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REPRESENTATIONS OF LATTICE-ORDERED GROUPS HAVING A BASIS

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A convex l-subgroup C of a lattice-ordered group G is said to be a prime subgroup provided the collection $L(C)$ of left cosets of G by C is totally-ordered by the relation: $xC \leqq yC$ if and only if there exists $c \in C$ such that $xc \leqq y$. A collection $\bar{C}$ of prime subgroups of G is called a representation for G if $\bigcap \bar{C}$ contains no proper l-ideal of G. A representation $\bar{C}$ is said to be irreducible if the intersection of any proper subcollection of $\bar{C}$ does contain a proper l-ideal of G. $\bar{C}$ is a minimal representation if each element of $\bar{C}$ is a minimal prime subgroup. A representation $\bar{C}$ is *-irreducible if $\bigcap \bar{C} = \{1\}$ while $\bigcap (\bar{C} - \{C\}) \neq \{1\}$ for every $C \in \bar{C}$. In this paper it is shown that an l-group with a basis admits a minimal irreducible representation and that such a representation can be chosen in essentially only one way. In particular, an l-group with a normal basis has a unique minimal irreducible representation. In addition, two properties equivalent to the existence of a basis are derived; namely the existence of a representation $\bar{C}$ such that each element of $\bar{C}$ has a nontrivial polar and the existence of a *-irreducible representation.

For a linearly-ordered set L, let $P(L)$ denote the collection of all order-preserving permutations of L. $P(L)$ is a group under the operation of composition of functions, and is an l-group if $f \in P(L)$ is defined to be positive provided $f(x) \geqq x$ for all $x \in L$. C. Holland [2] has related an arbitrary l-group G to l-groups of the form $P(L)$ in the following way: Letting C be a prime subgroup of G, the collection $L(C)$ of left cosets of G by C is totally-ordered (by the relation mentioned above) and the map $g \rightarrow \bar{g}$ where $\bar{g}(xC) = gxC$ for all $xC \in L(C)$ is an l-homomorphism from G into $P(L(C))$. This map is called the *natural* l-homomorphism. If $C = \{C_i \mid i \in I\}$ is a represention for G and if δ_i denotes the natural l-homomorphism of G into $P(L(C_i))$, then the large cardinal product $\prod$ of the $\delta_i(G)$ contains an l-isomorphic copy of G as an l-subgroup and subdirect product. (This l-isomorphism is defined by $g \rightarrow (\cdots, \delta_i(g), \cdots)$.) It is for this reason that $\bar{C}$ is called a representation. The main result of [2] is that every l-group has a representation. If $\bar{C} = \{C_i \mid i \in I\}$ is a representation for G and if each

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C_i is an l-ideal of G, then $\bar{C}$ is called a *realization* of G. In this case, each $\delta_i(G)$ is a totally-ordered group and G is l-isomorphic to an l-subgroup and subdirect product of a cardinal product of o-groups. If $\bar{C}$ is an irreducible representation consisting of l-ideals, then $\bar{C}$ is called an *irreducible realization.*

2. Minimal irreducible representations of l-groups with basis.

An element s of an l-group G is *basic* provided $s>1$ and $\{x \in G \mid 1 \leqq x \leqq s\}$ is totally-ordered by the order relation in G. A basic element s of G is *normal* if s and $g^{-1}sg$ are comparable ($g^{-1}sg \geqq s$ or $s > g^{-1}sg$) for all $g \in G$. For $x \in G$, the *absolute value* of x is defined by $|x| = x \vee x^{-1}$. Two elements x and y of G are said to be *disjoint* if $|x| \wedge |y| = 1$. A subset S of G is a *basis* for G if S is a maximal set of pairwise disjoint elements and each element of S is basic. A basis S is *normal* if each element of S is normal.

P. Lorenzen [4] has shown that an l-group G has a realization if and only if no positive element of G is disjoint from one of its conjugates. P. Jaffard [3] has proven that an abelian l-group has an irreducible realization if and only if it has a basis. F. Sik [5] generalized this result by showing that for an l-group G, the possession of a normal basis is equivalent to the existence of an irreducible realization of G. Using this result along with Lorenzen's, it is easily seen that an l-group G with a basis has a realization if and only if it has a normal basis.

It will now be shown that an l-group G with a basis has a minimal irreducible representation which can be chosen in essentially only one way. The construction depends upon those prime subgroups of G having nontrivial polars and not upon the choice of a basis. It will be shown further that the concept of a minimal irreducible representation is a direct generalization of the concept of an irreducible realization.

LEMMA 2.1. (*P. Conrad, unpublished*) *A convex l-subgroup C of an l-group G is prime if and only if the conditions $a, b \in G$ and $a \wedge b = 1$ imply $a \in C$ or $b \in C$.*

For an element x of an l-group G, let $D(x) = \{y \in G \mid |x| \wedge |y| = 1\}$. For a subset B of G, let $D(B) = \bigcap D(x)$ $(x \in B)$. Since each $D(x)$ is a convex l-subgroup of G, $D(B)$ is also a convex l-subgroup of G.

LEMMA 2.2. *Let C be a prime subgroup of the l-group G where $D(C) \neq \{1\}$. Let $s \in D(C)$, $s > 1$. Then s is basic, $C = D(s)$ and C is minimal prime. Conversely, if s is basic, then $D(s)$ is a prime subgroup of G and $s \in DD(s)$.*

Proof. Let $s \geqq x \geqq 1$ and $s \geqq y \geqq 1$. Then $x(x \wedge y)^{-1}, y(x \wedge y)^{-1} \in D(C)$ and $x(x \wedge y)^{-1} \wedge y(x \wedge y)^{-1} = 1$. It follows from Lemma 2.1 that $x(x \wedge y)^{-1} \in C \cap D(C)$ or $y(x \wedge y)^{-1} \in C \cap D(C)$ and so $x(x \wedge y)^{-1} = 1$ or $y(x \wedge y)^{-1} = 1$. Thus $y \geqq x$ or $x \geqq y$ and so s is basic. Since $s \notin C$, Lemma 2.1 implies that $D(s) \subseteqq C$ and since $s \in D(C)$ it is immediate that $C \subseteqq D(s)$. Thus $C = D(s)$. Since $D(s)$ is contained in any prime subgroup which does not contain s, it is clear that C is minimal.

Suppose s is basic and let $a, b \in G$ be such that $a \wedge b = 1$. If $a, b \notin D(s)$, then $s \geqq a \wedge s > 1$ and $s \geqq b \wedge s > 1$. Since s is basic, $a \wedge s \geqq b \wedge s$ or $b \wedge s > a \wedge s$. In either case it follows that $a \wedge b \wedge s > 1$, contradicting the assumption that $a \wedge b = 1$. Thus $a \wedge b = 1$ implies that $a \in D(s)$ or $b \in D(s)$ and so $D(s)$ is prime by Lemma 2.1. It is clear that $s \in DD(s)$.

LEMMA 2.3. *Let C_1 and C_2 be distinct prime subgroups of the l-group G. Then $D(C_1) \cap D(C_2) = \{1\}$.*

Proof. If $D(C_1) \cap D(C_2) \neq \{1\}$, let $s \in D(C_1) \cap D(C_2)$ where $s > 1$. By Lemma 2.2., $C_1 = D(s) = C_2$; and this contradicts the supposition that $C_1 \neq C_2$.

THEOREM 2.1. *Let C' be the collection of all prime subgroups C of the l-group G such that $D(C) \neq \{1\}$. For each $C \in C'$, let $s(C) \in D(C)$ where $s(C) > 1$. Then the following are equivalent:*

(a) *$\{s(C) \mid C \in C'\}$ is a basis for G.*

(b) *$\bigcap C' = \{1\}$.*

(c) *C' is a representation for G.*

In case any of these conditions hold, a subset $\bar{C}$ of C' is an irreducible representation if and only if $\bar{C}$ contains exactly one group from each conjugate class in C'.

Proof. Suppose that (a) holds and let $x \in G$ where $x > 1$. Then there exists $C \in C'$ such that $x \wedge s(C) > 1$. By Lemma 2.2., $D(s(C)) = C$ and so $x \notin C$. Thus $\bigcap C' = \{1\}$. (b) implies (c) by definition. If (c) holds and if $1 < x \in G$, then there exists $g \in G$ and $C \in C'$ such that $g^{-1}xg \notin C$. Thus $x \notin gCg^{-1}$ while $gCg^{-1} \in C'$. It follows from Lemma 2.1. that $x \wedge s(gCg^{-1}) > 1$. Therefore $\{s(C) \mid C \in C'\}$ is a basis for G.

It is clear that an irreducible representation cannot contain distinct conjugate subgroups. Suppose then that $\bar{C}$ contains exactly one group from each conjugate class in C'. Let $1 < x \in G$ and let $C \in C'$ be such that $x \wedge s(C) > 1$. There exists $g \in G$ such that $g^{-1}Cg \in \bar{C}$ and since

$x \notin C$ it follows that $g^{-1}xg \notin \bigcap \bar{C}$. Thus $\bigcap \bar{C}$ contains no proper l-ideal of G and so $\bar{C}$ is a representation of G. If E is a proper subcollection of $\bar{C}$, let $C \in C'$ be such that no conjugate of C is in E. If there exists $C_1 \in E$ and $g \in G$ such that $g^{-1}s(C)g \notin C_1$, then $s(C) \notin gC_1g^{-1}$ while $gC_1g^{-1} \in C'$. The only element of C' not containing $s(C)$ is C and so $gC_1g^{-1} = C$ contradicting the supposition that no conjugate of C is in E. Thus $\bigcap E$ does contain a proper l-ideal and so $\bar{C}$ is an irreducible representation of G.

COROLLARY 2.1. *If S is a basis of the l-group G, then $\{D(s) \mid s \in S\}$ is the set C' of all prime subgroups C of G which satisfy $D(C) \neq \{1\}$.*

Proof. By Lemma 2.2. $\{D(s) \mid s \in S\} \subseteqq C'$. Thus $\bigcap C' = \{1\}$ and so it follows from the Theorem that $\{D(s) \mid s \in S\} = C'$.

COROLLARY 2.2. *Every l-group with a basis admits a minimal irreducible representation.*

COROLLARY 2.3. *An l-group G has a representation $\bar{C}$ such that $D(C) \neq \{1\}$ for each $C \in \bar{C}$ if and only if G has a basis.*

The above results show one way in which a minimal irreducible representation can be chosen for an l-group with a basis. The following shows that this is the only way in which such a representation can be chosen.

THEOREM 2.2. *If an l-group G has a basis S and if $\bar{C}$ is a minimal irreducible represntation for G, then $\bar{C} \subseteqq C' = \{D(s) \mid s \in S\}$. Thus $\bar{C}$ contains exactly one group from each conjugate class in C'.*

Proof. Let $C \in \bar{C}$. Then $\bigcap (\bar{C} - \{C\})$ contains a proper l-ideal N of G. (For the purpose of the following argument, let $N = G$ in case $\bar{C}$ has only one element.) Let $1 < g \in N$ and choose $s \in S$ such that $1 < g \wedge s \leqq s$. Then $g \wedge s$ is basic and since $1 < g \wedge s \leqq g$, $h^{-1}(g \wedge s)h \in N$ for all $h \in G$. Since $\bigcap \bar{C}$ does not contain an l-ideal of G, there exists $k \in G$ such that $k^{-1}(g \wedge s)k \notin C$. Moreover, $k^{-1}(g \wedge s)k$ is basic. Since C is prime, it follows that $D(k^{-1}(g \wedge s)k \subseteqq C$. The minimality of C implies that $D(k^{-1}(g \wedge s)k) = C$. It follows from Corollary 2.1. that $C \in C'$.

It is easily seen that a basic element s is normal if and only if $D(s)$ is an l-ideal. The following is then immediate.

COROLLARY 2.4. *An l-group with a normal basis has a unique minimal irreducible representation $\bar{C}$ and each element of $\bar{C}$ is an l-ideal. Thus $\bar{C}$ is an irreducible realization.*

THEOREM 2.3. *A representation $\bar{C}$ of an l-group G is *-irreducible if and only if G has a basis S and $\bar{C} = \{D(s) \mid s \in S\}$.*

Proof. If G has a basis S and if $\bar{C} = \{D(s) \mid s \in S\}$, it is clear that $\bar{C}$ is a *-irreducible representation of G.

Suppose then that $\bar{C}$ is a *-irreducible representation and let C' denote the collection of prime subgroups C of G such that $D(C) \neq \{1\}$. Let $C_1 \in \bar{C}$ and let $1 < g \in \bigcap(\bar{C} - \{C_1\})$. If $1 < h \in C_1$ then $g \wedge h \in \bigcap \bar{C}$ and so $g \wedge h = 1$. Thus $D(C_1) \neq \{1\}$ and so $\bar{C} \subseteqq C'$. It follows that $\bigcap C' = \{1\}$ and therefore by Theorem 2.1. that $\{s(C) \mid C \in C'\}$ is a basis for G. By Corollary 2.1., $C' = \{D(s(C)) \mid C \in C'\}$. Since the intersection of any proper subcollection of C' is nontrivial, it follows that $\bar{C} = C'$.

COROLLARY 2.5. (*F. Sik* [5]) *An l-group G has a normal basis if and only if it has an irreducible realization.*

Proof. If G has a normal basis S then $C' = \{D(s) \mid s \in S\}$ is an irreducible realization.

If $\bar{C}$ is an irreducible realization of G, then $\bar{C}$ is a *-irreducible representation of G. It follows from the Theorem that G has a basis S and $\bar{C} = \{D(s) \mid s \in S\}$. Thus each $D(s)$ is an l-ideal of G and so S is a normal basis for G.

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