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A NONNORMAL BLASCHKE-QUOTIENT

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We shall call the meromorphic functions of the form $F(z) = B_1(z)/B_2(z)$ Blaschke-quotients, where $B_1(z)$ and $B_2(z)$ are Blaschke products in $|z| < 1$ with zeros at $\{a_n\}$ and $\{b_k\}$ respectively. Although there is a characterization of meromorphic functions which are normal there is no characterization of the Blaschke-quotients which are normal in terms of the non-Euclidean (hyperbolic) distances between the zeros $\{a_n\}$ and $\{b_k\}$. In this paper we show by construction that even if the zeros of a Blaschke-quotient are separated by a positive non-Euclidean distance the Blaschke-quotient need not be normal.

We shall be concerned in this paper with the boundary behavior of meromorphic functions. The concept of a normal function was introduced by K. Noshiro in [2]. Several properties of normal meromorphic functions are developed in a paper of O. Lehto and K. I. Virtanen [2]. Their definition of a normal meromorphic function is:

DEFINITION A meromorphic function $f(z)$ is called normal in a simply-connected domain G if the family $\{f(S(z))\}$ is normal in G (in the sense of Montel), where $z' = S(z)$ denotes an arbitrary one-to-one conformal mapping of G onto itself.

In the same paper they gave the following characterization of a normal meromorphic function:

THEOREM 1. *A nonconstant function $f(z)$, meromorphic in a domain G, is normal if and only if*

$$\rho(f(z))\,|\,dz\,| \leqq C\, d\sigma(z)$$

at every point of G. $\rho(f)$ is the spherical derivative of f, $d\sigma$ is the hyperbolic element of length and C is an absolute constant.

The class of normal functions includes the bounded functions, schlicht functions and functions omitting three values. Moreover, it is known that the class of normal functions interselects the class of functions of bounded characteristic. Functions of bounded characteristic. Functions of bounded characteristic have the following form

$$F(z) = e^{i\lambda} \frac{B_1(z)}{B_2(z)} \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\psi(\theta)\right).$$

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$\psi(\theta)$ is a function of bounded variation, λ is real and the $B_i(z)$ are infinite products.

$$B(z, a_n) = B_1(z) = \prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \left(\frac{a_n - z}{1 - \bar{a}_n z}\right),$$

$$\sum_{n=1}^{\infty} (1 - |a_n|) < \infty .$$

For brevity we refer to a quotient of the form $B_1(z)/B_2(z)$ as a Blaschke-quotient. There are functions which are normal which are not of bounded characteristic and functions of bounded characteristic which are not normal. In particular F. Bagemihl and W. Seidel [1] have constructed a holomorphic function of bounded characteristic which is not normal.

If z_1 and z_2 are two points of the unit disk $D = \{z \mid |z| < 1\}$, the non-Euclidean (hyperbolic) distance between z_1 and z_2 is given by the formula

$$\rho(z_1, z_2) = \frac{1}{2} \log\left(\frac{1 + |h|}{1 - |h|}\right), \quad h = \frac{z_1 - z_2}{1 - \bar{z}_1 z_2}$$

The following theorems of F. Bagemihl and W. Seidel [1] show that the non-Euclidean metric is in a sense a natural one for discussing the boundary behavior of a normal function.

THEOREM 2. *Let $\{z_n\}$ be a sequence of points in $D = \{z \mid |z| < 1\}$ which converges to a point $\zeta \in C = \{z \mid |z| = 1\}$ and is such that lim $\rho(z_n, z_{n+1}) = 0$ as $n \to \infty$, and let $f(z)$ be a normal meromorphic function in D for which* $\lim f(z_n) = c$ *as $n \to \infty$, where c is finite or infinite. Then $f(z)$ has the angular limit c at ζ.*

THEOREM 3. *Let $\{z_n\}$ be a sequence of points in $D = \{|z| < 1\}$ for which $|z_n| \to 1$ and $f(z)$ is a normal, meromorphic function in D such that* $\lim_{n\to\infty} f(z_n) = C$. *If $\{z_n'\}$ denotes any sequence of points in D for which* $\lim_{n\to\infty} \rho(z_n, z_n') = 0$, *then also* $\lim_{n\to\infty} f(z_n') = C$.

One problem remaining in the study of normal functions is to determine those Blaschke-quotients which are normal. That is, if $F(z) = B_1(z)/B_2(z) = B_1(z; a_n)/B_2(z; b_n)$, to determine conditions on the non-Euclidean distances between the zeros of B_1 and B_2 which will make F normal. We note a few immediate results and then construct an example of a Blaschke-quotient such that the zeros of B_1 and those of B_2 are separated by a positive non-Euclidean distance and such that $B_1(z)/B_2(z)$ is not normal.

2. The construction. The first comment to be made is that a Blaschke-product has radial limits a.e. That is, $\lim B(re^{i\theta})$, $r \to 1$, exists and equals a complex uni-modular number $e^{i\varphi}$ [3]. The only singularities of $B(z) = B(z; a_n)$ in $|z| \leqq 1$ are those points z_0, $|z_0| = 1$, such that there is a subsequence $\{a_{n_k}\}$ of the zeros of $B(z; a_n)$ with $a_{n_k} \to z_0$ as $k \to \infty$. If we denote by $A(a_n)$ the set of limit points of the set $\{a_n\}$, then we have the following:

LEMMA 1. *Let* $F(z) = B_1(z)/B_2(z) = B_1(z; a_n)/B_2(z; b_n)$. *Then if* $A(a_n) \cap A(b_n)$ *is empty*, F *is normal.*

Proof. The condition of Theorem 1 for $F = B_1/B_2$ with domain the unit disk is

$$\rho(F(z))\,|\,dz\,| = \frac{|\,B_1(z)B_2'(z) - B_1'(z)B_2(z)\,|}{|\,B_1(z)\,|^2 + |\,B_2(z)\,|^2}\,|\,dz\,| \leqq \frac{C\,|\,dz\,|}{(1 - |\,z\,|^2)}\,.$$

For $P = e^{i\varphi}$ it suffices to show

$$\varlimsup_{\substack{z \to P \\ |z|<1}} \frac{|\,B_1(z)B_2'(z) - B_1'(z)B_2(z)\,|}{|\,B_1(z)\,|^2 + B_2(z)\,|^2}\,(1 - |\,z\,|^2) < \infty\,.$$

Now

$$\frac{|\,B_1(z)B_2'(z) - B_1'(z)B_2(z)\,|\,(1 - |\,z\,|^2)}{|\,B_1(z)\,|^2 + |\,B_2(z)\,|^2}$$
$$\leqq \frac{|\,B_2'(z)\,|\,(1 - |\,z\,|^2) + |\,B_1'(z)\,|\,(1 - |\,z\,|^2)}{|\,B_1(z)\,|^2 + |\,B_2(z)\,|^2}\,.$$

The lemma follows from the fact that

$$\varliminf_{\substack{z \to P \\ |z|<1}} (|\,B_1(z)\,|^2 + B_2(z)\,|^2) \geqq 1$$

and a result of W. Seidel and J. L. Walsh [5] which states that $|\,B_i'(z)\,|\,(1 - |\,z\,|)$ is bounded for $|\,z\,| < 1$.

The following lemma is a restatement of Theorem 3.

LEMMA 2. *If* $F(z) = B_1(z; a_n)/B_2(z; b_n)$, *with* $\{a_n\}$ *and* $\{b_n)$ *disjoint, and if there exist subsequences* $\{a_{n_k}\}$ *and* $\{b_{n_k}\}$, *with* $\rho(a_{n_k}, b_{n_k})$ *tending to zero as* $k \to \infty$, *then* F *is not normal.*

We turn now to the example.

THEOREM 4. *There exists a Blaschke-quotient* $F(z) = B_1(z; a_n)/B_2(z; b_n)$ *with* $\rho(a_n, b_k) \geqq \delta_1 > 0$ *for all positive integers* n *and* k *such that* F *is not normal.*

Proof. The construction proceeds as follows. We exhibit a Blaschke-product $B_2(z; b_n)$ whose zeros lie in an angle at $z = 1$ and which tends to zero in angular approach to 1. We can then construct $B_1(z; a_n)$ such that the zeros $\{a_n\}$ tend to 1 and have the property expressed in the theorem. $B_1(z)$ has the additional property that there is a nontangential path Γ such that $\lim B_1(z)$ for z on Γ is positive. This implies

$$\lim_{\substack{z \to 1 \\ z \in \Gamma}} \frac{B_1(z)}{B_2(z)} = \lim_{\substack{z \to 1 \\ z \in \Gamma}} \frac{B_1(z; a_n)}{B_2(z; b_n)} = \infty$$

whereas $B_1(a_n)/B_2(a_n) = 0$. Applying Theorem 2, we see that F cannot be normal.

First define B_2 as follows:

$$B_2(z; b_n) = B_2(z; 1 - \frac{1}{n_2} = \prod_{n=1}^{\infty} \frac{\left(1 - \frac{1}{n^2}\right) - z}{1 - \left(1 - \frac{1}{n^2}\right) z}$$

$B_2(z)$ is well defined and holomorphic in $|z| < 1$. The function ρ_n,

$$\rho_n = \rho(b_n, b_{n+1}) = \frac{1}{2} \log\left(\frac{1 + |h_n|}{1 - |h_n|}\right)$$

where

$$|h_n| = \left| \frac{\left(1 - \frac{1}{n^2}\right) - \left(1 - \frac{1}{(n+1)^2}\right)}{1 - \left(1 - \frac{1}{n^2}\right)\left(1 - \frac{1}{(n+1)^2}\right)} \right| = \frac{2n + 1}{2n^2 + 2n}$$

tends to zero as $n \to \infty$. Theorem 2 shows that B_2 has angular limit zero at $z = 1$.

For the remainder of the construction we find it useful to do some of the work in a half plane. The linear transformation T,

$$T(z) = i\left(\frac{1 + z}{1 - z}\right) = \omega = u + iv$$

maps $|z| < 1$ onto $v > 0$, preserving the non-Euclidean metric. The point $z = 1$ corresponds to $\omega = \infty$. As a nontangential path tending to infinity we choose the hypercycle $u = v$, $v > 0$. This curve has fixed non-Euclidean distance $1/2 \log(\sqrt{2} + 1)$ from $u = 0$, $v > 0$. Select points a_n' as follows:

$$a_n' = \alpha_n e^{i\pi/4} = (2e^n - 1)e^{i\pi/4} .$$

The corresponding points a_n of $|z| < 1$ are

$$T^{-1}(a_n') = a_n = \frac{a_n' - i}{a_n' + i}\,.$$

Define $B_1(z)$ by using the points $\{a_n\}$ as the zeros of $B_1(z)$. To show $B_1(z)$ is well defined, it suffices to show $\sum_{n=1}^{\infty}(1 - |a_n|) < \infty$.

$$\sum_{n=1}(1\ |a_n|) = \sum_{n=1}^{\infty}\left(1 - \frac{(\alpha_n^2 + 1) - \sqrt{2}\,\alpha_n}{[(\alpha_n + 1)^2 - 2\alpha_n^2]^{1/2}}\right) \leqq \sum_{n=1}^{\infty}\frac{\sqrt{2}}{\alpha_n}\,.$$

This last series has terms which are order of magnitude of e^{-n} and so converges.

We note that the curve $u = v$ maps under T^{-1} into a hypercycle h_α having a fixed non-Euclidean distance from the real axis. It also forms the fixed angle $\pi/4$ with the real axis at $z = 1$. Thus the zeros of $B_1(z)$ and those of $B_2(z)$ are separated by a positive non-Euclidean distance.

In the plane $v > 0$, $B_1(z)$ has the form

$$B_1(z) = B_1(T^{-1}(\omega)) = \pi_1(\omega) = \prod_{u=1}^{\infty} e^{ip_n}\left(\frac{\omega - a_n'}{\bar{a}_n' - \omega}\right)$$

where p_n is real. Now select points $i\beta_k = i(2k^2 - 1)$ on $u = 0$, $v > 0$.

$$T^{-1}(i\beta_k) = C_k = 1 - \frac{1}{k^2}\,.$$

If we show that $\underline{\lim}_{k\to\infty}|\pi_1(i\beta_k)| \geqq \delta > 0$, this implies

$$\underline{\lim}_{k\to\infty}\left|B_1\left(1 - \frac{1}{k^2}\right)\right| \geqq \delta > 0\,.$$

However, we know $\rho(C_k, C_{k+1})$ tends to zero as $k \to \infty$. Thus by Theorem 3, if $\{t_k\}$ is any real increasing sequence $t_k < 1$, $t_k \to 1$ with $B_1(t_k) \to \alpha$, then we can choose a subsequence $\{C_{n_k}\}$ such that

$$C_{n_k} \leqq t_k < C_{n_k+1}, \quad \rho(C_{n_k}, t_k) < \rho(C_{n_k}, C_{n_k+1})\,.$$

This implies $B_1(C_{n_k}) \to \alpha$ and so $|\alpha| \geqq \delta$. This proves $\underline{\lim}_{x\to 1}|B_1(x)|$ is positive. It remains to prove $\underline{\lim}_{k\to\infty}|\pi_1(i\beta_k)| \geqq \delta$.

The calculation for $\pi_1(\omega)$ is as follows:

$$|\pi_1(i\beta_k)|^2 = \prod_{n=1}^{\infty}\left|\frac{\beta_k e^{i(\pi/2)} - \alpha_n e^{i(\pi/4)}}{\alpha_n e^{-i(\pi/4)} - \beta_k e^{i(\pi/2)}}\right|$$

$$= \prod_{n=1}^{\infty}\left(1 - \frac{2\sqrt{2}\,\alpha_n\beta_k}{\alpha_n^2 + \beta_k^2 + \sqrt{2}\,\alpha_n\beta_k}\right).$$

Consider the following function of the positive real variables ξ and η,

$$g(\xi,\eta)=\frac{2\sqrt{2}(2e^{\xi}-1)(2\eta^2-1)}{(2e^{\xi}-1)^2+(2\eta^2-1)^2+\sqrt{2}(2e^{\xi}-1)(2\eta^2-1)}\,.$$

Setting $x=[(2e^{\xi}-1)/2\eta^2-1]^2$ we have

$$g(\xi,\eta)=\frac{2\sqrt{2}}{x+\dfrac{1}{x}+\sqrt{2}}\leqq\frac{2\sqrt{2}}{2+\sqrt{2}}<1$$

since $x+(1/x)\geqq 2$ if $x>0$.

We are considering the product $\prod_{n=1}^{\infty}(1-g(n,k))$, where $0\leqq g(n,k)\leqq\sigma<1$. We can choose a positive number A so large that $1+x\geqq e^{Ax}$ for $-\sigma\leqq x\leqq 0$. The number $A=-\log(1-\sigma)/\sigma$ will suffice ($A\geqq 2$).

For each positive integer k let $N(k)=N$ be the positive integer satisfying $N<2\log k\leqq N+1$. $g(y,k)$ is monotone increasing on $1\leqq y<2\log k$ and monotone descreasing on $y>2\log k$, with

$$g(2\log k,k)=\frac{1}{2(2k^2-1)}\,.$$

Thus we have for fixed k

$$\begin{aligned}\sum_{n=1}^{\infty}g(n,k)&\leqq 2\sqrt{2}(2k^2-1)\sum_{n=1}^{\infty}\frac{(2e^n-1)}{(2e^n-1)^2+(2k^2-1)^2}\\&\leqq 2\sqrt{2}(2k^2-1)\Bigg[\int_1^N\frac{(2e^y-1)dy}{(2e^y-1)^2+(2k^2-1)^2}\\&\quad+\int_{N+1}^{\infty}\frac{(2e^y-1)dy}{(2e^y-1)^2+(2k^2-1)^2}\\&\quad+\frac{(2e^N-1)}{(2e^N-1)^2+(2k^2-1)^2}+\frac{(2e^{N+1}-1)}{(2e^{N+1}-1)^2+(2k^2-1)^2}\Bigg]\\&\leqq 2\sqrt{2}(2k^2-1)\left[\int_1^{\infty}\frac{(2e^y)dy}{(2e^y-1)^2+(2k^2-1)^2}+\frac{2}{2(2k^2-1)}\right]\\&=2\sqrt{2}\arctan\left(\frac{2e^y-1}{2k^2-1}\right)\Bigg|_1^{\infty}+2\sqrt{2}\,.\end{aligned}$$

Now evaluating the infinite product using the above estimates, we obtain

$$\begin{aligned}\prod_{n=1}^{\infty}(1-g(n,k))&>\exp\Big(-A\sum_{n=1}^{\infty}g(n,k)\Big)\\&\geqq\exp\Big(-2\sqrt{2}A\Big(\frac{\pi}{2}-\arctan\Big(\frac{2e-1}{2k^2-1}\Big)\Big)-2\sqrt{2}A\Big)\,.\end{aligned}$$

Thus $\underline{\lim}_{k\to\infty}|\pi_1(i\beta_k)|$ is positive. This completes the construction.

The function $F(z) = B_1(z)/B_2(z)$ tends to ∞ on the real axis as $x \longrightarrow 1$ and has a sequence of zeros on the nontangential curve (hypercycle) h_α. Thus F is not normal.

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