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CONTRACTIVE PROJECTIONS ON AN $\mathfrak{L}_1$ SPACE

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This paper discusses the class of contractive (operator norm one) projections on the complex $\mathfrak{L}_1$ space of a probability measure. In particular there is a characterization of such projections and of their range spaces, and also of the closed vector sublattices of $\mathfrak{L}_1$ and the subspaces of $\mathfrak{L}_1$ that are isometrically isomorphic to some $\mathfrak{L}_1$ space. Further results include an extension of the above results to more general measure spaces and several results about contraction operators on $\mathfrak{L}_1$.

Let $(X, \mathscr{S}, \mu)$ be a probability space. Let $E^{\mathscr{T}}$ denote the conditional expectation for the σ-subalgebra $\mathscr{T}$ of $\mathscr{S}$. A nonnegative function $k \in \mathfrak{L}_1(X, \mathscr{S}, \mu)$ is said to be a weight function for $\mathscr{T}$ if $E^{\mathscr{T}}k = \chi_T$ where χ_T is the characteristic function of some $T \in \mathscr{T}$. The weighted conditional expectation $E_k^{\mathscr{T}}$ is defined to be: $E_k^{\mathscr{T}} = k \cdot E^{\mathscr{T}}$. For each measurable function ϕ of modulus one, let U_ϕ denote the multiplication operator $U_\phi f = \phi \cdot f$. Let $\mathfrak{R}[P]$ denote the range of P. A projection P is said to satisfy $(*)$ if $P\{f \mid (f) \cdot \mathfrak{R}[P] = (0)\} = (0)$.

THEOREM 1. *An operator P on $\mathfrak{L}_1$ is a contractive projection satisfying $(*)$ if and only if $P = U_{\bar{\phi}} E_k^{\mathscr{T}} U_\phi$ for some σ-subalgebra $\mathscr{T}$ of $\mathscr{S}$, weight function k for $\mathscr{T}$ and measurable function ϕ of modulus one.*

The characterization is completed by showing that each contractive projection splits canonically into a contractive projection that satisfies $(*)$ and an "arbitrary" contraction operator with certain properties.

COROLLARY. *An operator Q on $\mathfrak{L}_1$ is a conditional expectation if and only if*
(1) $Q^2 = Q$, (2) $\|Q\| \leqq 1$, *and* (3) $Q1 = 1$.

THEOREM 3. *For a subspace $\mathfrak{M}$ of $\mathfrak{L}_1$ the following statements are equivalent*:
(1) *$\mathfrak{M}$ is the range of a contractive projection,*
(2) *$U_\phi \mathfrak{M}$ is a closed vector sublattice for some measurable function ϕ of modulus one, and*
(3) *$\mathfrak{M}$ is isometrically isomorphic to some $\mathfrak{L}_1$ space.*

LEMMA 1. *Let $\mathfrak{M}$ be a closed vector sublattice of $\mathfrak{L}_1$ (a closed*

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self adjoint subspace in which the real functions form a lattice). Then there exists a σ-subalgebra $\mathscr{T}$ and a weight function k for $\mathscr{T}$ such that $\mathfrak{M} = k \cdot \mathfrak{L}_1(X, \mathscr{T}, \mu_{\mathscr{T}})$, where $\mu_{\mathscr{T}}$ denotes the restriction of μ to $\mathscr{T}$.

An operator P defined on a Banach space is said to be a projection if $P^2 = P$. Attention is confined to a subclass of projections. The particular subclass under study is the class of contractive (operator norm one) projections on an $\mathfrak{L}_1$ space. The methods developed shed some light on general contraction operators defined on an $\mathfrak{L}_1$ space; in particular, a relation between such operators and positive operators is shown.

The main theorem in this paper gives a characterization of contractive projections on an $\mathfrak{L}_1$ space. This result is obtained in several steps each of which is stated as a separate proposition and the final result is then summarized in Theorem 1. The "concrete model" of a contractive projection is given in Proposition 1 and is the conjugation (in the group theoretic sense) of a "weighted conditional expectation" by a multiplication operator where the "multiplier function" has modulus one. It is further shown in Proposition 1 that this "concrete model" satisfies a certain "regularity hypothesis". In Proposition 2 it is shown that a general contractive projection splits canonically into a "regular" contractive projection and an arbitrary contraction with a fixed range and domain. In Propositions 3 and 4, the characterization is completed.

Several corollaries to Theorem 1 are given; one of these states that a contractive projection that takes the function 1 into 1 is a conditional expectation. This is related to results of Moy [6], Bahadur [1], Rota [7], and Sidák [8]. Also in the proof of Proposition 3, it is necessary to determine the structure of those closed subspaces of an $\mathfrak{L}_1$ space that are also sublattices; this result is stated as Lemma 1 and is related to results that appear in Moy [6], Bahadur [1], and Brunk [2].

In § 3 the problem of determining which subspaces of an $\mathfrak{L}_1$ space are the range of a contractive projection is raised. Two solutions to this problem are given in Theorem 3. The second solution further solves the problem of determining which subspaces of an $\mathfrak{L}_1$ space are themselves $\mathfrak{L}_1$ spaces (in the sense of being isometrically isomorphic to an $\mathfrak{L}_1$ space). In this sense Theorem 3 can be regarded as a "Stone-Weierstrass type theorem" for $\mathfrak{L}_1$ spaces.

In §§ 2 and 3 only $\mathfrak{L}_1$ spaces defined relative to finite measure spaces are considered. The extension of these results to more general measure spaces is taken up in § 4. In § 5 several results about contraction operators on an $\mathfrak{L}_1$ space (including a mean ergodic theorem for such operators) are stated and proved. Finally, a few concluding remarks appear in § 6.

The results of this paper represent an extension and refinement of certain results from the author's doctoral dissertation which was submitted to the Graduate Faculty of Louisiana State University in August, 1962. The author wishes to acknowledge his indebtedness to P. Porcelli for his guidance and assistance in preparing that dissertation. The author is also grateful to the referee for many helpful suggestions.

1. Technicalities and definitions. The standard work of Halmos [3] will be used as a basic reference. All measures considered in this paper are assumed to be countably additive.

Let $(X, \mathscr{S}, \mu)$ be a fixed probability space, that is, X is a set, $\mathscr{S}$ is a σ-algebra of subsets of X, and μ is a measure defined on $\mathscr{S}$ for which $\mu(X) = 1$. $\mathfrak{L}_1 = \mathfrak{L}_1(X, \mathscr{S}, \mu)$ will denote the usual Banach space of complex valued integrable functions defined on X, in which two functions are regarded as equal if they are equal almost everywhere relative to μ, and in which the norm of a function f is defined to be $\|f\| = \int_X |f|\, d\mu$. Hereafter, the relation $f = g$ will be interpreted to mean that the functions f and g are equal almost everywhere relative to μ, and the relation $S = T$, to mean that $\mu(S \cap T') + \mu(S' \cap T) = 0$.[1] The relations of inequality and containment will be interpreted similarly. Further, for each subset $S \in \mathscr{S}$, let χ_S denote the *characteristic function* of S, that is, $\chi_S(x) = 1$ if $x \in S$ or 0 if $x \notin S$. Lastly, define the *support* of a measurable function f to be the measurable subset $S(f) = \{x \mid f(x) \neq 0\}$.

For the σ-subalgebra $\mathscr{T}$ of $\mathscr{S}$, let $E^{\mathscr{T}}$ denote the *conditional expectation* for $\mathscr{T}$, which is defined for $f \in \mathfrak{L}_1$ as follows: $E^{\mathscr{T}}f$ is the unique $\mathscr{T}$-measurable function having the property that $\int_T E^{\mathscr{T}} f d\mu = \int_T f d\mu$ for every subset $T \in \mathscr{T}$. That such a function exists and is unique follows from the Radon-Nikodym Theorem. A nonnegative measurable function k is said to be a *weight function for* $\mathscr{T}$ if $\int_T k d\mu = \int_T \chi_{T_0} d\mu$ for every $T \in \mathscr{T}$, where $T_0 = S(k)$ and is a set in $\mathscr{T}$ for which $S \in \mathscr{S}$ and $T_0 \subset S$ imply $S \in \mathscr{T}$. (The essential property is that $E^{\mathscr{T}}(k)$ is a characteristic function; the additional hypothesis is imposed for uniqueness considerations later on.) For a σ-subalgebra $\mathscr{T}$ of $\mathscr{S}$ and weight function k for $\mathscr{T}$, the *weighted conditional expectation* $E_k^{\mathscr{T}}$ is defined to be: $E_k^{\mathscr{T}} = k \cdot E^{\mathscr{T}}$.

An example of a nontrivial non-$\mathscr{T}$-measurable weight function can

[1] Note, $S' = X - S$ for every subset $S \in \mathscr{S}$.

be obtained as follows. Let X be the unit square $[0, 1] \times [0, 1]$, $\mathscr{S}$ be the σ-algebra of Lebesgue measurable subsets of X, ahd $\mathscr{T}$ be the σ-subalgebra of $\mathscr{S}$ consisting of sets of the form $E \times [0, 1]$. It can be easily verified that the function $k(x, y) = 2y$ is a weight function for $\mathscr{T}$ that is not $\mathscr{T}$-measurable. Moreover, in this case $S(k) = X$. This example also yields the nontrivial weighted conditional expectation $E_k^{\mathscr{T}}$.

Lastly, for each measurable function ϕ of modulus one (that is, $|\phi(x)| = 1$ for $x \in X$), the operator U_ϕ is defined as follows: $U_\phi f = \phi f$ for every $f \in \mathfrak{L}_1$. Notice that U_ϕ is an isometry on $\mathfrak{L}_1$ and that $U_\phi U_{\bar{\phi}}$ is equal to the identity operator on $\mathfrak{L}_1$.

All operators considered in this paper are assumed to be bounded and linear, and to have $\mathfrak{L}_1$ as domain, and to have range contained in $\mathfrak{L}_1$. An operator P is said to be a *contraction* (or to be *contractive*) if $\|P\| \leqq 1$; to be *positive* if $Pf \geqq 0$ for every nonnegative function $f \in \mathfrak{L}_1$; and to be a *projection* if $P^2 = P$. Let $\mathfrak{R}[P]$ denote the range of P (that is, $\mathfrak{R}[P] = P(\mathfrak{L}_1)$), and $\mathfrak{K}[P]$ the subspace $\{f \mid f \in \mathfrak{L}_1$ and $f \cdot \mathfrak{R}[P] = (0)\}$. Notice that each of $\mathfrak{R}[P]$ and $\mathfrak{K}[P]$ is a closed subspace of $\mathfrak{L}_1$. A projection P is said to *satisfy* (*) if $P\{\mathfrak{K}[P]\} = (0)$. (This is the regularity condition referred to in the introduction.)

2. Characterization of contractive projections.

In this section the characterization of contractive projections is obtained. It is first shown that a certain class of operators (the "concrete models") consists of contractive projections that satisfy (*).

PROPOSITION 1. *Let $\mathscr{T}$ be a σ-subalgebra of $\mathscr{S}$, k a weight function for $\mathscr{T}$, and ϕ a measurable function of modulus one. The operator $U_{\bar{\phi}} E_k^{\mathscr{T}} U_\phi$ is a contractive projection that satisfies (*).*

Proof. The following useful equation will be derived first: (†) For each $\mathscr{T}$-measurable function $h \in \mathfrak{L}_1$, the equality $\int_T khd\mu = \int \chi_{S(k)} hd\mu$ holds for every $T \in \mathscr{T}$.

To see this observe that the special case

$$\int_T k\chi_S d\mu = \int_{S \cap T} kd\mu = \int_{S \cap T} \chi_{S(k)} d\mu = \int_T \chi_{S(k)} \chi_S d\mu$$

where $S \in \mathscr{T}$, follows from the definition of weight function. From this special case it further follows that (†) holds for each $\mathscr{T}$-simple function h. Now for h a nonnegative $\mathscr{T}$-measurable function in $\mathfrak{L}_1$, there exists an increasing sequence $\{h_n\}_{n=1}^\infty$ of nonnegative $\mathscr{T}$-simple functions converging pointwise to h. The sequence $\{kh_n\}_{n=1}^\infty$ is also an increasing sequence of $\mathfrak{L}_1$ functions that converges pointwise to the function kh, and this and the fact that

$$\int_X kh_n d\mu \leqq \int_X h_n d\mu \leqq \int_X h d\mu$$

implys that kh is in $\mathfrak{L}_1$ and that

$$\int_T khd\mu = \lim_{n\to\infty}\int_T kh_n d\mu = \lim_{n\to\infty}\int_T \chi_{S(k)}h_n d\mu = \int_T \chi_{S(k)}hd\mu$$

for every $T \in \mathscr{T}$. Thus (†) has been proved.

The case $\phi = 1$ is now treated. For every $f \in \mathfrak{L}_1$, the inequality

$$\begin{aligned}||E_k^{\mathscr{T}}f|| &= \int_K |E_k^{\mathscr{T}}f|\,d\mu = \int_X k\,|E^{\mathscr{T}}f|\,d\mu \leqq \int_X |E^{\mathscr{T}}f|\,d\mu \\ &= \sup\left\{\int_T E^{\mathscr{T}}fd\mu - \int_{T'}E^{\mathscr{T}}fd\mu \mid T \in \mathscr{T}\right\} \\ &= \sup\left\{\int_T fd\mu - \int_{T'}fd\mu \mid T\in\mathscr{T}\right\} \leqq \int_X |f|\,d\mu = ||f||\end{aligned}$$

is obtained, and thus $E_k^{\mathscr{T}}$ is a contraction. Further, for every $\mathscr{T}$-measurable function $h \in \mathfrak{L}_1$, the equality $E^{\mathscr{T}}(kh) = h\chi_{S(k)}$ is a restatement of (†). Therefore, $(E_k^{\mathscr{T}})^2 = kE^{\mathscr{T}}(kE^{\mathscr{T}}) = kE^{\mathscr{T}} = E_k^{\mathscr{T}}$ and so $E_k^{\mathscr{T}}$ is a projection. Lastly, it is clear that a necessary condition for $f \in \mathfrak{K}[E_k^{\mathscr{T}}]$ is that $fk = fE_k^{\mathscr{T}}1 = 0$, and so $E_k^{\mathscr{T}}f = kE^{\mathscr{T}}f = 0$ because $S(k) \in \mathscr{T}$ and $S(f) \cap S(k) = \phi$ imply $S(E^{\mathscr{T}}f) \cap S(k) = \phi$. Thus $E_k^{\mathscr{T}}\{\mathfrak{K}[E_k^{\mathscr{T}}]\} = (0)$ and $E_k^{\mathscr{T}}$ is a contractive projection that satisfies (*). (It is well known that $E^{\mathscr{T}}$ is a contractive projection (see e.g. [6, pp. 48–49] and [1, pp. 565–566])).

The operator $U_{\bar\phi}E_k^{\mathscr{T}}U_\phi$ is also a contractive projection:

$$(U_{\bar\phi}E_k^{\mathscr{T}}U_\phi)^2 = U_{\bar\phi}E_k^{\mathscr{T}}U_\phi U_{\bar\phi}E_k^{\mathscr{T}}U_\phi = U_{\bar\phi}E_k^{\mathscr{T}}E_k^{\mathscr{T}}U_\phi = U_{\bar\phi}E_k^{\mathscr{T}}U_\phi$$

and

$$||U_{\bar\phi}E_k^{\mathscr{T}}U_\phi|| \leqq ||U_{\bar\phi}||\,||E_k^{\mathscr{T}}||\,||U_\phi|| = 1\ .$$

Moreover, because $\mathfrak{N}[U_{\bar\phi}E_k^{\mathscr{T}}U_\phi] = U_{\bar\phi}\mathfrak{N}[E_k^{\mathscr{T}}]$, then $\mathfrak{K}[U_{\bar\phi}E_k^{\mathscr{T}}U_\phi] = \mathfrak{K}[E_k^{\mathscr{T}}]$. Therefore a necessary condition for $f \in \mathfrak{K}[U_{\bar\phi}E_k^{\mathscr{T}}U_\phi]$ is that $fk = 0$, and so $(U_{\bar\phi}E_k^{\mathscr{T}}U_\phi)\{\mathfrak{K}[U_{\bar\phi}E_k^{\mathscr{T}}U_\phi]\} = (0)$. Thus $U_{\bar\phi}E_k^{\mathscr{T}}U_\phi$ is a contractive projection that satisfies (*).

Next it is shown that every contractive projection can be written as the sum of a contractive projection having the same range space that satisfies (*) and of a contraction that is nilpotent of order two. This decomposition is canonical and reduces the study of contractive projections to the study of those that satisfy (*).

PROPOSITION 2. Let P be a contractive projection on $\mathfrak{L}_1$. There exists a unique subset $T_0 \in \mathscr{S}$ (called the support of P) such that $S(f) \subset T_0$ for every $f \in \mathfrak{R}[P]$ and for which $\mathfrak{K}[P] = \chi_{T_0'} \cdot \mathfrak{L}_1$. If A denotes the operator defined to be $Af = P(\chi_{T_0'} f)$ for every $f \in \mathfrak{L}_1$, then

(1) A is a contraction,

(2) $\mathfrak{R}[A] \subset \mathfrak{R}[P]$,

(3) $A^2 = 0$,

(4) $A\{\chi_{T_0} \mathfrak{L}_1\} = (0)$, and

(5) $P - A$ is a contractive projection with the same range as P that satisfies $(*)$.

Proof. Select a sequence of functions $\{f_n\}_{n=0}^{\infty}$ from the range of P as follows: set $f_0 = 0$ and assuming that the functions $\{f_n\}_{n=0}^{N-1}$ have been chosen, select f_N such that $\mu[S(f_N) - \bigcup_{n=0}^{N-1} S(f_n)] > 1/N$ if this is possible or set $f_N = 0$ if it is not. Set $T_0 = \bigcup_{n=0}^{\infty} S(f_n)$. It is clear that $T_0 \in \mathscr{S}$ and so it remains to prove that $S(f) \subset T_0$ for every $f \in \mathfrak{R}[P]$. For each $g \in \mathfrak{R}[P]$ either $S(g) \subset T_0$ or there is a positive integer M for which $\mu(S(g) - T_0) > 1/M$. The existence of such an M is, however, impossible because g would have been selected as f_N for some N. The alternative to this is that it was possible to select $f_N \neq 0$ for $N \geqq M$, which would imply that

$$\mu(T_0) \geqq \sum_{n=M}^{\infty} \mu\Big[S(f_n) - \bigcup_{i=0}^{n-1} S(f_i)\Big] \geqq \sum_{n=M}^{\infty} 1/n = \infty .$$

This is a contradiction and thus $S(f) \subset T_0$ for every $f \in \mathfrak{R}[P]$. (This construction will be used several times in this paper. For each subspace $\mathfrak{M}$ of $\mathfrak{L}_1$ the set T_0 obtained from this construction is called the *support* of $\mathfrak{M}$.)

If the operator A is defined as in the statement of the proposition, then the properties attributed to it follow readily.

The operator A is unique in the sense that it is the unique operator for which $P - A$ is a contractive projection with the same range as P that satisfies $(*)$. The proof of this depends on the fact that a contractive projection is determined by its range (see Corollary 3).

The structure of the closed vector sublattices of $\mathfrak{L}_1$ is determined in the following lemma. (Note that by vector sublattice is meant a selfadjoint subspace—that is, a subspace such that $\bar{f}$ is in it whenever f is—for which the subspace of real valued functions forms a sublattice.) This lemma is very important to the proofs that follow and will be used also in § 3. It is related to results that appear explicitly in Bahadur [1, pp. 565–566] and Brunk [2, Theorems 4 and 5, p. 302] and implicitly in Moy [6, pp. 51–58].

LEMMA 1. *Let $\mathfrak{M}$ be a closed subspace of $\mathfrak{L}_1$ that is also a vector* *sublattice. Then there exists a σ-subalgebra $\mathscr{T}$ of $\mathscr{S}$ and a weight function k for $\mathscr{T}$ for which $\mathfrak{M} = k \cdot \mathfrak{L}_1(X, \mathscr{T}, \mu_{\mathscr{T}})$. Moreover, the pair $(\mathscr{T}, k)$ is unique. (The measure $\mu_{\mathscr{T}}$ is the restriction of μ to $\mathscr{T}$.)*

Proof. There exists a nonnegative function $h \in \mathfrak{M}$ such that $S(f) \subset S(h)$ for every $f \in \mathfrak{M}$. To see this select a sequence of real valued functions $\{f_n\}_{n=0}^{\infty}$ from $\mathfrak{M}$ as in the proof of Proposition 2, discarding now the zero terms. The function $h = \sum_{n=0}^{\infty} (2^n \|f_n\|)^{-1} |f_n|$ is in $\mathfrak{M}$ and has the desired property.

If $\mathscr{T}_h = \{T \in \mathscr{S} \mid h\chi_T \in \mathfrak{M}\}$, then $\mathscr{T}_h$ is a σ-subalgebra of $\mathscr{S}$ because $\mathfrak{M}$ is a closed vector sublattice of $\mathfrak{L}_1$. Further, the following argument shows that there exists a function $k \in \mathfrak{M}$ such that

$$\int_T \chi_{S(h)} d\mu = \int_T k d\mu$$

for every $T \in \mathscr{T}_h$. The positive measures defined on $(X, \mathscr{T}_h)$ by the indefinite integrals $\int \chi_{S(h)} d\mu$ and $\int h d\mu$ are mutually absolutely continuous. Thus there exists a $\mathscr{T}_h$-measurable nonnegative function g by the Radon-Nikodym Theorem such that $\int_T \chi_{S(h)} d\mu = \int_T ghd\mu$ for every $T \in \mathscr{T}_h$. Moreover, g is the pointwise limit of an increasing sequence $\{g_n\}_{n=1}^{\infty}$ of $\mathscr{T}_h$-simple nonnegative functions, and because the summable function gh dominates each function $g_n h$, the sequence $\{g_n h\}_{n=1}^{\infty}$ must converge in norm to gh. Thus gh is in $\mathfrak{M}$ because each of the functions $g_n h$ is in $\mathfrak{M}$ and $\mathfrak{M}$ is closed. Therefore, the function $k = gh$ is a weight function for $\mathscr{T}_h$ which is in $\mathfrak{M}$.

Further, it can be shown readily that the σ-subalgebra $\mathscr{T}_k$ is the same as $\mathscr{T}_h$. To see this suppose that $T \in \mathscr{T}_k$. Then $k\chi_T \in \mathfrak{M}$ and so also is each of the functions $(h - nk\chi_T)^+$. Now because $S(h) = S(k)$ it follows that the sequence $\{(h - nk\chi_T)^+\}_{n=1}^{\infty}$ converges in norm to the function $h\chi_{T'}$ which must also be in $\mathfrak{M}$. Thus $T' \in \mathscr{T}_h$ and so $\mathscr{T}_k \subset \mathscr{T}_h$. The proof that $\mathscr{T}_h \subset \mathscr{T}_k$ proceeds similarly. If $\mathscr{T}_k$ is denoted now by just $\mathscr{T}$, then what remains is to prove that $\mathfrak{M} = k\mathfrak{L}_1(X, \mathscr{T}, \mu_{\mathscr{T}})$.

First, the inclusion $k\mathfrak{L}_1(X, \mathscr{T}, \mu_{\mathscr{T}}) \subset \mathfrak{M}$ is immediate. This follows because each nonnegative $f \in \mathfrak{L}_1(X, \mathscr{T}, \mu_{\mathscr{T}})$ is the pointwise limit of an increasing sequence $\{f_n\}_{n=1}^{\infty}$ of $\mathscr{T}$-simple nonnegative functions. Hence, because

$$\|kf_n\| = \int_X kf_n d\mu \leqq \int_X f_n d\mu = \|f_n\| \leqq \|f\| ,$$

the function kf is summable and thus kf_n converges in norm to kf. Therefore, $k\mathfrak{L}_1(X, \mathscr{T}, \mu_{\mathscr{T}}) \subset \mathfrak{M}$.

Assume that f is a nonnegative function in $\mathfrak{M}$. Because for each $\lambda > 0$ the sequence $\{h_n\}_{n=1}^{\infty}$, where $h_n = [n(f - \lambda k)^+] \wedge k$, is increasing and dominated by the summable function k, it converges in norm to a function $h \in \mathfrak{M}$. Moreover, $h = k\chi_{S[(f-\lambda k)^+]}$ and thus $S[(f-\lambda k)^+] \in \mathscr{T}$. It is now easy to see that f is the limit in norm of a sequence $\{kg_n\}_{n=1}^{\infty}$, where each g_n is a $\mathscr{T}$-simple function for which $S(g_n) \subset S(k)$. Thus, because $\|kg_n - kg_m\| = \|g_n - g_m\|$ for every n and m, the sequence $\{g_n\}_{n=1}^{\infty}$ converges in norm to a function $g \in \mathfrak{L}_1(X, \mathscr{T}, \mu_{\mathscr{T}})$ and $f = kg$. Therefore $\mathfrak{M} \subset k\mathfrak{L}_1(X, \mathscr{T}, \mu_{\mathscr{T}})$ and the proof is complete. The uniqueness of the pair $(\mathscr{T}, k)$ is proved easily.

The next lemma states that the range of a positive contractive projection is a closed vector sublattice of $\mathfrak{L}_1$. This result is isolated as a lemma because it will be used again in § 3.

LEMMA 2. *If P is a positive contractive projection, then $\mathfrak{R}[P]$ is a closed vector sublattice of $\mathfrak{L}_1$.*

Proof. Because P is a positive projection, the range of P is a closed self-adjoint subspace. Therefore, to prove that $\mathfrak{R}[P]$ is a closed vector sublattice of $\mathfrak{L}_1$ it is sufficient in light of the identity $h \vee g = 1/2\{h + g + |h-g|\}$ to prove that $f^+ \in \mathfrak{R}[P]$ for every real valued $f \in \mathfrak{R}[P]$. If f is a real valued function in $\mathfrak{R}[P]$, then because P is positive and $f^+ - f \geqq 0$, the inequality $P(f^+) \geqq Pf = f$ is obtained and thus also $P(f^+) \geqq f^+ \geqq 0$. This implies

$$0 \leqq \|P(f^+) - f^+\| = \|P(f^+)\| - \|f^+\| \leqq 0$$

or that $P(f^+) = f^+$. Therefore, $f^+ \in \mathfrak{R}[P]$ and $\mathfrak{R}[P]$ has been shown to be a closed vector sublattice of $\mathfrak{L}_1$.

PROPOSITION 3. An operator P is a positive contractive projection that satisfies $(*)$ if, and only if, there exists a σ-subalgebra $\mathscr{T}$ of $\mathscr{S}$ and a weight function k for $\mathscr{T}$ for which $P = E_k^{\mathscr{T}}$.

Proof. That a weighted conditional expectation is a positive contractive projection that satisfies $(*)$ follows from Proposition 1. Therefore, assume that P is a positive contractive projection. As a result of Lemmas 1 and 2 there exists a σ-subalgebra $\mathscr{T}$ of $\mathscr{S}$ and a weight function k for $\mathscr{T}$ for which $\mathfrak{R}[P] = k\mathfrak{L}_1(X, \mathscr{T}, \mu_{\mathscr{T}})$. Thus because $\mathfrak{R}[E_k^{\mathscr{T}}] = k\mathfrak{L}_1(X, \mathscr{T}, \mu_{\mathscr{T}})$ also and in view of equation (†), to prove that $P = E_k^{\mathscr{T}}$, it is sufficient to show that $\int_T Pf d\mu = \int_T E_k^{\mathscr{T}} f d\mu$ for every $f \in \mathfrak{L}_1(X, \mathscr{S}, \mu)$ and $T \in \mathscr{T}$. Moreover, because each of P and $E_k^{\mathscr{T}}$ is continuous and satisfies $(*)$ and $\mathfrak{K}[P] = \mathfrak{K}[E_k^{\mathscr{T}}] = \chi_{S(k)'}\mathfrak{L}_1$, to establish this it is sufficient to show $\int_T P(k\chi_S) d\mu = \int_T E_k^{\mathscr{T}}(k\chi_S) d\mu$ for every $S \in \mathscr{S}$ and $T \in \mathscr{T}$. To see this observe first that for every $f \in \mathfrak{L}_1$ and $T \in \mathscr{T}$, the following identities hold:

$$\int_T Pf d\mu = \int_T P(\chi_{S(k)}f) d\mu + \int_T P(\chi_{S(k)'}f) d\mu = \int_T P(\chi_{S(k)}f) d\mu$$

because $P(\chi_{S(k)'}f) \in P\{\mathfrak{N}[P]\} = (0)$, and $\int_T E_k^{\mathscr{T}} f d\mu = \int_T E_k^{\mathscr{T}}(\chi_{S(k)}f) d\mu$ because $E_k^{\mathscr{T}}(\chi_{S(k)'}f) = 0$. Now the sufficiency is obvious because the function $\chi_{S(k)}f$ can be approximated by linear combinations of functions $k\chi_{S(k)}$.

Assume that $S \in \mathscr{S}$ and $T \in \mathscr{T}$; then $k\chi_T = P(k\chi_T) \geqq P(k\chi_{S\cap T})$ and $k\chi_{T'} = P(k\chi_{T'}) \geqq P(k\chi_{S\cap T'})$ because P is positive. Further, the relations $k\chi_{S\cap T} + k\chi_{S\cap T'} = k\chi_S$ and $0 \leqq \int_T P(k\chi_{S\cap T'}) d\mu \leqq \int_T k\chi_{T'} d\mu = 0$ imply that

$$\begin{aligned}\int_T P(k\chi_S)\, d\mu &= \int_T P(k\chi_{S\cap T})\, d\mu + \int_T P(k\chi_{S\cap T'}) d\mu \\ &= \int_T P(k\chi_{S\cap T}) d\mu\ .\end{aligned}$$

Thus the inequality

$$\begin{aligned}\int_T k\chi_S d\mu &= \| k\chi_{S\cap T} \| \\ &\geqq \| P(k\chi_{S\cap T}) \| \geqq \int_T P(k\chi_{S\cap T}) d\mu = \int_T P(k\chi_S) d\mu\end{aligned}$$

or $\int_T k\chi_S d\mu \geqq \int_T P(k\chi_S) d\mu$ holds for every $T \in \mathscr{T}$ and $S \in \mathscr{S}$. Hence

$$\begin{aligned}\int_T k d\mu &= \int_T k\chi_S d\mu + \int_T k\chi_{S'} d\mu \\ &\geqq \int_T P(k\chi_S) d\mu + \int_T P(k\chi_{S'}) d\mu \geqq \int_T Pk d\mu = \int_T k d\mu\end{aligned}$$

and so $\int_T k\chi_S d\mu = \int_T P(k\chi_S) d\mu$ for every $T \in \mathscr{T}$ and $S \in \mathscr{S}$. Thus for each $S \in \mathscr{S}$ the relation $\int_T P(k\chi_S) d\mu = \int_T k\chi_S d\mu = \int_T E_k^{\mathscr{T}}(k\chi_S) d\mu$ is obtained for every $T \in \mathscr{T}$ from equation (†) and therefore by the previous remarks $P = E_k^{\mathscr{T}}$.

The characterization of contractive projections that satisfy (*) can now be proved.

PROPOSITION 4. Let P be a contractive projection that satisfies (*). Then there exists a σ-subalgebra $\mathscr{T}$ of $\mathscr{S}$, a weight function k for $\mathscr{T}$ and a measurable function ϕ of modulus one such that $P = U_{\bar{\phi}} E_k^{\mathscr{T}} U_\phi$. Moreover, the pair $(\mathscr{T}, k)$ is unique.

Proof. For every $f \in \mathfrak{L}_1$, define the function $\theta(f)(x) = f(x)/|f(x)|$ if $f(x) \neq 0$ or 1 if $f(x) = 0$. Then for every $f \in \mathfrak{R}[P]$ and nonnegative

$h \in \mathfrak{L}_1$ such that $|f| \geqq h$, the following inequalities hold:

$$\begin{aligned}\|f\| - \|\theta(f)h\| = \|f - \theta(f)h\| &\geqq \|Pf - P[\theta(f)h]\| \\ &\geqq \|f\| - \|P[\theta(f)h]\| \geqq \|f\| - \|\theta(f)h\| .\end{aligned}$$

Thus

$$\|f - P[\theta(f)h]\| = \|f\| - \|P[\theta(f)h]\| ,$$

which implies

$$\|\overline{\theta(f)}f - \overline{\theta(f)}P[\theta(f)h]\| = \|\overline{\theta(f)}f\| - \|\overline{\theta(f)}P[\theta(f)h]\| .$$

This implies that

$$S(P[\theta(f)h]) \doteq S(\overline{\theta(f)}P[\theta(f)h]) \subset S(\overline{\theta(f)}f) = S(f)$$

and also because $\overline{\theta(f)}f \geqq 0$, that $\overline{\theta(f)}P[\theta(f)h] \geqq 0$. Therefore, because every nonnegative function $h \in \mathfrak{L}_1$ such that $S(h) \subset S(f)$ can be approximated by nonnegative functions $h_n \in \mathfrak{L}_1$ such that $(1/n)h_n \leqq |f|$, it is readily seen that $U_{\overline{\theta(f)}}PU_{\theta(f)}h \geqq 0$ and $S(Ph) \subset S(f)$ for every nonnegative $h \in \mathfrak{L}_1$ such that $S(h) \subset S(f)$. Moreover, this implies further that for $h \in \mathfrak{L}_1$ and $S(h) \subset S(f)$ the inclusion $S(Ph) \subset S(f)$ follows.

It is proved next that $h\chi_{S(f)'} \in \mathfrak{R}[P]$ for every f and h in $\mathfrak{R}[P]$. The equation $h = Ph = P(h\chi_{S(f)}) + P(h\chi_{S(f)'})$ and the fact that the support of the function $P(h\chi_{S(f)})$ is contained in $S(f)$ implies that

$$h\chi_{S(f)'} = P(h\chi_{S(f)})\chi_{S(f)'} + P(h\chi_{S(f)'})\chi_{S(f)'} = P(h\chi_{S(f)'})\chi_{S(f)'} .$$

Thus $h\chi_{S(f)'} = P(h\chi_{S(f)'})$ because

$$\|h\chi_{S(f)'}\| = \|P(h\chi_{S(f)'})\chi_{S(f)'}\| \leqq \|P(h\chi_{S(f)'})\| \leqq \|h\chi_{S(f)'}\|$$

implies $P(h\chi_{S(f)'})\chi_{S(f)'} = P(h\chi_{S(f)'})$. Therefore $h\chi_{S(f)'} \in \mathfrak{R}[P]$.

This result and an argument as in Proposition 2 show that a sequence of functions $\{f_n\}_{n=1}^\infty$ can be chosen from $\mathfrak{R}[P]$ such that $\bigcup_{n=1}^\infty S(f_n) = T_0$ (the support of P) and such that the sets $S(f_n)$ are pairwise disjoint. Each operator $U_{\overline{\theta(f_n)}}PU_{\theta(f_n)}$ is positive on functions supported on $S(f_n)$. Therefore $U_\phi PU_{\overline{\phi}}$ is positive on $\mathfrak{L}_1$, where $\phi = \prod_{n=1}^\infty \overline{\theta(f_n)}$, because for each nonnegative $h \in \mathfrak{L}_1$ the equality

$$h = \sum_{n=1}^\infty h\chi_{S(f_n)} + h\chi_{T_0'}$$

shows that

$$\begin{aligned}(U_\phi PU_{\overline{\phi}})h &= \sum_{n=1}^\infty (U_\phi PU_{\overline{\phi}})(h\chi_{S(f_n)}) \\ &= \sum_{n=1}^\infty (U_{\overline{\theta(f_n)}}PU_{\theta(f_n)})(h\chi_{S(f_n)}) \geqq 0 .\end{aligned}$$

By Proposition 3 there exists a σ-subalgebra $\mathscr{T}$ of $\mathscr{S}$ and a weight function k for $\mathscr{T}$ such that $U_\phi P U_{\bar\phi} = E_k^{\mathscr{T}}$, and therefore, since $U_{\bar\phi} U_\phi = U_\phi U_{\bar\phi}$ is the identity operator, such that $P = U_{\bar\phi} E_k^{\mathscr{T}} U_\phi$.

The proof of the uniqueness of the pair $(\mathscr{T}, k)$ proceeds as follows. Suppose $P = U_{\bar\phi} E_k^{\mathscr{T}} U_\phi = U_{\bar\psi} E_h^{\mathscr{W}} U_\psi$, where $\mathscr{W}$ is a σ-subalgebra of $\mathscr{S}$, h is a weight function for $\mathscr{W}$ and ψ is a measurable function of modulus one. Then

$$\begin{aligned}\bar\phi k \mathfrak{L}_1(X, \mathscr{T}, \mu_{\mathscr{T}}) &= \mathfrak{R}[U_{\bar\phi} E_k^{\mathscr{T}} U_\phi] \\ &= \mathfrak{R}[U_{\bar\psi} E_h^{\mathscr{W}} U_\psi] = \bar\psi h \mathfrak{L}_1(X, \mathscr{W}, \mu_{\mathscr{W}})\end{aligned}$$

or $\bar\phi k \mathfrak{L}_1(X, \mathscr{T}, \mu_{\mathscr{T}}) = \bar\psi h \mathfrak{L}_1(X, \mathscr{W}, \mu_{\mathscr{W}})$. Now it is clear that the support of each function in $\bar\phi k \mathfrak{L}_1(X, \mathscr{T}, \mu_{\mathscr{T}})$ is in $\mathscr{W}$ and vice versa. Hence it follows from this observation and the definition of weight function that $\mathscr{T} = \mathscr{W}$. Further, there exists a $\mathscr{T}$-measurable function f in $\mathfrak{L}_1$ such that $S(f) = S(k)$ and $\bar\phi k = \bar\psi h f$, which implies that $k = |\bar\phi k| = |\bar\psi h f| = h\,|f|$. Now by (†)

$$\begin{aligned}\chi_{S(k)} &= E^{\mathscr{T}}(k) = E^{\mathscr{T}}(h\,|f|) \\ &= |f|\, E^{\mathscr{T}}(h) = |f|\, \chi_{S(h)} = |f|\, \chi_{S(k)}\end{aligned}$$

or $|f| = \chi_{S(k)}$ and $h = k$.

REMARK. The measurable function of modulus one is not unique. *It can be easily shown that necessary and sufficient for the contractive* projections $U_{\bar\phi} E_k^{\mathscr{T}} U_\phi$ and $U_{\bar\psi} E_k^{\mathscr{T}} U_\psi$ to be equal is that $\phi\bar\psi$ be $\mathscr{T}$-measurable.

The results of the preceding propositions are collected in Theorem 1.

THEOREM 1. *Let P be a contractive projection on $\mathfrak{L}_1$. There exists a unique $T_0 \in \mathscr{S}$ such that if the operators Q and A are defined to be $Qf = P(\chi_{T_0} f)$ and $Af = P(\chi_{T_0'} f)$ for every $f \in \mathfrak{L}_1$, then*

(1) *Q is a contractive projection having the same range as P that satisfies $(*)$; and*

(2) *A is a contraction that is nilpotent of order two for which $\mathfrak{R}[A] \subset \mathfrak{R}[P]$ and $A\{\chi_{T_0} \mathfrak{L}_1\} = (0)$.*

Further, there exists a σ-subalgebra $\mathscr{T}$ of $\mathscr{S}$, a weight function k for $\mathscr{T}$ such that $T_0 = S(k)$ and a measurable function ϕ of modulus one for which $Q = U_{\bar\phi} E_k^{\mathscr{T}} U_\phi$, and the pair $(\mathscr{T}, k)$ is unique. Moreover, an operator $P = U_{\bar\phi} E_k^{\mathscr{T}} U_\phi + A$ is a contractive projection.

COROLLARY 1. *An operator Q on $\mathfrak{L}_1$ is a conditional expectation ($Q = E^{\mathscr{T}}$ for some σ-subalgebra $\mathscr{T}$ of $\mathscr{S}$) if, and only if,*

(1) $Q^2 = Q$,

(2) $Q1 = 1$, *and*

(3) $\|Q\| = 1$.

Proof. From Proposition 1 it follows that a conditional expection satisfies (1), and (2) and (3) which are well known properties of a conditional expectation. Assume that Q is an operator that satisfies (1), (2), and (3). Conditions (2) and (3) imply that Q is positive as follows. For every function $h \in \mathfrak{L}_1$ such that $1 \geqq h \geqq 0$, the inequality

$$\begin{aligned} \|1\| - \|h\| &= \|1 - h\| \\ &\geqq \|Q1 - Qh\| \geqq \|1\| - \|Qh\| \geqq \|1\| - \|h\| \end{aligned}$$

holds and implies that $\|1\| - \|Qh\| = \|1 - Qh\|$ or that $1 \geqq Qh \geqq 0$. Thus Q is positive. Further, $Q1 = 1$ implies that the support of Q is X and that Q satisfies $(*)$. Therefore it follows from Proposition 3 that there exists a σ-subalgebra $\mathscr{T}$ of $\mathscr{S}$ and a weight function k for $\mathscr{T}$ such that $Q = E_k^{\mathscr{T}}$. But $1 = Q1 = E_k^{\mathscr{T}}1 = kE^{\mathscr{T}}1 = k$ and hence $Q = E^{\mathscr{T}}$. Therefore Q is a conditional expectation.

REMARK. This is actually a corollary to Proposition 3. It is related to results of Moy [6, Theorem 2.2, p. 61], Bahadur [1, Corollary 2, p. 566], and Rota [7, Theorem 1, p. 58], Sidak [8].

COROLLARY 2. *A contractive projection that satisfies $(*)$ is positive, if and only if, its range is a closed vector sublattice of $\mathfrak{L}_1$.*

Proof. If P is a positive contractive projection, then $\mathfrak{R}[P]$ is a closed vector sublattice by Lemma 2. Assume that $P = U_{\bar{\phi}} E_k^{\mathscr{T}} U_\phi$ and $\mathfrak{R}[P]$ is a closed vector sublattice. Then $P(\bar{\phi}k) = \bar{\phi}k$ and because $k = \sup_{0 \leqq \theta \leqq 2\pi} 1/2\{e^{i\theta}\bar{\phi}k + e^{-i\theta}\phi k\}$ and each of the functions $e^{i\theta}\phi k + e^{-i\theta}\phi k$ is in the vector sublattice $\mathfrak{R}[P]$, it follows that $k \in \mathfrak{R}[P]$ and thus $Pk = k$. The same argument used in the preceding corollary can now be used with k in place of 1 to show that P is positive.

COROLLARY 3. *A contractive projection that satisfies $(*)$ is determined by its range.*

Proof. Suppose $P = U_{\bar{\phi}} E_k^{\mathscr{T}} U_\phi$ and $Q = U_{\bar{\psi}} E_h^{\mathscr{W}} U_\psi$ are contractive projections satisfying $(*)$ such that $\mathfrak{R}[P] = \mathfrak{R}[Q]$. Then

$$\bar{\phi}k\mathfrak{L}_1(X, \mathscr{T}, \mu_{\mathscr{T}}) = \bar{\psi}h\mathfrak{L}_1(X, \mathscr{W}, \mu_{\mathscr{W}})$$

and as in the proof of Proposition 4 it follows that $\mathscr{T} = \mathscr{W}$ and $k = h$. Thus $\bar{\phi}\psi$ is a $\mathscr{T}$-measurable function. Now the proof given for (†) also shows that $\bar{\phi}\psi$ a bounded $\mathscr{T}$-measurable function and f an $\mathfrak{L}_1$ function implies $E^{\mathscr{T}}(\bar{\phi}\psi f) = \bar{\phi}\psi E^{\mathscr{T}}(f)$. Therefore, for f in $\mathfrak{L}_1$

it follows that $(U_{\psi\bar{\phi}}E_k^{\mathscr{T}}U_{\bar{\psi}\phi})(f) = \psi\bar{\phi}E_k^{\mathscr{T}}(\bar{\psi}\phi f) = \bar{\psi}\bar{\phi}\psi\phi E_k^{\mathscr{T}}(f) = E_k^{\mathscr{T}}(f)$ and so $(U_{\bar{\phi}}E_k^{\mathscr{T}}U_{\phi}) = U_{\bar{\psi}}[U_{\psi\bar{\phi}}E_k^{\mathscr{T}}U_{\bar{\psi}\phi}]U_{\psi} = U_{\bar{\psi}}E_k^{\mathscr{T}}U_{\psi}$.

There is a certain subclass of contractive projections which have a very simple structure. A different characterization of these is contained in the following proposition.

PROPOSITION 5. *Let P be a positive projection $\mathfrak{L}_1$ for which $I - P$ is also positive. Then there exists a subset $T_0 \in \mathscr{S}$ such that $Pf = \chi_{T_0}f$ for every $f \in \mathfrak{L}_1$.*

Proof. Set $k = P1$; then each of k and $1 - k$ is a nonnegative function. Further set $h = k \wedge (1 - k)$; then

$$0 = (I - P)k = (I - P)(k - h) + (I - P)h ,$$

where each of the functions $(I - P)(k - h)$ and $(I - P)h$ is nonnegative. Thus $(I - P)h = 0$ and similarly $Ph = 0$. Therefore $k \wedge (1 - k) = h = Ph + (I - P)h = 0$ and thus $S(k) \cap S(1 - k) = \phi$.

If f is assumed to be any nonnegative function in $\mathfrak{L}_1$, then $nk \geqq P(f \wedge n1)$ for every integer n because $n1 \geqq f \wedge n1$. Thus $Pf = \chi_{S(k)}f$, because $S(Pf) \subset S(k)$, $S[(I - P)f] \subset S(1 - k)$, and $Pf + (I - P)f = f$. Therefore $Pf = \chi_{S(k)}f$ for every $f \in \mathfrak{L}_1$.

REMARK. This proposition is also valid for positive projections on $\mathfrak{L}_p(X, \mathscr{S}, \mu)$, for $1 \leqq p < \infty$.

The following further characterization of this class of projections on $\mathfrak{L}_p(X, \mathscr{S}, \mu)$ for $p \neq 2$ can be obtained as a corollary to Proposition 5 using a result due to Lamperti [5, Corollary 2.1, p. 460]: Let P be a projection on $\mathfrak{L}_p$ such that $\|f\|_p^p = \|Pf\|_p^p + \|(I - P)f\|_p^p$ for every $f \in \mathfrak{L}_p$. Then there exists a subset $T_0 \in \mathscr{S}$ such that $Pf = \chi_{T_0}f$ for every $f \in \mathfrak{L}_p$.

3. The range of a contractive projection.

In this section the problem of what subspaces of $\mathfrak{L}_1$ are the range of a contractive projection is raised and solved. Two characterizations are given; for each, the problem of determining the subspaces that are the range of a positive contractive projection is first considered, and then the more general case is reduced to it. An obvious necessary condition is that the subspace be closed.

PROPOSITION 6. *A subspace is the range of a positive contractive projection if, and only if, it is a closed vector sublattice of $\mathfrak{L}_1$.*

Proof. That the range of a positive contractive projection is a closed sublattice of $\mathfrak{L}_1$ is a result of Lemma 2.

If the subspace $\mathfrak{M}$ if assumed to be a closed vector sublattice of $\mathfrak{L}_1$, then by Lemma 1 there exists a σ-subalgebra $\mathscr{T}$ of $\mathscr{S}$ and a weight function k for $\mathscr{T}$ for which $\mathfrak{M} = k\mathfrak{L}_1(X, \mathscr{T}, \mu_{\mathscr{T}})$. Thus $\mathfrak{M} = E_k^{\mathscr{T}}(\mathfrak{L}_1(X, \mathscr{S}, \mu))$ and the proposition is proved.

PROPOSITION 7. A subspace $\mathfrak{M}$ is the range of a contractive projection if, and only if, there exists a measurable function ϕ of modulus one for which $U_\phi\mathfrak{M}$ is a closed vector sublattice of $\mathfrak{L}_1$.

Proof. If P is a contractive projection and $\mathfrak{M} = P(\mathfrak{L}_1)$, then $\mathfrak{M} = (U_{\bar{\phi}}QU_\phi)(\mathfrak{L}_1)$ for some measurable function ϕ of modulus one and positive contractive projection Q; thus $U_\phi\mathfrak{M} = Q(\mathfrak{L}_1)$ is a closed vector sublattice of $\mathfrak{L}_1$ by Proposition 6.

If for a subspace $\mathfrak{M}$ and measurable function ϕ of modulus one, the subspace $U_\phi\mathfrak{M}$ is a closed vector sublattice of $\mathfrak{L}_1$, then by Proposition 6, there is a contractive projection P for which $P(\mathfrak{L}_1) = U_\phi\mathfrak{M}$. But then $\mathfrak{M}$ is the range of the contractive projection $U_{\bar{\phi}}PU_\phi$.

PROPOSITION 8. A subspace is the range of a positive contractive projection if, and only if, there exists a positive isometrical isomorphism from some $\mathfrak{L}_1$ space onto it.

Proof. If the subspace $\mathfrak{M}$ is the range of a positive contractive projection, then $\mathfrak{M} = k\mathfrak{L}_1(X, \mathscr{T}, \mu_{\mathscr{T}})$ for some σ-subalgebra $\mathscr{T}$ of $\mathscr{S}$ and weight function k for $\mathscr{T}$. Further, the mapping from $\mathfrak{L}_1(X, \mathscr{T}, \mu_{\mathscr{T},k})$ to $\mathfrak{L}_1(X, \mathscr{S}, \mu)$ that takes a function h to kh is in view of equation (†) a positive isometrical isomorphism onto $\mathfrak{M}$, where $\mu_{\mathscr{T},k}$ is the measure defined on $\mathscr{T}$ by the indefinite integral $\int \chi_{S(k)} d\mu_{\mathscr{T}}$.

Assume that $(Y, \mathscr{U}, \nu)$ is a measure space and that Φ is a positive isometrical isomorphism from $\mathfrak{L}_1(Y, \mathscr{U}, \nu)$ onto the subspace $\mathfrak{M}$. The proof of the proposition is completed by showing that $\mathfrak{M}$ is self-adjoint and $f^+ \in \mathfrak{M}$ for every real valued $f \in \mathfrak{M}$. Since $\mathfrak{L}_1(Y, \mathscr{U}, \nu)$ is self-adjoint and Φ takes real valued functions onto real valued functions, $\mathfrak{M}$ is self-adjoint. Further, if the function $f \in \mathfrak{M}$ and is real valued, then $f = \Phi\{[\Phi^{-1}f]^+\} - \Phi\{[\Phi^{-1}f]\}$, where each of $\Phi\{[\Phi^{-1}f]^+\}$ and $\Phi\{[\Phi^{-1}f]^-\}$ is nonnegative. Moreover, the equality

$$\begin{aligned} \|f\| = \|\Phi^{-1}f\| &= \|[\Phi^{-1}f]^+\| + \|[\Phi^{-1}f]^-\| \\ &= \|\Phi\{[\Phi^{-1}f]^+\}\| + \|\Phi\{[\Phi^{-1}f]^-\}\| \end{aligned}$$

implies that $f^+ = \Phi\{[\Phi^{-1}f]^+\}$ and thus $f^+ \in \mathfrak{M}$.

PROPOSITION 9. A subspace is the range of a contractive projection if, and only if, it is isometrically isomorphic to some $\mathfrak{L}_1$ space.

Proof. It is clear that a subspace $\mathfrak{M}$ is isometrically isomorphic to $U_\phi\mathfrak{M}$ for each measurable function ϕ of modulus one. Therefore, it follows from Propositions 7 and 8 that the range of a contractive projection is isometrically isomorphic to an $\mathfrak{L}_1$ space.

Assume that $(Y, \mathscr{U}, \nu)$ is a finite measure space and Φ is an isometrical isomorphism from $\mathfrak{L}_1(Y, \mathscr{U}, \nu)$ onto the subspace $\mathfrak{M}$. Set $h = \Phi 1$ and let Ψ denote the composite map $U_{\overline{\theta(h)}}\Phi$ from $\mathfrak{L}_1(Y, \mathscr{U}, \nu)$ onto $U_{\overline{\theta(h)}}\mathfrak{M}$. (Recall that $\theta(h) = h/|h|$.) Then for every nonnegative $f \in \mathfrak{L}_1(Y, \mathscr{U}, \nu)$, the equality $\|f + 1\| = \|f\| + \|1\|$ implies that

$$\|\Psi f + \Psi 1\| = \|\Psi f\| + \|\Psi 1\| .$$

Thus because $\Psi 1 = \overline{\theta(h)}h$ is nonnegative and $S(\Psi f) \subset S(h)$, the function Ψf is nonnegative and Ψ is a positive isometrical isomorphism from $\mathfrak{L}_1(Y, \mathscr{U}, \nu)$ onto $U_{\overline{\theta(h)}}\mathfrak{M}$. Therefore it follows from Propositions 7 and 8 that $\mathfrak{M}$ is the range of a contractive projection.

Assume that $(Y, \mathscr{U}, \nu)$ is a measure space and Φ is an isometrical isomorphism from $\mathfrak{L}_1(Y, \mathscr{U}, \nu)$ onto $\mathfrak{M}$; then ν is totally σ-finite. If ν were not totally σ-finite, then there would exist an uncountable set of nonzero functions $\{f_\alpha\}_{\alpha \in A}$ in $\mathfrak{L}_1(Y, \mathscr{U}, \nu)$ such that

$$\|f_\alpha + f_\beta\| = \|f_\alpha - f_\beta\| = \|f_\alpha\| + \|f_\beta\|$$

for every distinct pair of α and β in A. But then

$$\begin{aligned}\|\Phi(f_\alpha) + \Phi(f_\beta)\| &= \|\Phi(f_\alpha) - \Phi(f_\beta)\| \\ &= \|\Phi(f_\alpha)\| + \|\Phi(f_\beta)\|\end{aligned}$$

which implies $S(\Phi(f_\alpha)) \cap S(\Phi(f_\beta)) = \phi$ for every distinct pair α and β in A, which is impossible since $\mathfrak{M}$ is a subspace of $\mathfrak{L}_1(X, \mathscr{S}, \mu)$ and μ is finite. Thus, ν is totally σ-finite and there exists a finite measure ξ on $(Y, \mathscr{U})$ such that $\mathfrak{L}_1(Y, \mathscr{U}, \nu)$ is isometrically isomorphic to $\mathfrak{L}_1(Y, \mathscr{U}, \xi)$. Therefore there is an isometrical isomorphism from $\mathfrak{L}_1(Y, \mathscr{U}, \xi)$ onto $\mathfrak{M}$ and the proposition is proved.

These results are summarized in the following two theorems.

THEOREM 2. *For each subspace $\mathfrak{M}$ of $\mathfrak{L}_1$ the following statements are equivalent*:

(1) *$\mathfrak{M}$ is the range of a positive contractive projection,*

(2) *$\mathfrak{M}$ is a closed vector sublattice of $\mathfrak{L}_1$, and*

(3) *there exists a positive isometrical isomorphism from some $\mathfrak{L}_1$ space onto $\mathfrak{M}$.*

THEOREM 3. *For each subspace $\mathfrak{M}$ of $\mathfrak{L}_1$ the following statements are equivalent*:

(1) $\mathfrak{M}$ *is the range of a contractive projection,*

(2) $U_\phi\mathfrak{M}$ *is a closed vector sublattice for some measurable function* ϕ *of modulus one, and*

(3) $\mathfrak{M}$ *is isometrically isomorphic to some* $\mathfrak{L}_1$ *space.*

4. Extension to general measure spaces. The results about contractive projections on an $\mathfrak{L}_1$ space defined for a probability measure can be extended quite satisfactorily to the case of a totally σ-finite measure. With the use of an isometrical isomorphism, information about contractive projections for the totally σ-finite case can be obtained from the results in §§ 2 and 3.

In certain special cases a more direct attack is appropriate. Let $(X, \mathscr{S}, \mu)$ be a totally σ-finite measure space; then there exists a probability measure ν defined on $\mathscr{S}$ that is equivalent to μ (that is, μ and ν are mutually absolutely continuous). Let ψ denote the Radon-Nikodym derivative $d\mu/d\nu$, that is, ψ is the unique positive measurable function for which the equality $\int_S \psi d\nu = \int_S d\mu$ holds for every $S \in \mathscr{S}$. Then the map Ψ that takes a function $f \in \mathfrak{L}_1(X, \mathscr{S}, \mu)$ to the function $f\psi$ is a positive isometrical isomorphism from $\mathfrak{L}_1(X, \mathscr{S}, \mu)$ onto $\mathfrak{L}_1(X, \mathscr{S}, \nu)$.

If P is a contractive projection on $\mathfrak{L}_1(X, \mathscr{S}, \mu)$, then $\Psi P \Psi^{-1}$ is a contractive projection on $\mathfrak{L}_1(X, \mathscr{S}, \nu)$ and thus is susceptible to the analysis of §§ 2 and 3. An attempt at using this analysis on P directly will not succeed in general, because it is impossible to define the conditional expectation for certain σ-subalgebras of $\mathscr{S}$ in the usual manner. The hypothesis needed on a σ-subalgebra in order that the conditional expectation can be defined is that the restriction of μ to the σ-algebra be locally σ-finite. If it can somehow be ascertained that the σ-subalgebra involved in the study of a givnn contractive projection P satisfies this condition then the characterization $P = U_{\bar\phi} E_k U_\phi + A$ is valid. Further, in any case the conclusions of Theorems 2 and 3 remain valid and Lemma 1 can be stated as follows: a subspace $\mathfrak{M}$ of $\mathfrak{L}_1(X, \mathscr{S}, \mu)$ is a closed vector sublattice if, and only if, there exists a σ-subalgebra $\mathscr{T}$ of $\mathscr{S}$ and a weight function k for $\mathscr{T}$ *with respect to* ν such that

$$\mathfrak{M} = \frac{d\nu}{d\mu} k \mathfrak{L}_1(X, \mathscr{T}, \nu_{\mathscr{T}}) .$$

If $(X, \mathscr{S}, \mu)$ is just assumed to be a measure space, then information about contractive projections on $\mathfrak{L}_1(X, \mathscr{S}, \mu)$ can be obtained using the results of §§ 2 and 3 and the preceding remarks, but the statement of precise theorems is difficult. The following remarks give an indication as to how results about the totally σ-finite case can be applied to a more general measure space.

If P is a contractive projection on $\mathfrak{L}_1(X, \mathscr{S}, \mu)$ and $f \in \mathfrak{R}[P]$, then μ restricted to $S(f)$ is totally σ-finite. Moreover, if P satisfies (*) the subspace $\mathfrak{M} = \chi_{S(f)}\mathfrak{L}_1(X, \mathscr{S}, \mu)$ is a reducing subspace of P, that is, $P\mathfrak{M} \subset \mathfrak{M}$ and $P[\mathfrak{L}_1 \ominus \mathfrak{M}] \subset [\mathfrak{L}_1 \ominus \mathfrak{M}]$. Further, it is still true (although the proof is technically more difficult) that an arbitrary contractive projection can be decomposed into a regular part and an arbitrary part. Therefore, in a sense, the study of a contractive projection on $\mathfrak{L}_1(X, \mathscr{S}, \mu)$ can be reduced to the study of contractive projections on totally σ-finite $\mathfrak{L}_1$ spaces. Difficulties arise in trying to assemble the information obtained to give global information. The problem of "sets of measure zero" becomes acute and only in certain cases can it be resolved satisfactorily.

REMARK. There have been attempts at generalizing the concept of conditional expectation to the contexts just considered (see e.g., [2, Theorem 2, pp. 300–301]). This is achieved by choosing certain of the properties of conditional expectation and then showing the existence of an operator having these properties. These results appear, however, to be relevant to the study of contractive projections only in those instances where the preceding procedure already yields satisfactory results.

The preceding procedure is actually applicable in a more general context. Let $(X, \mathscr{S})$ be a measurable space and $\boldsymbol{M}(X, \mathscr{S})$ denote the Banach space of finite complex valued measures defined on $\mathscr{S}$, in which the norm of a measure is defined to be the total variation. Let P be a contractive projection defined on $\boldsymbol{M}(X, \mathscr{S})$, and ν be a measure in $\boldsymbol{M}(X, \mathscr{S})$ for which $P\nu = \nu$. Assume that $\mathfrak{L}_1(X, \mathscr{S}, |\nu|)$ has been identified as a subspace of $\boldsymbol{M}(X, \mathscr{S})$ in the obvious way. Then the containment $P\{\mathfrak{L}_1(X, \mathscr{S}, |\nu|)\} \subset \mathfrak{L}_1(X, \mathscr{S}, |\nu|)$ can be obtained from the results of §§ 2 and 3. Moreover, $\boldsymbol{M}(X, \mathscr{S})$ can be written as the direct sum of normal subspaces (a subspace $\boldsymbol{N}$ of $\boldsymbol{M}(X, \mathscr{S})$ is said to be normal if it is a closed vector sublattice having the further property that $\eta \in \boldsymbol{N}$ and $\xi \in \boldsymbol{M}(X, \mathscr{S})$ such that $|\eta| \geqq |\xi|$ implies $\xi \in \boldsymbol{N}$) such that all of the subspaces but one are reducing and can be identified as an $\mathfrak{L}_1$ space and the remaining subspace is the "complement of the support of P". Further, if $\mathfrak{K}[P]$ is now defined to be the subspace of $\boldsymbol{M}(X, \mathscr{S})$ consisting of measures that are singular relative to each measure in $\mathfrak{R}[P]$, then $\mathfrak{K}[P]$ is the "complement of the support of P". It is clear, however, that the possibility of being able to organize global results in this context is extremely remote.

5. **Contractions on $\mathfrak{L}_1(X, \mathscr{S}, \mu)$.** Certain arguments used in the proofs of §§ 2 and 3 are applicable in the more general context of a

contraction operator defined on an $\mathfrak{L}_1$ space. The results obtained with the applicable argument are now stated.

PROPOSITION 10. If $(X, \mathscr{S}, \mu)$ is a finite measure space and A is a contraction on $\mathfrak{L}_1(X, \mathscr{S}, \mu)$ for which $A1 = 1$, then A is positive, $A\{\mathfrak{L}_\infty(X, \mathscr{S}, \mu)\} \subset \mathfrak{L}_\infty(X, \mathscr{S}, \mu)$ and $\| A \|_\infty = 1$.

Proof. That A is positive follows from the proof of Corollary 1 and the rest is now obvious.

PROPOSITION 11. If $(X, \mathscr{S}, \mu)$ is a totally σ-finite measure space, A is a contraction on $\mathfrak{L}_1(X, \mathscr{S}, \mu)$ and T_0 is the support of the subspace $\{f \mid Af = f\}$, then $A\{\chi_{T_0} \cdot \mathfrak{L}_1(X, \mathscr{S}, \mu)\} \subset \chi_{T_0} \cdot \mathfrak{L}_1(X, \mathscr{S}, \mu)$ and there exists a measurable function ϕ of modulus one such that $U_{\bar{\phi}} A U_\phi$ is positive on $\chi_{T_0} \cdot \mathfrak{L}_1(X, \mathscr{S}, \mu)$.

Proof. This result follows from the first three paragraphs of the proof of Proposition 4 in which only the fact that $Pf = f$ is used.

PROPOSITION 12. If $(X, \mathscr{S}, \mu)$ is a measure space, A is a contraction on $\mathfrak{L}_1(X, \mathscr{S}, \mu)$, and f is a function for which $\| Af \| = \| f \| \neq 0$, then the operator $U_{\overline{\theta(Af)}} A U_{\theta(f)}$ is positive on $\chi_{S(f)} \mathfrak{L}_1(X, \mathscr{S}, \mu)$.

Proof. If $\| Af \| = \| f \|$, then for each nonnegative function $h \in \mathfrak{L}_1(X, \mathscr{S}, \mu)$ for which $| f | \geqq h$, the inequality

$$\begin{aligned} \| f \| - \| \theta(f)h \| = \| f - \theta(f)h \| &\geqq \| Af - A(\theta(f)h) \| \\ &\geqq \| Af \| - \| A(\theta(f)h) \| \geqq \| f \| - \| \theta(f)h \| \end{aligned}$$

implies that

$$\| Af - A(\theta(f)h) \| = \| Af \| - \| A(\theta(f)h) \|$$

or that $U_{\overline{\theta(Af)}} A U_{\theta(f)} h$ is positive. The proposition now follows.

PROPOSITION 13. If $(X, \mathscr{S}, \mu)$ is a measure space and A is a positive contraction on $\mathfrak{L}_1(X, \mathscr{S}, \mu)[\boldsymbol{M}(X, \mathscr{S})]$, then the subspace of fixed points $\{f \mid Af = f\}[\{\nu \mid A\nu = \nu\}]$ is a closed vector sublattice.

Proof. This result follows using the proof of Lemma 2.

Using techniques developed in this paper it is possible to prove the following Mean Ergodic Theorem for contractions on $\mathfrak{L}_1(X, \mathscr{S}, \mu)$. It is a generalization of a theorem of Kakutani [4, Theorem 9, p. 534] and like it, is proved using the Yoshida-Kakutani Mean Ergodic Theorem.

PROPOSITION 14. Let $(X, \mathscr{S}, \mu)$ be a measure space, A be a contraction on $\mathfrak{L}_1(X, \mathscr{S}, \mu)$, A_n denote the operator $(1/n)\sum_{j=0}^{n-1} A^j$, and $\mathfrak{M}$ denote the normal subspace of $\mathfrak{L}_1(X, \mathscr{S}, \mu)$ generated by the subspace $\{f \mid Af = f\}$. Then for every $h \in \mathfrak{M}$, the sequence $\{A_n h\}_{n=1}^{\infty}$ converges in norm to Pf, where P is the unique contractive projection defined on $\mathfrak{M}$ having range $\{f \mid Af = f\}$.

Proof. If f and h are functions in $\mathfrak{L}_1(X, \mathscr{S}, \mu)$ for which $Af = f$ and $|f| \geqq |h|$, then the sublemma that appears in the first paragraph of the proof of Proposition 4, shows that $|f| \geqq |Ah|$, and so $|f| \geqq |A_n h|$ for $n = 1, 2, 3, \cdots$ Thus, it follows from the Yoshida-Kakutani Mean Ergodic Theorem [10, Theorem 1, p. 192] that the sequence $\{A_n f\}_{n=1}^{\infty}$ is a norm Cauchy sequence. Further, if $\{A_n f_N\}_{n=1}^{\infty}$ is a norm Cauchy sequence for each function f_N, $N = 1, 2, 3$, where f_N converges in norm to a function f, then $\{A_n f\}_{n=1}^{\infty}$ is also a norm Cauchy sequence. Therefore, for every h in the normal subspace $\mathfrak{M}$ generated by the subspace $\{f \mid Af = f\}$, the sequence $\{A_n h\}_{n=1}^{\infty}$ is a norm Cauchy sequence. Moreover, it is obvious that the map that takes a function $h \in \mathfrak{M}$ to the limit function is the contractive projection defined on $\mathfrak{M}$ that has range $\{f \mid Af = f\}$.

This proposition is also valid for contractions defined on $\boldsymbol{M}(X, \mathscr{S})$; that is, for a contraction A, the sequence $\{(1/n)\sum_{j=0}^{n-1} A^j \nu\}_{n=1}^{\infty}$ is a norm Cauchy sequence for every ν in the normal subspace of $\boldsymbol{M}(X, \mathscr{S})$ spanned by the subspace $\{\xi \mid A\xi = \xi\}$.

6. Concluding remarks. Although only complex $\mathfrak{L}_1$ spaces have been considered in this paper, the results are also valid for real $\mathfrak{L}_1$ spaces.

With the aid of a lemma stating that a contractive projection on a Hilbert space is Hermitean and with the Spectral Theorem for Hermitean operators it can be shown that the "universal model" of a contractive projection on a Hilbert space is multiplication by a characteristic function. Similarly, it is an easy exercise to prove using Theorem 1 that the "universal model" of a contractive projection on $\mathfrak{L}_1$ that satisfies $(*)$ is a conditional expectation followed with multiplication by a characteristic function.

The class of contractive projections on an $\mathfrak{L}_1$ space is related to certain special projections on $\mathfrak{L}_\infty$. Let $(X, \mathscr{S}, \mu)$ be a probability space. If P is a contractive projection on $\mathfrak{L}_1(X, \mathscr{S}, \mu)$, then the adjoint operator P^* is a contractive projection on $\mathfrak{L}_\infty(X, \mathscr{S}, \mu)$. (Recall that P^* is the unique operator on $\mathfrak{L}_\infty(X, \mathscr{S}, \mu) = \mathfrak{L}_1^*(X, \mathscr{S}, \mu)$ that satisfies the relation $\int_X (Ph) g \, d\mu = \int_X h(P^* g) d\mu$ for every $h \in \mathfrak{L}_1(X, \mathscr{S}, \mu)$ and $g \in \mathfrak{L}_\infty(X, \mathscr{S}, \mu)$.) If $P = U_{\bar{\Phi}} E_k^{\mathscr{T}} U_\Phi$, then because the conditional expectation is self adjoint on $\mathfrak{L}_2(X, \mathscr{S}, \mu)$ (see e.g. [1, Corollary 2, p. 566]), it follows that $P^*g = U_{\bar{\phi}}E^{\mathscr{T}}(k\phi g)$ for every $g \in \mathfrak{L}_\infty(X, \mathscr{S}, \mu)$. If P is positive, then $P^*g = E^{\mathscr{T}}(kg)$ is an averaging operator on $\mathfrak{L}_\infty(X, \mathscr{S}, \mu)$, that is, $P^*f \cdot P^*g = P^*(f \cdot P^*g)$ for every f and g in $\mathfrak{L}_\infty(X, \mathscr{S}, \mu)$. Further, if $E^{\mathscr{T}}(k) = 1$, then P^* is a conditional mean in the sense of Wright [9, p. 199]. Although all conditional means are not obtained in this way, in general, this is an interesting subclass of conditional means.

Added in proof. The proof of the following corollary to Theorem 1 is immediate:

COROLLARY 4. *Let P be a contractive projection on $\mathscr{L}_1$. Then $P\mathscr{L}_\infty \subset \mathscr{L}_\infty$ and $\|P\|_\infty = 1$ if and only if P satisfies* (*) *and there exists a σ-subalgebra $\mathscr{T}$ of $\mathscr{S}$, a $T_0 \in \mathscr{T}$, and a measurable function ϕ of modulus one for which $P = U_{\bar{\phi}}\chi_{T_0}E^{\mathscr{T}}U_\phi$.*

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