the best polynomial. We shall now consider a subset $\pi(\beta)$ of the set of polynomials π. We shall then answer the question of existence, uniqueness and constructability of the best polynomial approximation in the maximum norm to zero within this class $\pi(\beta)$ on the cube

$$-1 \leqq x_1 \leqq 1, \cdots, -1 \leqq x_k \leqq 1, -1 \leqq y \leqq 1 .$$

It is apparent from Theorem 1, that if we want uniqueness independent of σ, it is necessary to consider some subset of π.

DEFINITION. A polynomial

$$Q(x, y) = p_m(x_1, x_2, \cdots, x_k)y^m + p_{m-1}(x_1, x_2, \cdots, x_k)y^{m-1} + \cdots + p_0(x_1, x_2, \cdots, x_k)$$

which is in π and for which

$$x_1^{n_1}x_2^{n_2}\cdots x_k^{n_k} - p_m(x_1, x_2 \cdots x_k) = \tilde{T}_{n_1}(x_1)\tilde{T}_{n_2}(x_2)\cdots \tilde{T}_{n_k}(x_k)$$

is said to be in $\pi(\beta)$.

LEMMA. *Let $q(y)$ be a polynomial in y, let $y_0 > y_1 > \cdots > y_m$ be any set of real numbers for which*

$$q(y_0) \leqq 0, q(y_1) \geqq 0, q(y_2) \leqq 0, \cdots (-1)^m q(y_m) \leqq 0 .$$

Then $q(y)$ has m zeros including multiplicities on $[y_0, y_m]$.

Proof. (by induction): For $m = 1$ obvious. Assume theorem to be true for $m \leqq k$. Let $y_0 > y_1 > y_2 > \cdots > y_{k+1}$ be any set of real numbers such that

$$q(y_0) \leqq 0, q(y_1) \geqq 0, \cdots (-1)^k q(y_k) \leqq 0, (-1)^{k+1}q(y_{k+1}) \leqq 0 .$$

Case 1. $q(y_s) \neq 0$ for some $1 \leqq s \leqq k$. Then by the induction hypothesis $q(y)$ has s zeros on $[y_0, y_s]$ and has $k + 1 - s$ zeros on $[y_s, y_{k+1}]$. But $q(y_s) \neq 0$ thus $q(y)$ has s zeros on $y_0 \leqq y \leqq y_s$ and thus $q(y)$ has $s + (k + 1 - s) = k + 1$ zeros on $[y_0, y_{k+1}]$.

Case 2. $q(y_0) < 0$. Then unless $q(y_s) = 0$ for $1 \leqq s \leqq k$ we are in Case 1 and we are finished. Therefore, assume $q(y_s) = 0, 1 \leqq s \leqq k$.

We may as well assume $q(y) < 0$ on (y_0, y_1) since if not then $q(y)$ has a zero there because $q(y_0) < 0$, and we are finished. Also, we may as well assume $q(y) > 0$ on (y_1, y_2) since if not and $q(y)$ has no zeros on (y_1, y_2) (if does have a zero then we are finished) then since $q(y_0) < 0$ and $q(y_1) = 0$, we must have that $q(y)$ has 2 zeros in (y_0, y_2), continuing in this way we see that we may as well assume that $(-1)^s q(y) < 0$ on (y_s, y_{s+1}) for $0 \leqq s \leqq k$. In particular $(-1)^k q(y) < 0$ for y on (y_k, y_{k+1}). But by assumption $(-1)^{k+1} q(y_{k+1}) \leqq 0$. Thus by the continuity of $q(y)$, we have $q(y_{k+1}) = 0$ and $q(y_s) = 0$ for $1 \leqq s \leqq k+1$ i.e. $q(y)$ has $k+1$ zeros on $[y_0, y_{k+1}]$.

Case 3. $q(y_0) = 0$ proof is obvious making use of Case 1.

THEOREM 3. There exists a unique $Q^*(x, y)$ in $\pi(\beta)$ such that

$$\max_{\substack{-1 \leqq x_s \leqq 1 \\ -1 \leqq y \leqq 1}} | x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} y^m - Q^*(x, y) |$$

is a minimum. Moreover:

$$Q^*(x, y) = -\widetilde{T}_{n_1}(x_1)\widetilde{T}_{n_2}(x_2) \cdots \widetilde{T}_{n_k}(x_k)\widetilde{T}_m(y) + x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} y^m .$$

Proof. Existence by construction. Let the σ of Theorem 1 be the special set of vectors

$$\sigma(\beta) = \{(x_{1j_1}, x_{2j_2}, \cdots, x_{kj_k})\}$$

where

$$x_{1j_1} = \cos(j_1 \pi / n_1),\ x_{2j_2}, \cdots, x_{kj_k} = \cos(j_k \pi / n_k)$$
$$0 \leqq j_1 \leqq n_1,\ 0 \leqq j_2 \leqq n_2, \cdots, 0 \leqq j_k \leqq n_k .$$

Then

$$\begin{aligned} A[p_m, \pi(\beta), \sigma(\beta)] &= \min_{x \text{ in } \sigma(\beta)} | x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} - p_m(x_1, x_2, \cdots, x_k) | \\ &= \min_{x \text{ in } \sigma(\beta)} | \widetilde{T}_{n_1}(x_1)\widetilde{T}_{n_2}(x_1) \cdots \widetilde{T}_{n_k}(x_k) | \\ &= 2^{1-n_1} 2^{1-n_2} \cdots 2^{1-n_k} . \end{aligned}$$

Thus by Theorem 1

$$\max_{\substack{-1 \leqq x_j \leqq 1 \\ -1 \leqq y \leqq 1}} | x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} y^m - Q(x, y) | \geqq 2^{1-n_1} 2^{1-n_2} \cdots 2^{1-n_k} 2^{1-m} .$$

But the polynomial

$$Q^*(x, y) = x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} y^m - \widetilde{T}_{n_1}(x_1)\widetilde{T}_{n_2}(x_2) \cdots \widetilde{T}_{n_k}(x_k)\widetilde{T}_m(y)$$

clearly belongs to $\pi(\beta)$ and

$$\max_{\substack{-1 \leqq x_s \leqq 1 \\ -1 \leqq y \leqq 1}} | x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} y^m - Q^*(x, y) | = 2^{1-n_1} 2^{1-n_2} \cdots 2^{1-n_k} 2^{1-m} .$$

Thus $Q^*(x, y)$ is a best approximation from the set $\pi(\beta)$

Uniqueness. Let $Q^*(x, y)$ in $\pi(\beta)$ be a polynomial of best approximation and let

$$\begin{aligned} P(x, y) &= x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} y^m - Q^*(x, y) - \widetilde{T}_{n_1}(x_1) \cdots \widetilde{T}_{n_k}(x_k) \widetilde{T}_m(y) \\ &= [x_1^{n_1} x_2^{n_1} \cdots x_k^{n_k} - p_m(x)] y^m - p_{m-1}(x) y^{m-1} - \cdots p_0(x) \\ &\quad - \widetilde{T}_{n_1}(x_1) \widetilde{T}_{n_2}(x_2) \cdots \widetilde{T}_{n_k}(x_k) \widetilde{T}_m(y) \\ &= q_{m-1}(x) y^{m-1} + q_{m-2}(x) y^{m-2} + \cdots + q_0(x) \end{aligned}$$

where $q_{m-1}(x), \cdots, q_0(x)$ are polynomials of degree $\leqq n_s$ in x_s $0 \leqq s \leqq k$ since $Q^*(x, y)$ is in $\pi(\beta)$.

Let $x^* = (x_1^*, x_2^*, \cdots, x_k^*)$ be a fixed but arbitrary element of $\sigma(\beta)$. Then we claim that $P(x^*, y)$ has m zeros including multiplicities in $[-1, 1]$. To see this let $y_s = \cos(s\pi/m)$, $0 \leqq s \leqq m$, then since

$$| x_1^{*n_1} x_2^{*n_2} \cdots x_k^{*n_k} y^m - Q^*(x^*, y) | \leqq 2^{1-n_1} 2^{1-n_2} \cdots 2^{1-n_k} 2^{1-m} ,$$

$$P(x^*, y_0) \leqq 0, P(x^*, y_1) \geqq 0, \cdots (-1)^m P(x^*, y_m) \leqq 0 .$$

By the lemma $P(x^*, y)$ has m zeros counting multiplicities for $-1 \leqq y \leqq 1$.

Thus $P(x^*, y)$ has m zeros but is only a polynomial of degree $m - 1$, thus $P(x^*, y) \equiv 0$. But this holds for all x^* in $\sigma(\beta)$, thus $P(x, y) \equiv 0$ and the theorem is proved.

We could formulate Theorem 3 in the following way. Let $\pi(k)$, $k \geqq 1$, be the set of polynomials of the form

$$Q(x, y) = p_m(x_1, \cdots, x_k) x_{k+1}^m + p_{m-1}(x) x_{k+1}^{m-1} + \cdots + p_0(x)$$

which is of degree $\leqq n_s$ in x_s, $1 \leqq s \leqq k$ and for which $p_m(x_1 \cdots x_k)$ is a polynomial that best approximates zero, if such exists, on the cube $I_1 \times I_2 \times \cdots \times I_k$, $I_s = [-1, 1]$, $1 \leqq s \leqq k$.

Theorem 3 alternate. For $k = 2, 3, 4 \cdots$, the following is true:

Statement k. $\pi(k-1)$ is not empty and there exists a unique $M_k(x_1, x_2, \cdots, x_k, x_{k+1})$ in $\pi(k)$ such that:

$$\max_{\substack{-1 \leqq x_s \leqq 1 \\ -1 \leqq y \leqq 1}} | M_k(x_1, x_2, \cdots, x_k, x_{k+1}) |$$

is a minimum. Moreover:

$$M_k(x_1, x_2, \cdots, x_k, x_{k+1}) = \widetilde{T}_{n_1}(x_1)\widetilde{T}_{n_2}(x_2) \cdots \widetilde{T}_{n_k}(x_k)\widetilde{T}_{n_{k+1}}(x_{k_{n+1}}) .$$

Proof. Obvious.

Finally we wish to prove:

THEOREM 4. *There exists a monic polynomial*

$$P(x_1, \cdots, x_k, y) = x_1^{n_1} \cdots x_k^{n_k} y^m - Q(x_1, \cdots, x_k, y)$$

where $Q(x, y)$ *belongs to* π_0 *that best approximates zero on the cube* $I_1 \times I_2 \times \cdots \times I_{k+1}, I_s = [-1, 1]$. *The polynomial is*

$$x_1^{n_1} \cdots x_k^{n_k} \widetilde{T}_m(y) .$$

Proof. By Theorem 2

$$\max_{\substack{-1 \leqq x_s \leqq 1 \\ -1 \leqq y \leqq 1}} |P(x_1, \cdots, x_k^{n_k}, y)| \geqq 2^{1-m} .$$

But $x_1^{n_1} \cdots x_k^{n_k} \widetilde{T}_m(y)$ is a monic polynomial of the correct form with

$$\max_{\substack{-1 \leqq x_s \leqq 1 \\ -1 \leqq y \leqq 1}} |x_1^{n_1} \cdots x_k^{n_k} \widetilde{T}_m(y)| = 2^{1-m} .$$

Thus the theorem is correct.

The question of uniqueness in this case is an open one.

REFERENCE

1. R. C. Buck, *Linear spaces and approximation theory, On numerical approximation*, Edited by R. E. Langer, Published by the University of Wisconsin Press, 1959.

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