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A CHARACTERIZATION OF FREE PROJECTIVE PLANES

L. C. SIEBENMANN

A characterization of free projective planes is given that is more symmetrical than the original definition of M. Hall. *It tends to very simple proofs of two fundamental theorems* due to M. Hall and L. I. Kopejkina-one being the result that every subplane of a free plane is free.

In a fundamental paper Marshall Hall defines a free plane to be a projective plane which either is degenerate or is generated as follows from a 'basis configuration,' π_0, consisting of at least two points on a line together with two isolated points. For each pair of points not already joined in π_0 create a distinct line that joins them and add it to π_0. In the resulting configuration, π_1, consider pairs of lines that do not intersect, and for each create a distinct intersection point and add it to π_1, thus forming π_2. Continuing, construct $\pi_3, \pi_4, \pi_5, \pi_6$, etc. adding alternately lines and points as indicated above. Then $\pi = \bigcup_n \pi_n$ (with the obvious incidence relation) is a projective plane. It is by definition a free plane. Hall proved that a free plane contains no confined configuration, that is, no finite configuration that, like the Desargues configuration, has $\geqq 3$ points on each line and $\geqq 3$ lines through each point. Further, using a complicated argument, he showed that, if a finitely generated plane contains no confined configuration, it is free. It follows that any finitely generated subplane of a free plane is free.

L. I. Kopejkina [2] proved, shortly after, that an arbitrary subplane of a free plane is free. (Of interest is the analogy with free groups.) An exposition of Kopejkina's theorem appears in [3].

Because it suggests a more symmetrical definition of free plane that leads to very direct proofs of the above theorems, we introduce the notion of an extension process.

DEFINITION 1. An *extension process* is a well ordered nested sequence of partial planes $\pi_0 \subset \pi_1 \subset \cdots \subset \pi_n \subset \cdots$ (the subscripts $0, 1, \cdots, n, \cdots$ belonging to a well ordered set) such that if a point p and a line l appear in a term π_n, $n > 0$, and in no earlier term then p is not incident with l—in other words, the *new* elements in π_n may be incident in π_n with elements appearing in earlier terms, but have no incidences among themselves. From this point we adopt as definition the characterization of free plane we aim to justify.

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DEFINITION 2. A *free plane* is a (possibly degenerate) projective plane, π, for which there exists an extension process $\pi_0 \subset \pi_1 \subset \cdots \subset \pi_n \subset \cdots$ with $\pi_0 = \varnothing$ (the empty plane) and $\pi = \bigcup_n \pi_n$ such that every new point [or line] in a term π_n is incident in π_n with at most two lines [points].

Immediately we can prove

THEOREM I (Kopejkina). *Every subplane of a free plane is free.*

Proof. If π is a free plane and $\pi_0 \subset \pi_1 \subset \cdots \subset \pi_n \subset \cdots$ is an extension process as above for π, then, given any subplane π', the sequence $\pi_0 \cap \pi' \subset \pi_1 \cap \pi' \subset \cdots \subset \pi_n \cap \pi' \subset \cdots$ is visibly such an extension process for π'. So π' is free. ∎

Some definitions and notations are collected in § 2. In § 3 the result of Hall is proved. Our definition of a free plane is apparently broader than M. Hall's. In § 4 we prove that the definitions are equivalent.

2. A set of elements consisting of points and lines, together with a relation of incidence between points and lines is said to form a *partial plane* (or *configuration*) if every two distinct lines [points] are together incident with at most one point [respectively line], (which when it exists we call their *join*). If every two distinct lines [points] are together incident with exactly one point [respectively line] the system becomes a (projective) *plane*. A plane is said to be *non-degenerate* if it contains 4 points no 3 of which are incident with the same line, and otherwise is said to be *degenerate*.

If ρ and σ are subpartial planes of the partial plane π, then $\rho + \sigma$ (or $\rho \cup \sigma$), $\rho \cap \sigma$, and $\rho - \sigma$ are subpartial planes of π defined in the obvious way.

A configuration ρ in a plane π is said to *generate* the least subplane containing ρ. This subplane is denoted by $[\rho]_\pi$ (or $[\rho]$) and is called the *completion* of ρ in π. The plane π is *finitely generated* if, for some finite configuration $\rho \subset \pi$, $[\rho] = \pi$.

An extension process, $\mathscr{E}$, is regularly presented in the form $\mathscr{E} = \{\pi_n;\, n \in N\}$ where $N = \{0, 1, \cdots, n, \cdots\}$ is a well ordered set. π_{n-} will denote $\bigcup_{m<n} \pi_m$. $\mathscr{E}$ is said to *act on* π_0 and have the *result* $\mathscr{E}(\pi_0) = \bigcup_n \pi_n$. It should be pointed out that the partial plane $\mathscr{E}(\pi_0)$ need not be a full projective plane. The $\mathscr{E}$*-stage* (or simply *stage*) of an element $x \in \mathscr{E}(\pi_0)$ is the least n such that $x \in \pi_n$. If $n > 0$, x is said to *appear at* $\mathscr{E}$*-stage* n; it is incident in π_n with certain elements called its $\mathscr{E}$*-bearers* (or *bearers*), and these must all lie in $\pi_{n-} = \bigcup_{m<n} \pi_m$. Elements in π_0 (by convention) have no bearers. Observe that if $x \notin \pi_0$

is incident with y, and y is *not* a bearer of x, then x must be a bearer of y, for they cannot both appear at the same stage.

If ρ is a subpartial plane of $\pi = \mathscr{E}(\pi_0)$, there is a naturally defined *restriction* of $\mathscr{E}$ to ρ, denoted $\mathscr{E} \cap \rho$, viz. $\{\pi_n \cap \rho; n \in N\}$. This is apparently a process that acts on $\pi_0 \cap \rho$ with result ρ. Also there is a naturally defined *saturation* of $\mathscr{E}$ by ρ, denoted $\mathscr{E} + \rho$, viz. $\{\pi_n + \rho; n \in N\}$. It apparently acts on $\pi_0 + \rho$ with result π. We make two simple but important observations.

(1) The bearers in $\mathscr{E} \cap \rho$ of an element $x \in \rho$ are just those $\mathscr{E}$-bearers that lie in ρ.

(2) The bearers in $\mathscr{E} + \rho$ of an element $x \notin \pi_0 + \rho$ include its $\mathscr{E}$-bearers and in addition any elements of ρ incident with x in π.

If $\mathscr{E}$ and $\mathscr{F}$ are extension processes, $\mathscr{E}$ acting on π_0 and $\mathscr{F}$ acting on $\mathscr{E}(\pi_0)$, then there is a naturally defined composition of $\mathscr{F}$ with $\mathscr{E}$ denoted $\mathscr{F} \circ \mathscr{E}$.

We call a process, $\mathscr{E}$, (a) *bound*; (b) *free*; (c) *hyperfree* if for all $n > 0$ every new element of π_n has (a) $\geqq 2$; (b) 2; (c) $\leqq 2$ bearers. Of course an extension process need not fall into any of these categories. A *free plane* is by definition a plane which is the result of a hyperfree process acting on the empty plane, $\varnothing$.

A bound process $\mathscr{E}$, whose result, $\mathscr{E}(\pi_0)$, is a full projective plane, is called a *completion process* for π_0. If ρ is a configuration in a plane π, there is a canonical completion process $\mathscr{E} = \{\rho_n; n \in J^+\}$, indexed on the integers $\geqq 0$, that acts on ρ with result $[\rho]_\pi$. In fact $\rho_0 = \rho$ and ρ_n is defined inductively as the subpartial plane of π whose elements are those of ρ_{n-1} together with all points [lines] of π that are joins of lines [or points] of ρ_{n-1} resp. as n is even or odd.

3. Now we prove Hall's result. Suppose π is a finite partial plane having P points, L lines, and I incidences between the points and lines. The *rank* of π as defined by Hall is

$$r(\pi) = 2(P + L) - I .$$

LEMMA 1. *Any finite partial plane, π, which contains no confined configurations is the result of a hyperfree process.*

Proof. Set $\pi_m = \pi$ where m is the number of points and lines in π. Since π_m is not confined, there exists some element $x_m \in \pi_m$ which is incident with $\leqq 2$ elements in π_m. Define $\pi_{m-1} \subset \pi_m$ to be π_m less the element x_m. Since π_{m-1} is not confined, the process may be repeated, and after exactly m steps we obtain $\pi_0 = \varnothing$. Then $\mathscr{F} = \{\pi_i; i = 0, 1, \cdots, m\}$ is hyperfree and $\mathscr{F}(\varnothing) = \pi$.

COROLLARY. *The rank of a finite partial plane containing no confined configuration is nonnegative.*

Proof. In fact $r(\pi) \geqq r(\pi_m) \geqq \cdots \geqq r(\pi_0) = 0$. ▌

THEOREM II (Hall). *A free plane contains no confined configuration. A finitely generated plane that contains no confined configuration is free.*

REMARK. Kopejkina [2] constructed a plane (not finitely generated) that contains no confined configuration, but is not free.

Proof. Suppose first that π is a free plane and $\mathscr{F} = \{\pi_n; n \in N\}$ is a hyperfree process such that $\mathscr{F}(\varnothing) = \pi$. If ρ is any finite configuration in π, there is an element $x \in \rho$ having maximal $\mathscr{F}$-stage, m. Since x is incident with at most two elements of π_m, it obviously cannot be incident with at least three elements of $\rho \subset \pi_m$. Thus ρ cannot be a confined configuration.

The proof of the second assertion depends essentially on our definition of free plane. Suppose that the plane π is generated by a finite configuration π_0 and that π contains no confined configuration. Let $\mathscr{C} = \{\pi_n; n \in J^+\}$ be the canonical completion process for π_0. Observe that each partial plane π_n is finite and $\mathscr{C}(\pi_0) = \pi$. Since $\mathscr{C}$ is bound $r(\pi_0) \geqq r(\pi_1) \geqq \cdots \geqq r(\pi_n)$. But by the corollary above $r(\pi_n) \geqq 0$ for all n. Hence for some integer m the minimal rank be attained, and thereafter $r(\pi_n)$ will have this minimal value. But this means that in the bound process $\mathscr{S} = \{\pi_n; n \geqq m\}$ every element has exactly 2 bearers i.e., $\mathscr{S}$ is free. Now, by Lemma 1, $\pi_m = \mathscr{T}(\pi_0)$ where $\mathscr{T}$ is hyperfree. So composing $\mathscr{S}$ with $\mathscr{T}$ we obtain a hyperfree process with $\mathscr{S} \circ \mathscr{T}(\varnothing) = \pi$. ▌

4. This last section is devoted to proving that the adopted definition of free plane is equivalent to Hall's.

LEMMA 2. *Suppose $\mathscr{F}$ is hyperfree and $\pi = \mathscr{F}(\pi_0)$ is a plane. Then*

(1) *$\mathscr{F}_1 = \mathscr{F} \cap [\pi_0]$ is a free completion process for π_0 in π; in fact, for $x \in [\pi_0]$, the $\mathscr{F}$-bearers lie in $[\pi_0]$ and coincide with those in any completion process for π_0 in π.*

(2) *$\mathscr{F}_2 = \mathscr{F} + [\pi_0]$ is still hyperfree; in fact, for $x \notin [\pi_0]$, the $\mathscr{F}_2$-bearers coincide with the $\mathscr{F}$-bearers.*

Proof. (1) Let $\mathscr{C}$ be any completion process for π_0 in π. If the first assertion is false, let x be an element of least $\mathscr{C}$-stage, m, for which the $\mathscr{G}$-bearers do not conincide with the $\mathscr{F}$-bearers. Clearly $m > 0$. Then at least one of the $\geqq 2$ $\mathscr{G}$-bearers of x, say y, must fail to be an $\mathscr{F}$-bearer. Since $x \notin \pi_0$, y must have x as an $\mathscr{F}$-bearer. But y doesn't have x as an $\mathscr{G}$-bearer and y has $\mathscr{G}$-stage $< m$. This is a contradiction.

(2) Supposing the second assertion false, we have some $x \notin [\pi_0]$ with an $\mathscr{F}_2$-bearer y that is not an $\mathscr{F}$-bearer and (hence) lies in $[\pi_0] - \pi_0$. But x is incident with $y \notin \pi_0$; so $x \notin [\pi_0]$ must be an $\mathscr{F}$-bearer of $y \in [\pi_0]$ in contradiction to (1). ∎

REMARK. This lemma has a useful generalization in which π_0 is replaced in (1) and (2) by a partial plane $\rho \subset \pi_0$ that is 'complete' in π_0 (see [1, 4.3]). Then, in the proof of (1), the possibility that $x \in \pi_0$ must be eliminated.

DEFINITION 3. A *free completion* of a partial plane, π_0, is a plane π which is the result of a free process acting on π_0.

Again this seems less restrictive than Hall's definition, but

THEOREM III. *Any two free completions of a partial plane π_0 are related by a unique isomorphism that fixes π_0.*

Proof. Let $\mathscr{F}$ be a free completion process for π_0, and let $\mathscr{G}$ be the canonical completion process for π_0 in $\mathscr{F}(\pi_0)$. Clearly $\mathscr{G}(\pi_0) = [\pi_0] = \mathscr{F}(\pi_0)$; and according to assertion (1) of the above lemma, $\mathscr{G}$ is free. The theorem now follows from the fact that, if $\mathscr{G}$ and $\mathscr{G}'$ are two canonical free completion processes for π_0, there is a unique isomorphism of $\mathscr{G}(\pi_0)$ onto $\mathscr{G}'(\pi_0)$ that fixes π_0. ∎

In a free extension process $\{\pi_0, \pi_1\}$ of just two terms we say that π_1 is derived from π_0 by a *free addition*, and π_0 from π_1 by a *free subtraction*. Two partial planes are *free equivalent* if the one can be derived from the other by a finite sequence of free additions and subtractions. Free equivalent partial planes evidently have isomorphic free completions.

Recall that a basis configuration consists of a number of points on a 'base' line and two isolated points.

THEOREM IV. *Every nondegenerate free plane contains a basis configuration of which it is a free completion.*

Proof. Suppose $\pi = \mathscr{F}(\varnothing)$ is a free plane, where $\mathscr{F} = \{\pi_n; n \in N\}$ is a hyperfree process. We may assume without loss of generality that at each stage one element and no more is added, i.e., $\pi_n = \pi_{n-} + x_n$, where x_n is an element that has 0, 1, or 2 bearers.

Applying Lemma 2 to $\mathscr{F}$ for stages $\geqq n$ we find that $\mathscr{F}' = \mathscr{F} + [\pi_{n-}]$ is hyperfree; applying the same lemma to $\mathscr{F}'$ for stages $>n$ we find that $\mathscr{F}' \cap [\pi_n]$ free completes $[\pi_{n-}] + x_n$ to $[\pi_n]$.

Now let a be the first stage such that $[\pi_a]$ is nondegenerate. Then $[\pi_{a-}]$ must be a degenerate plane, and, as we have observed, $[\pi_a]$ is a free completion of $[\pi_{a-}] + x_a$. By an inspection of the various cases one shows that $[\pi_{a-}] + x_a$ is in every case free equivalent to some basis configuration π_0^α. This implies that $[\pi_a]$ is a free completion of π_0^α.

For $n > a$, one readily shows (cf. [1] or [3]) that $[\pi_{n-}] + x_n$ is free equivalent to $[\pi_{n-}]$ with a set μ_n of points adjoined to the base line, l, of π_0^α. (The set μ_n consists of 2 or 1 points if x_n is incident with 0 or 1 elements in $[\pi_{n-}]$, and of course μ_n is empty when $x_n \in [\pi_{n-}]$.) Thus $[\pi_n]$ is a free completion of $[\pi_{n-}] + \mu_n$.

We will show that π is a free completion of the basis configuration $\pi_0^\beta = \pi_0^\alpha + \bigcup_{n>a} \mu_n$. Let $\{([\pi_{n-}] + \mu_n)^k; k \in J^+\}$ be the canonical process that free completes $[\pi_{n-}] + \mu_n$ to π_n, and form the composition of all these processes to obtain $\mathscr{S} = \{([\pi_{n-}] + \mu_n)^k; (n, k) \in N \times J^+, n \geqq a\}$ where $([\pi_{a-}] + \mu_a)$ is to be read as π_0^α, and $N \times J^+$ is well ordered lexicographically. This is a hyperfree process acting on π_0^α with result π. The saturation $\bar{\mathscr{S}} = \mathscr{S} + \pi_0^\beta$ is the desired free completion process.

Clearly $\bar{\mathscr{S}}(\pi_0^\beta) = \pi$; and $\bar{\mathscr{S}}$ is a bound process since all elements with <2 $\mathscr{S}$-bearers lie in π_0^β. To show that $\bar{\mathscr{S}}$ is actually free observe that every new element, x, of $\bar{\mathscr{S}}$ appeared in $\mathscr{S}$ with exactly 2 bearers. If x has an extra bearer, y, in $\bar{\mathscr{S}}$, then $y \in \mu_n$ where x has $\bar{\mathscr{S}}$-stage $(m, k) \in N \times J^+$ and $n > m$. But y is incident with both x and base line l of π_0^α, i.e., y is the join of x and l. Then $y \in [\pi_m]$ in contradiction to $y \in \mu_n$. So $\bar{\mathscr{S}}$ must be free. ∎

This completes the proof that the definition of free plane we have proposed coincides with M. Hall's definition.

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