PACIFIC JOURNAL OF MATHEMATICS
Vol. 15, No. 2, 1965

CERTAIN ALGEBRAS OF DEGREE ONE

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In this note the following is proved: Suppose R is a finite-dimensional algebra over an algebraically closed field F of characteristic 0 whose associator satisfies $4(y, x, x) = 4(x, y, x) + [[y, x], x]$ and $(x, x, x) = 0$. If R is simple and non-nil then R is iso-morphic to F.

We call it Theorem B, and prove it below.

In [3] nonassociative algebras satisfying identities of degree three were studied and it was shown that relative to quasi-equivalence any algebra satisfying such an identity (subject to some rather weak additional hypotheses) must in fact satisfy at least some one of seven particular identities; each of degree three. In this note we concern ourselves with one of these seven residual cases; namely the identity

$$(1) \qquad 4(y, x, x) = 4(x, y, x) + [[y, x], x]$$

where the associator (x, y, z) is defined by $(x, y, z) = (xy)z - x(yz)$ and the commutator $[x, y]$ by $[x, y] = xy - yx$ for elements x, y, z of the algebra.

Throughout the remainder of this note R will be a ring of characteristic not two or three which satisfies (1) in addition to the following identity:

$$(2) \qquad (x, x, x) = 0 .$$

The following result was established in [3]:

THEOREM A. *Suppose R has an idempotent $e \neq 0,1$. Then R is not simple.*

This reduces the study of simple rings to the consideration of rings whose only nonzero idempotent is the identity element.

Ideals and Simple rings. A well-known consequence of (2) is

$$(3) \qquad (x, x, y) + (x, y, x) + (y, x, x) = 0 .$$

We define $x \circ y = xy + yx$ and proceed to simplify (1). We rewrite (1) as

$$(4) \qquad 4yx^2 = 4x \circ yx - 3(xy)x - x(xy) - (yx)x + x(yx)$$

and (3) as

Received April 6, 1964.

$$2yx^2 = x^2 \circ y + (xy)x + (yx)x - x(yx) - x(xy) . \tag{5}$$

Adding (4) and (5) we obtain

$$6yx^2 = x^2 \circ y + 4x \circ yx - 2x \circ xy . \tag{6}$$

Finally we add and subtract $2x \circ yx$ to the right-hand member of (6) giving us

$$6yx^2 = 6x \circ yx - 2x \circ (x \circ y) + x^2 \circ y . \tag{7}$$

Replacing x by $x_1 + x_2$ in (7) and then using (7) to simplify the result we find

$$\begin{aligned} 6y(x_1 \circ x_2) = {} & 6x_1 \circ yx_2 + 6x_2 \circ yx_1 - 2x_1 \circ (x_2 \circ y) \\ & - 2x_2 \circ (x_1 \circ y) + (x_1 \circ x_2) \circ y . \end{aligned} \tag{8}$$

We define the ring R^+ to be the same additive group as R but the multiplication in R^+ is given by $(x, y) = 1/2x \circ y$. We set $(x, y, z)^+ = (x \circ y) \circ z - x \circ (y \circ z)$ and note that R^+ is associative if and only if $(x, y, z)^+ = 0$ for all $x, y, z \in R$.

LEMMA 1. *Let L be the additive group generated by all $(x, y, z)^+$ where $x, y, z \in R$. Then L is a left ideal of R.*

Proof. First of all we consider $y[(x_1 \circ x_2) \circ x_3]$. Then (8) (with x_1 replaced by $x_1 \circ x_2$ and x_2 by x_3) becomes

$$\begin{aligned} 6y[(x_1 \circ x_2) \circ x_3] = {} & 6(x_1 \circ x_2) \circ yx_3 + 6x_3 \circ y(x_1 \circ x_2) - 2(x_1 \circ x_2) \circ (x_3 \circ y) \\ & -2x_3 \circ [(x_1 \circ x_2) \circ y] + [(x_1 \circ x_2) \circ x_3] \circ y . \end{aligned} \tag{9}$$

We use (8) to rewrite the second term of the right-hand member of (9) as:

$$\begin{aligned} 6x_3 \circ y(x_1 \circ x_2) = {} & 6x_3 \circ (x_1 \circ yx_2) + 6x_3 \circ (x_2 \circ yx_1) - 2x_3 \circ [x_1 \circ (x_2 \circ y)] \\ & -2x_3 \circ [x_2 \circ (x_1 \circ y)] + x_3 \circ [(x_1 \circ x_2) \circ y] . \end{aligned}$$

A substitution of this into (9) results in

$$\begin{aligned} 6y[(x_1 \circ x_2) \circ x_3] = {} & 6(x_2 \circ x_2) \circ yx_3 + 6x_3 \circ (x_1 \circ yx_2) + 6x_3 \circ (x_2 \circ yx_1) \\ & - 2x_3 \circ [x_1 \circ (x_2 y)] - 2x_3 \circ [x_2 \circ (x_1 \circ y)] \\ & - 2(x_1 \circ x_2) \circ (x_3 \circ y) + (x_3, x_1 \circ x_2, y)^+ . \end{aligned} \tag{10}$$

If we interchange x_1 and x_2 in (10) we obtain

$$\begin{aligned} 6y[(x_3 \circ x_2) \circ x_1] = {} & 6(x_3 \circ x_2) \circ yx_1 + 6x_1 \circ (x_3 \circ yx_2) + 6x_1 \circ (x_2 \circ yx_3) \\ & - 2x_1 \circ [x_3 \circ (x_2 \circ y)] - 2x_1 \circ [x_2 \circ (x_3 \circ y)] \\ & - 2(x_3 \circ x_2) \circ (x_1 \circ y) + (x_1, x_3 \circ x_2, y)^+ . \end{aligned} \tag{11}$$

Then subtracting (11) from (10) yields

$$\begin{aligned}6y(x_1, x_2, x_3)^+ = {} & 6(x_1, x_2, yx_3)^+ + 6(x_1, yx_2, x_3)^+ + 6(yx_1, x_2, x_3)^+ \\ & - 2(x_1, x_2 \circ y, x_3)^+ + 2(x_3, x_2, x_1 \circ y)^+ \\ & - 2(x_1, x_2 \circ x_3, y)^+ + (x_3, x_1 \circ x_2, y) + (x_1, x_3 \circ x_2, y)^+ .\end{aligned}$$

Thus $yL \subseteq L$ and L is a left ideal of R.

THEOREM 1. *$L + LR$ is an ideal (two-sided) of R.*

Proof. As is immediate from Lemma 1 it suffices to show that $R(LR) + (LR)R \subseteq L + LR$. Suppose $x_1, x_2 \in R, y \in L$. Then $(x_1, x_2, y)^+ \in L$ so that $(x_1 \circ x_2) \circ y - x_1 \circ (x_2 \circ y) \in L$. But $(x_1 \circ x_2) \circ y$ and $x_1 \circ x_2 y$ belong to $L + LR$. Hence, $x_1 \circ y x_2 \in L + LR$. Next we interchange x_2 and y in (8), obtaining

$$\begin{aligned}6x_2(x_1 \circ y) = {} & 6x_1 \circ x_2 y + 6y \circ x_2 x_1 - 2x_1 \circ (x_2 \circ y) \\ & - 2y \circ (x_1 \circ x_2) + (x_1 \circ y) \circ x_2 .\end{aligned} \tag{12}$$

But Lemma 1 along with the preceding remarks implies that each term of the right-hand member belongs to $L + LR$. Hence, $x_2(x_1 \circ y) \in L + LR$ but $x_2(x_1 y) \in R(RL) \subseteq L$ so that $x_2(yx_1) \in L + LR$. Thus, we must also have $(yx_1)x_2 \in L + LR$, since $x_2 \circ yx_1 \in L + LR$. Therefore $R(LR) + (LR)R \subseteq L + LR$ and $L + LR$ is an ideal of R.

THEOREM 2. *L is an ideal of $L + LR$.*

Proof. Since L is a left ideal of R we need only show that $L(LR) \subseteq L$. Suppose $x_1, x_2 \in L, y \in R$. Then (8) implies

$$2x_1(x_2 y) + 2x_2(x_1 y) - (x_1 \circ x_2)y \in L . \tag{13}$$

Considering that $(x_1, x_2, y)^+$ and $(x_1, y, x_2)^+$ belong to L we find

$$(x_1 \circ x_2)y - x_1(x_2 y) \in L . \tag{14}$$

and

$$x_2(x_1 y) - x_1(x_2 y) \in L . \tag{15}$$

Adding (13) and (14) we obtain $x_1(x_2 y) + 2x_2(x_1 y) \in L$. This along with (15) implies that $x_2(x_1 y) \in L$ or $L(LR) \subseteq L$, as was to be shown.

COROLLARY. *If R is simple then either R^+ is associative or $L = R$.*

Proof. If R is simple then either $L + LR = 0$ or $L + LR = R$. In the first instance $L = 0$ so that R is associative while in the second

L is an ideal of R. Hence, either $L = 0$ or $L = R$.

Now suppose R is a simple finite-dimensional algebra over an angebraically closed field F of characteristic 0. Then R is power-associative [3, Lemma 2] so that if R is nonnil, R must possess a nonzero idempotent e. By Theorem A of the Introduction we must, in fact, have $e = 1$, the identity of R. A result of Albert's [1] states that $R = F\ 1 + N$ where all the elements of N are nilpotent and N is an ideal of R^+. From this it is immediate that $(x, y, z)^+ \in N$ for all $x, y, z \in R$ so that $L \subseteq N \neq R$. Hence, $L = 0$ and R^+ is associative. But then R satisfies

$$2(y, x, x) = 2(x, x, y) + [[y, x], x] \qquad \text{(See [3])}.$$

Subtracting this relation from (1) we have

$$2(y, x, x) = 4(x, y, x) - 2(x, x, y)$$

which along with (3) implies that $(x, y, x) = 0$. Hence, R is flexible and the results of Theorem B follow from [2].

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