PACIFIC JOURNAL OF MATHEMATICS
Vol. 15, No. 1, 1965

ON A LINEAR FORM WHOSE DISTRIBUTION IS IDENTICAL WITH THAT OF A MONOMIAL

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Several authors studied identically distributed linear forms in independently and identically distributed random variables. J. Marcinkiewicz considered finite or infinite linear forms and assumed that the random variables have finite moments of all orders. He showed that the common distribution of the random variables is then the Normal distribution. Yu. V. Linnik obtained some deep results concerning identically distributed linear forms involving only a finite number of random variables. The authors have investigated in a separate paper the case where one of the linear forms contains infinitely many terms while the other is a monomial. They obtained a characterization of the normal distribution under the assumption that the second moment of the random variable is finite. In the present paper we investigate a similar problem and do not assume the existence of the second moment.

1. We prove the following theorem:

THEOREM. *Let $\{X_j\}$ be a finite or denumerable sequence of independently and identically distributed nondegenerate random variables and let $\{a_j\}$ be a sequence of real numbers such that the sum $\sum_j a_jX_j$ exists*[1]. *Let $\alpha \neq 0$ be a real number such that*
(i) *the sum* $\sum_j a_jX_j$ *is distributed as* αX_1
(ii) $\sum_j a_j^2 \geqq \alpha^2$.
Then the common distribution of the X_j is normal.

REMARK. The converse statement is evidently true provided that $\sum_j a_j = \alpha$ if the sum $\sum_j a_jX_j$ contains more than two terms or $\mathcal{E}(X_j) = 0$ in case $\sum_j a_jX_j$ has only two terms.

In § 2 we prove three lemmas, the third of these has some independent interest. In § 3 the theorem is proved.

Received October 14, 1963, and in revised form March 10, 1964. The work of the first author was supported by the National Science Foundation under grant GP-96. The work of the second author was supported by the U.S. Air Force under grant AF-AFOSR-473-63.

[1] We say that the infinite sum $\sum_j a_jX_j$ exists, if it converges almost everywhere. It is known (see Loève [3] pg. 251) that for a series of independent random variables the concepts of convergence almost everywhere and weak convergence are equivalent.

2. Lemmas. We denote the common distribution of the random variable X_j by $F(x)$ and write $f(t)$ for the corresponding characteristic function.

LEMMA 1. *Suppose that all the conditions of the theorem except* (ii) *are satisfied. Then* $\sup_j |a_j| < |\alpha|$.

According to the assumptions we have

$$\prod_j f(a_j t) = f(\alpha t) . \tag{2.1}$$

We set $b_j = a_j/\alpha$ $(j = 1, 2, \cdots)$ and obtain

$$\prod_j f(b_j t) = f(t) . \tag{2.2}$$

The lemma is proven if we show that $|b_j| < 1$ for all j. First we note that if $|b_j| = 1$ for at least one value of j, then X_j has necessarily a degenerate distribution. We consider the case where $|b_k| > 1$ for at least one value k. We see then from (2.2) that

$$1 \geqq |f(b_k t)| \geqq |f(t)|$$

which means

$$1 \geqq |f(t)| \geqq |f(t/b_k)| \geqq |f(t/b_k^2)| \geqq \cdots \lim_{n\to\infty} |f(t/b_k^n)| = f(0) = 1 .$$

Therefore $|f(t)| \equiv 1$ and the distribution of X_j is again degenerate. We conclude therefore that

$$|b_j| < 1 \qquad (j = 1, 2 \cdots) \tag{2.3}$$

LEMMA 2. *Suppose that all the conditions of the theorem, except* (ii), *are satisfied then the function* $f(t)$ *has no real zeros.*

We first remark that the existence of the infinite sum $\sum_j a_j X_j$ implies that the sequence of random variables $S_N = \sum_{j=N+1}^{\infty} a_j X_j$ converges to zero (as $N \to \infty$) with probability 1. It follows from the continuity theorem that

$$\lim_{N\to\infty} \prod_{j=N+1}^{\infty} f(a_j t) = 1 \tag{2.4}$$

uniformly in every finite t-interval.

Let $\varepsilon > 0$ be an arbitrarily small number and let T be a positive number. It follows then from (2.4) that there exists an $N_0 = N_0(\varepsilon, T)$ such that for all $N \geqq N_0$ the inequality

$$\left| \prod_{j=N+1}^{\infty} f(b_j t) - 1 \right| \leqq \varepsilon \tag{2.5}$$

holds uniformly for $|t| \leqq T$.

We give an indirect proof of Lemma 2. Suppose that the function $f(t)$ has real zeros and let t_0 be one of the zeros of $f(t)$ which is closest to the origin. Then

$$\prod_j f(b_j t_0) = f(t_0) = 0 ,$$

so that either $f(b_j t_0) = 0$ for at least one value of j or the product is infinite and diverges to zero at the point $t = t_0$. The first case is impossible by virtue of (2.3) while the second contradicts the uniform convergence of the infinite product so that Lemma 2 is proven.

LEMMA 3. *Let $\{X_j\}$ be a finite or denumerable sequence of independently and identically distributed nondegenerate random variables and let $\{a_j\}$ be a sequence of real numbers such that the sum $\sum_j a_j X_j$ exists. Let $\alpha \neq 0$ be a real number such that $\sup_j |a_j| < |\alpha|$. Suppose that the sum $\sum_j a_j X_j$ has the same distribution as αX_1, then the common distribution of each X_j is infinitely divisible.*

To prove Lemma 3 we write (2.2) in the form[2]

$$f(t) = f(b_1 t) f(b_2 t) \cdots f(b_N t) \Phi_N(t) \tag{2.6}$$

where

$$\Phi_N(t) = \prod_{j=N+1}^{\infty} f(b_j t) \tag{2.7}$$

and where N is so large that the inequality (2.5) holds. Using (2.6) we see that

$$f(t) = \prod_{j=1}^{N} f(b_j^2 t) \prod_{\substack{j,k=1 \\ j>k}}^{N} [f(b_j b_k t)]^2 \left[\prod_{j=1}^{N} \Phi_N(b_j) \right] \cdot \Phi_N(t) . \tag{2.8}$$

We repeat this process n times and obtain

$$\begin{aligned} f(t) = & \left\{ \prod_{j_1 + \cdots + j_N = n} [f(b_1^{j_1} \cdots b_N^{j_N} t)]^{(n;j_1 \cdots j_N)} \right\} \\ & \cdot \left\{ \prod_{k=1}^{n} \prod_{j_1 + \cdots + j_N = n-k} [\Phi_N(b_1^{j_1} \cdots b_N^{j_N} t)]^{(n-k;j_1 \cdots j_N)} \right\} . \end{aligned} \tag{2.9}$$

Here all $j_k \geqq 0$ and $(m ; j_1 \cdots j_N) = m!/j_1! \cdots j_N!$. Formula (2.9) indicates that the random variable X, whose characteristic function is $f(t)$, is the sum of $k_n = N^n + N^{n-1} + \cdots + N^2 + N + 1$ independent random

[2] If the sequence $\{X_j\}$ is finite then N is equal to the number of variables X_j so that $\Phi_N(t) \equiv 1$.

variables $X_{n,k}(k - 1, 2, \cdots, k_n)$, that is $X = \sum_{k=1}^{k_n} X_{n,k}$ for every n.

Such sequences of sums of independent random variables occur in the study of the central limit theorem, and we give next a few results which we wish to apply.

We say that the summands $X_{n,k}$ are uniformly asymptotically negligible (u.a.n), if $X_{n,k}$ converges in probability to zero, uniformly in k, as n tends to infinity; this means that for any $\varepsilon > 0$

$$\lim_{n\to\infty} \max_{1\leqq k\leqq k_n} P(|X_{n,k}| \geqq \varepsilon) = 0 . \tag{2.10}$$

It is known (see Loève [3] pg. 302) that condition (2.10) is equivalent to

$$\lim_{n\to\infty} \max_{1\leqq k\leqq k_n} |f_{n,k}(t) - 1| = 0 \tag{2.11}$$

uniformly in every finite t-interval.

Let $X_{n,k}$ $(k = 1, 2, \cdots, k_n)$ be, for each n, a finite set of independent random variables and suppose that the $X_{n,k}$ are u.a.n. Then the limiting distribution (as n tends to infinity) of the sums $\sum_{k=1}^{k_n} X_{n,k}$ is infinitely divisible.

For the proof we refer the reader to Loève [3] (pg. 309).

We turn now to the proof of Lemma 3 and show that the factors of (2.9) satisfy condition (2.11).

Let $\varepsilon > 0$ be an arbitrarily small number and $T > 0$. We see from (2.5) and (2.7) that we can select a sufficiently large N such that

$$|\Phi_N(t) - 1| \leqq \varepsilon \tag{2.12}$$

uniformly in $|t| \leqq T$. Since $|b_j| < 1$ we have

$$|b_1^{j_1} \cdots b_N^{j_N} t| < T$$

so that, according to (2.12),

$$|\Phi_N(b_1^{j_1} \cdots b_N^{j_N} t) - 1| \leqq \varepsilon \tag{2.13}$$

uniformly in $|t| \leqq T$ for the chosen value of N.

We consider next a typical factor $f(b_1^{j_1} \cdots b_N^{j_N} t)$ of the product in the first brace of formula (2.9). Here $j_1 + j_2 + \cdots + j_N = n$ and $j_k \geqq 0$ so that at least one of the j_k is positive. We show now that it is possible to choose an $n_0 = n_0(\varepsilon, T)$ such that for $n \geqq n_0$

$$\gamma_{j_1 \cdots j_N}(t) = |f(b_1^{j_1} \cdots b_N^{j_N} t) - 1| \leqq \varepsilon \tag{2.14}$$

uniformly in $|t| \leqq T$.

Clearly,

$$\gamma_{j_1\cdots j_N}(t) \leqq \left|\int_{|x|\geqq A}\{\exp[ib_1^{j_1}\cdots b_N^{j_N}tx]-1\}dF(x)\right| + \left|\int_{|x|<A}\{\exp[ib_1^{j_1}\cdots b_N^{j_N}tx]-1\}dF(x)\right| . \tag{2.15}$$

We choose A so large that

$$\left|\int_{|x|\geqq A}\{\exp[ib_1^{j_1}\cdots b_N^{j_N}tx]-1\}dF(x)\right| \leqq 2\int_{|x|\geqq A}dF(x) \leqq \frac{\varepsilon}{2} . \tag{2.16}$$

We note that

$$\left|\int_{|x|<A}\{\exp[ib_1^{j_1}\cdots b_n^{j_N}tx]-1\}dF(x)\right| \leqq \left|b_1^{j_1}\cdots b_N^{j_N}\right| TA . \tag{2.17}$$

We select now an $n^* = n^*(j_1, \cdots, j_N; T, \varepsilon)$ so large that for $n \geqq n^*$ the inequality

$$|b_1^{j_1}\cdots b_N^{j_N}|\, TA \leqq \frac{\varepsilon}{2} \tag{2.18}$$

holds. This is possible in view of (2.3). There are altogether N^n terms of the form $f(b_1^{j_1}\cdots b_N^{j_N}t)$ in (2.14) and we choose

$$n_0 = n_0(\varepsilon, T) = \max_{j_1\cdots j_N} n^*(j_1, \cdots, j_N; T, \varepsilon) ; \tag{2.19}$$

then (2.14) follows from (2.16), (2.17), (2.18) and (2.19).

We see therefore that the set of independent random variables $X_{n,k}$ satisfies the u.a.n. condition (2.11). Therefore the distribution of X is infinitely divisible and Lemma 3 is proven.

Since $f(t)$ is an infinitely divisible characteristic function, it admits the Lévy-Khinchine representation

$$\ln f(t) = i\alpha t - \beta t^2/2 + \int_{-\infty}^{-0}\left(e^{itx} - 1 - \frac{itx}{1+x^2}\right)\frac{1+x^2}{x^2}dG(x) + \int_{+0}^{\infty}\left(e^{itx} - 1 - \frac{itx}{1+x^2}\right)\frac{1+x^2}{x^2}dG(x) \tag{2.20}$$

where α and β are real numbers, $\beta \geqq 0$, and where $G(x)$ is a non-decreasing, right-continuous function such that $G(-\infty) = 0$ and $G(+\infty) = K < \infty$. Let now $f(t)$ be the characteristic function of an infinitely divisible symmetric distribution, so that $f(t) = f(-t)$. In this case one sees after some elementary transformations of the integrals in (2.20) that

$$G(x) + G(-x-0) = C \tag{2.21}$$

for all $x \neq 0$. Using (2.20) and (2.21) we see that the characteristic function of a symmetric infinitely divisible distribution admits the representation

$$\ln f(t) = -\beta t^2/2 + \int_{+0}^{+\infty} (\cos tx - 1) \frac{1+x^2}{x^2} dH(x) \tag{2.22}$$

where

$$H(x) = \begin{cases} 2G(x) - C & \text{for } x > 0 \\ 0 & \text{for } x < 0 \, . \end{cases} \tag{2.22a}$$

Thus $H(x)$ is a non decreasing, right-continuous, bounded function and $H(x)$ and $G(x)$ determine each other uniquely.

3. Proof of the theorem. We introduce the function

$$g(t) = f(t)f(-t) \tag{3.1}$$

and conclude from (2.2) that the relation

$$\prod_j g(b_j t) = g(t) \tag{3.2}$$

holds for all real t. Here $g(t)$ is the characteristic function of a symmetric distribution and is therefore a real and even function. It is no restriction to assume that

$$0 \leqq b_j < 1 \qquad (j = 1, 2, \cdots) \tag{3.3a}$$

where

$$\sum_{j=1}^{\infty} b_j^2 \geqq 1 \, . \tag{3.3b}$$

According to (2.22) we have then the representation

$$\ln g(t) = -\beta t^2/2 + \int_{+0}^{\infty} (\cos tx - 1) \frac{1+x^2}{x^2} dH(x) \tag{3.4}$$

where $\beta \geqq 0$ and where $H(x)$ is a nondecreasing, right-continuous and bounded function. We use (3.4) and (3.3b) and obtain from (3.2) the relation

$$\begin{aligned} &\sum_{j=1}^{\infty} \int_{+0}^{\infty} (\cos b_j tx - 1) \frac{1+x^2}{x^2} dH(x) \\ &\qquad = K\frac{t^2}{2} + \int_{+0}^{\infty} (\cos tx - 1) \frac{1+x^2}{x^2} dH(x) \, . \end{aligned} \tag{3.5}$$

where

$$K = \beta\left[\sum_{j=1}^{\infty} b_j^2 - 1\right] \geqq 0 \, .$$

We define the sequence $\{\psi_\nu(t)\}$ by

$$\psi_\nu(t) = \sum_{j=1}^{\nu} \int_{+0}^{\infty} (\cos b_j tx - 1) \frac{1+x^2}{x^2} dH(x) \tag{3.6}$$

so that

$$\lim_{\nu\to\infty} \psi_\nu(t) = \psi(t) = K\frac{t^2}{2} + \int_{+0}^{\infty} (\cos tx - 1) \frac{1+x^2}{x^2} dH(x) \tag{3.7}$$

for every real t.

Since $\psi(t)$ is the characteristic function of an infinitely divisible distribution it follows that $K \leqq 0$, so that we conclude from assumption (ii) that $K = 0$ and $\sum_{j=1}^{\infty} b_j^2 = 1$.

By a change of the variable of integration in (3.6) we obtain

$$\psi_\nu(t) = \int_{+0}^{\infty} (\cos tx - 1) \frac{1+x^2}{x^2} \left[\sum_{j=1}^{\nu} \frac{b_j^2 + x^2}{1+x^2} dH(x/b_j) \right] .$$

We write

$$H_\nu(x) = \begin{cases} \int_{+0}^{x} \left[\sum_{j=1}^{\nu} \frac{b_j^2 + y^2}{1+y^2} dH(y/b_j) \right] & \text{for } x > 0 \\ 0 & \text{for } x < 0 . \end{cases} \tag{3.8}$$

Therefore we have, for every ν,

$$\psi_\nu(t) = \int_{+0}^{+\infty} (\cos tx - 1) \frac{1+x^2}{x^2} dH_\nu(x) . \tag{3.9}$$

It follows then from (3.7) and (3.8) that

$$\lim_{\nu\to\infty} H_\nu(x) = H(x) \tag{3.10}$$

for every x which is a continuity point of $H(x)$. The proof is carried in the same way in which the convergence theorem is proven (see Loève [3] pp. 300–301).

In view of (3.3a) we have

$$\frac{b_j^2 + y^2}{1+y^2} \geqq b_j^2 \qquad (j = 1, 2, \cdots)$$

so that we conclude from (3.8) that

$$H_\nu(x) \geqq \sum_{j=1}^{\nu} b_j^2 H\left(\frac{x}{b_j}\right) \tag{3.11}$$

for all ν.

It follows from (3.10) and (3.11) that

$$H(x) = \lim_{\nu \to \infty} H_\nu(x) \geqq \sum_{j=1}^{\infty} b_j^2 H\left(\frac{x}{b_j}\right)$$

for all $x > 0$ which are continuity points of $H(x)$.

Using equation (3.3b) we obtain

$$\sum_{j=1}^{\infty} b_j^2 \left[H(x) - H\left(\frac{x}{b_j}\right)\right] \geqq 0 \; . \tag{3.12}$$

Since $H(x)$ is a nondecreasing function, we see from (3.3a) that

$$H(x) \leqq H\left(\frac{x}{b_j}\right) . \tag{3.13}$$

It follows from (3.12) and (3.13) that

$$H(x) = H\left(\frac{x}{b_j}\right)$$

for every $x > 0$ which is a continuity point of $H(x)$. Therefore

$$H(x) = H(+\infty) = C$$

for $x > 0$. We now turn to equation (3.4) and get

$$\ln g(t) = -\beta t^2/2 \; . \tag{3.14}$$

The statement of the theorem is an immediate consequence of (3.1) and of Cramér's theorem.

References

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