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SUFFICIENT CONDITIONS FOR AN OPTIMAL CONTROL PROBLEM IN THE CALCULUS OF VARIATIONS

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An arc C is a collection of parameters b^ρ ($\rho = 1, \cdots, r$) on an open set B and sets of functions $y^i(x), a^h(x)$ ($i = 1, \cdots, n$; $h = 1, \cdots, m$) defined on an interval $x^1 \leqq x \leqq x^2$ with $y^i(x)$ continuous and $\dot{y}^i(x), a^h(x)$ piecewise continuous. The arc is admissible if it satisfies the differential equations

$$\dot{y}^i = P^i(x, y, a) \qquad (i = 1, \cdots, n)$$

on $x^1 \leqq x \leqq x^2$ and the end conditions

$$x^s = X^s(b), y^i(x^s) = Y^{is}(b) \qquad (s = 1, 2) .$$

The dot denotes differentiation with respect to x. The problem at hand is to find in a class of admissible arcs C, an arc C_0, which minimizes the integral

$$I(C) = g(b) + \int_{x^1}^{x^2} f(x, y, a)dx$$

where $P(x, y, a)$ and $f(x, y, a)$ are assumed to be class C'' for (x, y, a) in an open set R while $g(b), X^s(b), Y^{is}(b)$ are of class C'' on B. Under the added assumption that $P(x, y, a)$ is Lipschitzian in y and a, the indirect method of Hestenes is used to prove that the necessary conditions for relative minima of the problem above, strengthened in the usual manner, yield a set of sufficient conditions. This problem differs from that of Pontryagin in the choice of (x, y, a) to lie in an open set.

DEFINITIONS AND NOTATION. The arc C will be denoted by

$$C\colon b,\ y(x),\ a(x)$$

and the minimizing arc will be called C_0. A set of parameters β^ρ and functions $\eta^i(x), \alpha^h(x)$ is called a variation γ and denoted by

$$\gamma\colon \beta,\ \eta(x),\ \alpha(x)$$

if $\eta^i(x)$ are continuous and $\dot{\eta}^i(x), \alpha^h(x)$ are in L_2 on $x^1 \leqq x \leqq x^2$. The variation γ is differentially admissible if

$$\dot{\eta} = P_{y^j}\eta^j + P_{a^h}\alpha^h$$

along C_0 for almost all x on $x^1 \leqq x \leqq x^2$. Repeated indices indicate summation. It is admissible if in addition to being differentially admissible

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it also satisfies the variational end conditions

$$\eta^i(x^s) = \{Y^{is}_\rho - \dot{y}^i(x^s)X^s_\rho\}\beta^\rho = C^{is}_\rho\beta^\rho \qquad (s = 1, 2)$$

where the subscript ρ denotes the derivative with respect to b^ρ.

2. Condition S. An admissible arc

$$C_0 \colon b_0,\ y_0(x),\ a_0(x)$$

will be said to satisfy condition S if the following are true.

(a) $a_0(x)$ is continuous on $X^1(b_0) \leqq x \leqq X^2(b_0)$.

(b) C_0 satisfies the first necessary conditions, i.e., the Euler equations,

$$\dot{z}^i(x) = -H_{y^i},\ \dot{y}^i(x) = H_{z^i},\ H_{a^h} = 0$$

and the transversality condition

$$g_\rho - [H(x_0^s)X^s_\rho - z^i(x_0^s)Y^{is}_\rho]^{s=2}_{s=1} = 0$$

with $z^i(x)$ being continuous and having continuous derivatives on a neighborhood of C_0. The symbol $[f(x^s)]^{s=2}_{s=1}$ means $f(x^2) - f(x^1)$.

(c) C_0 is nonsingular, i.e., the determinant $|H_{a^h a^k}|$ is nonzero along C_0 where

$$H(x, y, a, z) = z^i(x)P^i(x, y, a) - f(x, y, a)\ .$$

(d) C_0 with $z^i(x)$ satisfies the strengthened condition II_N of Weierstrass, $E_H(x, y, p, q, z) \geqq 0$ whenever (x, y, p, z) is near those on C_0 and $(x, y, p) \neq (x, y, q)$ in R. The E-function is given by

$$\begin{aligned} E_H(x, y, p, q, z) = & -H(x, y, q, z) + H(x, y, p, z) \\ & + (q^h - p^h)H_{p^h}(x, y, p, z) \end{aligned}$$

(e) For every nonnull admissible variation γ, the second variation $I_2(\gamma)$ along C_0 is greater than zero where

$$\begin{aligned} I_2(\gamma) = \{ & g_{\rho\sigma} - [HX^s_{\rho\sigma} - z^iY^{is}_{\rho\sigma} \\ & + \{H_x - \dot{y}^iH_{y^i}\}X^s_\rho X^s_\sigma + H_{y^i}(Y^{is}_\rho X^s_\sigma + Y^{is}_\sigma X^s_\rho)]^{s=2}_{s=1}\}\beta^\rho\beta^\sigma \\ & - \int_{x^1}^{x^2} 2\omega(x, \eta, \alpha)dx\ , \\ 2\omega(x, & \eta, \alpha) = H_{y^iy^j}\eta^i\eta^j + 2H_{y^ia^h}\eta^i\alpha^h + H_{a^ha^k}\alpha^h\alpha^k\ . \end{aligned}$$

(f) There is a neighborhood of C_0 in xy-space such that

$$|P(x, y, a) - P(x, Y, A)| < c\{|y - Y|^2 + |a - A|^2\}^{1/2},\ c > 0$$

holds for all elements $(x, y, a), (x, Y, A)$ of R which have (x, y) in that neighborhood.

Unless otherwise specified it will be assumed that the arc denoted by C_0 will satisfy condition S. The principal theorem of this paper can now be stated and its proof will be given in § 7, using the results of the intervening sections.

THEOREM 2.1. *Let C_0 be an admissible arc on $x^1 \leqq x \leqq x^2$ satisfying condition S. There is a neighborhood N of C_0 in b y-space such that $I(C) > I(C_0)$ for all admissible arcs C with (b, y) in N and (x, y, a) in R.*

For future use it is convenient to state a theorem of Hestenes [8, Theorem 5.1] as

THEOREM 2.2. *Let C_0 be a nonsingular admissible minimizing arc satisfying condition II_N. There is a neighborhood N_0 of C_0 in b y a-space and a constant $h > 0$ such that*

$$E_H(x, y, p, q, z) \geqq hl(q - p)$$

for (x, y, p) in N_0 and (x, y, q) in R where

$$l(q - p) = \sqrt{1 + |q - p|^2} - 1$$

and $|q - p|$ = the length of the vector $q - p$.

3. $I^*(C)$. Let C_0 be a nonsingular minimizing arc and define

$$\begin{aligned} E_H^*(C) &= \int_{x^1}^{x^2} E_H(C)dx \\ &= -\int_{x^1}^{x^2} \{-H(a) + H(a_0) + (a^h - a_0^h)H_{a^h}(a_0)\}dx \end{aligned}$$

where the missing arguments are $(x, y(x), z(x))$. Choose a function $I^*(C)$ so that

$$I(C) = I^*(C) + E_H^*(C) .$$

It follows from the definitions of $I(C)$ and $E_H^*(C)$ that

$$\begin{aligned} I^*(C) &= g(b) + [z^i(x^s)y^i(x^s)]_{s=1}^{s=2} \\ &\quad - \int_{X^1(b)}^{X^2(b)} \{\dot{z}^i(x)y^i(x) + H(x, y, a_0, z) + \{a^h - a_0^h\}H_{a^h}(x, y, a_0, z)\}\,dx . \end{aligned}$$

Since $E_H^*(C_0) = 0$,

$$I(C) - I(C_0) = I^*(C) - I^*(C_0) + E_H^*(C) .$$

From the definition of $I^*(C)$,

$$
\begin{aligned}
I^*(C) - I^*(C_0) = \{g(b) - g(b_0)\}& \\
&+ [z^i(x^s)y^i(x^s) - z^i(x_0^s)y_0^i(x_0^s)]_{s=1}^{s=2} \\
&- \int_{X^1(b)}^{X^2(b)} \{\dot{z}^i\{y^i - y_0^i\} + H(y) \\
&\qquad - H(y_0) + \{a^h - a_0^h\}H_{a^h}(y)\}dx \\
&- \int_{X^2(b_0)}^{X^2(b)} \{\dot{z}^i y_0^i + H(y_0)\}dx \\
&+ \int_{X^1(b_0)}^{X^1(b)} \{\dot{z}^i y_0^i + H(y_0)\}dx
\end{aligned}
\tag{3.1}
$$

where the missing arguments in H are (x, a_0, z). The following result can now be proved.

THEOREM 3.1. *Let C_0 be a nonsingular admissible minimizing arc satisfying condition II_N. For every $\varepsilon > 0$ there exists a constant $\delta > 0$ and a neighborhood F of C_0 in by-space such that*

$$|I^*(C) - I^*(C_0)| < \varepsilon\{1 + E_H^*(C)\} ,$$

for every admissible arc C in F whose endpoints are in a δ-neighborhood of these on C_0.

Given $\varepsilon > 0$, δ and a neighborhood N_1 of C_0 in b y-space can be chosen such that from equation (3.1),

$$|I^*(C) - I^*(C_0)| < \left| \int_{X^1(b)}^{X^2(b)} \{a^h - a_0^h\}H_{g^h}(x, y, a_0, z)dx \right| + \frac{\varepsilon}{2} \tag{3.2}$$

for all arcs C with (b, y) in N_1. Since $H_{a^h}(x, y_0, a_0, z) = 0$, it follows that for $\varepsilon > 0$ a neighborhood N_2 of C_0 in b y-space can be chosen so that

$$|H_{a^h}(x, y, a_0, z)| < \varepsilon_1 \tag{3.3}$$

for all arcs C with (b, y) in N_2. From Theorem 2.2,

$$E_H(C) \geqq hl(q - p) > h\{|a - a_0| - 1\}$$

and

$$|a - a_0| \leqq \frac{1}{h}\{E_H(C) + h\} .$$

This together with inequality (3.3) yields

$$
\begin{aligned}
\left| \int_{X^1}^{X^2} \{a^h - a_0^h\}H_{a^h}(x, y, a_0, z)dx \right| &< \varepsilon_1 \int_{X^1}^{X^2} |a - a_0|\, dx \\
&< \frac{\varepsilon_1}{h}\{E_H^*(C) + h(x^2 - x^1)\} .
\end{aligned}
\tag{3.4}
$$

Choose ε_1 such that $\varepsilon_1(x^2 - x^1) < \varepsilon/2$ and $\varepsilon_1/h < \varepsilon$. If in addition F is taken to be the smaller of the neighborhoods N_1 and N_2, the theorem follows readily from inequalities (3.2) and (3.4).

THEOREM 3.2. *Given a constant $\sigma > 0$ there are positive constants δ, ρ and a neighborhood F of C_0 in b y-space such that for every admissible arc C in F satisfying theorem* 3.1, $I(C) > I(C_0) - \sigma$. *If $E_H^*(C) \leqq \rho$, then $I(C) < I(C_0) + \sigma$. If $E_H^*(C) \geqq 2\sigma$, then $I(C) > I(C_0) + \sigma$.*

The definition of $I(C)$ and Theorem 3.1 yield

$$-\varepsilon + \{1 - \varepsilon\}E_H^*(C) < I(C) - I(C_0) < \varepsilon + \{1 + \varepsilon\}E_H^*(C)$$

for all admissible arcs C with (b, y) in F. The theorem follows immediately from the proper choice of ε and ρ.

4. Extension of the arcs C_0 and C. We shall extend the arcs C_0, C to lie on a fixed interval $e^1 \leqq x \leqq e^2$ containing $X^1(b_0) \leqq x \leqq X^2(b_0)$ and $X^1(b) \leqq x \leqq X^2(b)$. The equation

$$H_{a^h}(x, y, a, z) = 0 \tag{4.1}$$

has a solution $y = y_0(x)$, $a = a_0(x)$ corresponding to the minimizing arc C_0. By the nonsingularity of C_0, there is a solution $a = a(x, y, z)$ of equation (4.1) which is continuous and has continuous derivatives in a neighborhood of C_0. Further, on $X^1(b_0) \leqq x \leqq X^2(b_0)$, $a(x, y_0, z) = a_0(x)$. By an imbedding theorem [2, pp. 196] the equations

$$\begin{aligned} \dot{y} &= H_z(x, y, a(x, y, z)) \\ \dot{z} &= -H_y(x, y, a(x, y, z)) \end{aligned}$$

have a solution $y = \bar{y}(x)$, $z = \bar{z}(x)$ on $e^1 \leqq x \leqq e^2$ such that $e^1 < X^1(b_0) < X^2(b_0) < e^2$ and $\bar{y}(x) = y_0(x)$, $\bar{z}(x) = z_0(x)$ on $X^1(b_0) \leqq x \leqq X^2(b_0)$. The arc $\bar{C}_0$,

$$\bar{C}_0\colon b_0,\ \bar{y}(x),\ \bar{a}(x) = a(x, \bar{y}(x), \bar{z}(x))$$

coincides with C_0 on $x^1 \leqq x \leqq x^2$, is defined on the larger interval $e^1 \leqq x \leqq e^2$ and is therefore an extension of the arc C_0. Since this extension is unique, the extended arc will be denoted by C_0,

$$C_0\colon b_0,\ y_0(x) = \bar{y}(x),\ a_0(x) = \bar{a}(x)\ .$$

If an admissible arc C lies in a sufficiently small neighborhood of C_0 then $e^1 \leqq X^1(b) < X^2(b) \leqq e^2$ and the arc C may be extended uniquely to the interval $e^1 \leqq x \leqq e^2$ by requiring that $a(x) = a_0(x)$ where it is undefined and that $\dot{y} = P(x, y, a(x))$ also holds on the extension. The extended arc will also be denoted by C.

This method of extension will be used throughout the rest of the paper. In the formulas for $I(C)$ and $I^*(C)$ it will be understood that the integrals will be evaluated on the interval $x^1 \leqq x \leqq x^2$ and not on the extended interval. An exception to this convention is made in the formula for $K(C, C_0)$ which is discussed in the next session.

5. **The function $K(C, C_0)$.** To measure the deviation of comparison arcs from the minimizing arc, we shall define a function $K(C, C_0)$ where C, C_0 are the unique extensions of admissible arcs given in the last section as

$$K(C, C_0) = |b - b_0|^2 + \max_{e^1 \leqq x \leqq e^2} |y(x) - y_0(x)|^2 + \int_{e^1}^{e^2} l(a - a_0)dx$$

with

$$l(a - a_0) = \sqrt{1 + |a - a_0|^2} - 1 .$$

Since $a(x) = a_0(x)$ on the extension,

$$\int_{e^1}^{e^2} l(a - a_0)dx = \int_{x^1}^{x^2} l(a - a_0)dx$$

and $E_H(C)$ is not changed by extending the interval.

THEOREM 5.1. *Let C, C_0 be extensions to $e^1 \leqq x \leqq e^2$ of an admissible arc and a nonsingular minimizing arc respectively. For every $\varepsilon > 0$ there is a b y-neighborhood of C_0 such that $K(C, C_0) < \varepsilon$ for all arcs C in that neighborhood satisfying $E_H^*(C) < \varepsilon/2$.*

By Theorem 2.2 and the hypothesis,

$$\frac{\varepsilon}{2} > E_H^*(C) > h\int_{x^1}^{x^2} l(a - a_0)dx .$$

Choose a neighborhood of C_0 in b y-space such that

$$|b - b_0|^2 + \max_{e^1 \leqq x \leqq e^2} |y(x) - y_0(x)|^2 < \frac{(2h - 1)\varepsilon}{2h} .$$

In that neighborhood,

$$K(C, C_0) < \frac{(2h - 1)\varepsilon}{2h} + \frac{\varepsilon}{2h} = \varepsilon$$

and the theorem is proved.

THEOREM 5.2. *Let C_q be the extension of an admissible arc and let the sequence $\{C_q\}$ of such extended arcs have the property that given* *a neighborhood F of C_0 in b y-space there is an integer q_0 such that C_q is in F for $q > q_0$. If* $\limsup_{q=\infty} I(C_q) \leqq I(C_0)$, *then* $\lim_{q=\infty} K(C_q, C_0) = 0$.

If F is the neighborhood in Theorem 3.2 and $E^*_H(C_q) \geqq 2\sigma$ for $q > q_0$, $\sigma > 0$, $I(C_q) > I(C_0) + \sigma$ which contradicts the hypothesis that $\limsup_{q=\infty} I(C_q) \leqq I(C_0)$. Hence, $E^*_H(C_q) \leqq 2\sigma < \varepsilon/4$. Theorem 5.1 asserts that $K(C_q, C_0) < \varepsilon$ for arbitrary $\varepsilon > 0$ and the theorem is proved.

THEOREM 5.3. *The sequence of arcs $\{C_q\}$ in Theorem 5.2 has the property that $\{b_q\}$ converges to b_0, $\{y_q(x)\}$ converges uniformly to $y_0(x)$ and $\{a_q(x)\}$ converges almost uniformly in subsequence to $a_0(x)$.*

Since $\lim_{q=\infty} K(C_q, C_0) = 0$, it follows that

$$\lim_{q=\infty} | b_q - b_0 |^2 = 0 ,$$

$$\lim_{q=\infty} \max_{e^1 \leqq x \leqq e^2} | y_q(x) - y_0(x) |^2 = 0 ,$$

and

$$\lim_{q=\infty} \int_{e^1}^{e^2} l(a_q - a_0) dx = 0 . \tag{5.1}$$

The first two of these equalities give the convergence properties of the sequences $\{b_q\}$ and $\{y_q(x)\}$ respectively. Suppose now that there is a subset S of $e^1 \leqq x \leqq e^2$ of positive measure, $m(S) > 0$, such that for any integer q_0 there is a $q > q_0$ for which $| a_q(x) - a_0(x) | > \sigma > 0$ for all x in S. Then, since $l(a_q - a_0) \geqq 0$ for all q, it follows that

$$\int_{e^1}^{e^2} l(a_q - a_0) dx \geqq \int_S l(a_q - a_0) dx > \{\sqrt{1 + \sigma^2} - 1\} m(S) > 0$$

for infinitely many q's. This contradicts equation (5.1) and the sequence $\{a_q(x)\}$ must converge in measure to $a_0(x)$ on $e^1 \leqq x \leqq e^2$. There is then a subsequence, call it $\{a_q(x)\}$, which converges almost uniformly to $a_0(x)$ on $e^1 \leqq x \leqq e^2$ and the theorem is proved.

THEOREM 5.4. *Let $\{C_q\}$ be a sequence of extended arcs having the convergence properties of the last theorem. Given a constant $\rho > 0$ there is a constant $\delta > 0$ and an integer q_0 such that if M is a subset of $e^1 \leqq x \leqq e^2$ of measure at most δ and $q \geqq q_0$ then*

$$0 \leqq \int_M l_q(x) dx < \rho$$

where $l_q(x) = l(a_q - a_0) + 2 = 1 + \sqrt{1 + | a_q - a_0 |^2}$.

By the definition of $l_q(x)$,

$$\int_M l_q(x)dx \leqq 2\delta + \int_M l(a_q - a_0)dx .$$

If q_0 is chosen so that $K(C_q, C_0) < \rho/2$ for all $q > q_0$ and δ is chosen to be $\rho/4$, the right side of the desired inequality is proved. The proof is completed by noting that $l_q(x) \geqq 0$. We have just proved that $\int_M l_q(x)dx$ is an absolutely continuous function of M uniformly with respect to q.

6. **Variations γ_q, γ_0.** Let k_q be the positive square root of $K(C_q, C_0)$ and define a variation γ_q as follows.

$$\gamma_q\colon \beta_q = \frac{b_q - b_0}{k_q}, \quad \eta_q(x) = \frac{y_q(x) - y_0(x)}{k_q}, \quad \alpha_q(x) = \frac{a_q(x) - a_0(x)}{k_q}.$$

For a sequence of arcs C_q with the property that $\lim_{q=\infty} K(C_q, C_0) = 0$ it will be shown that the sequence of variations $\{\gamma_q\}$ converges in subsequence to a variation γ_0 which is admissible on $x^1 \leqq x \leqq x^2$. From the definitions of γ_q and $K(C_q, C_0)$ it follows that

$$(6.1) \qquad |\beta_q|^2 + \max_{e^1 \leqq x \leqq e^2} |\eta_q(x)|^2 + \int_{e^1}^{e^2} \frac{|\alpha_q(x)|^2}{l_q(x)} dx = 1 .$$

Since each term is nonnegative.

$$(6.2) \qquad |\beta_q|^2 \leqq 1 ,$$

$$(6.3) \qquad \max_{e^1 \leqq x \leqq e^2} |\eta_q(x)|^2 \leqq 1 ,$$

and

$$(6.4) \qquad \int_{e^1}^{e^2} \frac{|\alpha_q(x)|^2}{l_q(x)} dx \leqq 1 .$$

Using these inequalities we shall obtain several theorems, the first of which is

THEOREM 6.1. *Let $\{C_q\}$ be a sequence of extended arcs for which $\lim_{q=\infty} K(C_q, C_0) = 0$ and $\beta_q = (b_q - b_0)/k_q$. The sequence $\{\beta_q\}$ converges in subsequence to a parameter β_0.*

This follows immediately from inequality (6.2) and the Bolzano-Weierstrass theorem.

THEOREM 6.2. *Let $\{C_q\}$ be the sequence of arcs in the previous theorem and $\alpha_q(x) = (a_q(x) - a_0(x))/k_q$. There is a function $\alpha_0(x)$ in L_2 on $e^1 \leqq x \leqq e^2$ such that the sequence $\{\alpha_q(x)\}$ converges weakly in* *subsequence to* $\alpha_0(x)$ *in* L_2 *on every measurable set* M *on which* $a_q(x)$ *converges uniformly to* a_0. *Moreover, for every bounded integrable function* $g(x)$,

$$\lim_{q=\infty}\int_{e^1}^{e^2} g(x)\alpha_q(x)dx = \int_{e^1}^{e^2} g(x)\alpha_0(x)dx \ . \tag{6.5}$$

From inequality (6.4) and the inequality of Schwarz,

$$\left|\int_M \alpha_q(x)dx\right|^2 \leqq \int_M \frac{|\alpha_q(x)|^2}{l_q(x)}dx \int_M l_q(x)dx \leqq \int_M l_q(x)dx$$

for all measurable subsets M of $e^1 \leqq x \leqq e^2$. Hence

$$\lim_{m(M)=0}\int_M \alpha_q(x)dx = 0$$

by Theorem 5.4 and $\int_M \alpha_q(x)dx$ is absolutely continuous in M uniformly with respect to q. In addition, equation (5.1) and the definition of $l_q(x)$ imply that there is an integer q_0 such that for $q > q_0$, $\int_{e^1}^{e^2} l_q(x)$ is bounded. Hence $\int_{e^1}^{e^2} |\alpha_q(x)|\,dx$ is bounded. Banach [1] proved that there is an integrable function $\alpha_0(x)$ such that the sequence $\{\alpha_q(x)\}$ satisfies equation (6.5) for all bounded integrable functions $g(x)$.

Now let M be a subset of $e^1 \leqq x \leqq e^2$ on which $\{a_q(x)\}$ converges uniformly to $a_0(x)$. For x in M there is an integer q_1 such that for $q > q_1$, $l_q(x) < 3$. Thus $\int_M |\alpha_q(x)|^2\,dx < 3$ for all $q > q_1$. Banach [1, p. 130] showed that for a sequence of functions $\{\alpha_q(x)\}$ in L_2 satisfying this last inequality, there is a function $\alpha_0(x)$ in L_2 to which $\{\alpha_q(x)\}$ converges weakly in L_2 in subsequence on M. Consequently,

$$3 \geqq \liminf_{q=\infty}\int_M |\alpha_q(x)|^2\,dx \geqq \int_M |\alpha_0(x)|^2\,dx \ .$$

Since this holds for every set M as above, we have $\int_{e^1}^{e^2} |\alpha_0(x)|^2\,dx \leqq 3$ and $\alpha_0(x)$ is in L_2 on $e^1 \leqq x \leqq e^2$. The theorem is thus proved.

THEOREM 6.3. *Let* $\{C_q\}$ *be the sequence of arcs in the previous theorem and let* $\eta_q(x) = (y_q(x) - y_0(x))/k_q$. *There exists a function* $\eta_0(x)$ *whose derivative* $\dot\eta_0(x)$ *is in* L_2 *such that the sequence* $\{\eta_q(x)\}$ *converges uniformly to* $\eta_0(x)$ *on* $e^1 \leqq x \leqq e^2$ *and* $\{\dot\eta_q(x)\}$ *converges weakly in* L_2 *to* $\dot\eta_0(x)$ *on every measurable set* M *on which* $\{a_q(x)\}$ *converges uniformly to* $a_0(x)$. *Moreover,*

$$\lim_{q=\infty}\int_{e^1}^{e^2} g(x)\dot\eta_q(x)dx = \int_{e^1}^{e^2} g(x)\dot\eta_0(x)dx$$

for every bounded measurable function g.

Applying the Lipschitz condition of condition S to equation (6.1), we get

$$|\beta_q|^2 + \max_{e\leqq x\leqq e^2} |\eta_q(x)|^2 + \frac{1}{c^2}\int_{e^1}^{e^2} \frac{|\dot{\eta}_q(x)|^2}{l_q(x)} dx \leqq 1 + \int_{e^1}^{e^2} \frac{|\eta_q(x)|^2}{l_q(x)} dx .$$

Since $\max_{e^1\leqq x\leqq e^2} |\eta_q(x)|^2 \leqq 1$ and $l_q(x) \geqq 2$,

$$\int_{e^1}^{e^2} \frac{|\eta_q(x)|^2}{l_q(x)} dx < \frac{1}{2}\int_{e^1}^{e^2} dx = \frac{1}{2}(e^2 - e^1) = c_1 ,$$

a constant. Hence,

$$|\beta_q|^2 + \max_{e^1\leqq x\leqq e^2} |\eta_q(x)|^2 + \frac{1}{c^2}\int_{e^1}^{e^2} \frac{|\dot{\eta}_q(x)|^2}{l_q(x)} dx \leqq 1 + c_1 .$$

By an argument similar to that for the sequence $\{\alpha_q(x)\}$ it follows that there is a function $\dot{\eta}_0(x)$ in L_2 to which the sequence $\{\dot{\eta}_q(x)\}$ converges weakly. Hence,

$$\lim_{q=\infty} \int_{e^1}^{x} \dot{\eta}_q(t)dt = \int_{e^1}^{x} \dot{\eta}_0(t)dt \tag{6.6}$$

uniformly on $e^1 \leqq x \leqq e^2$. Let

$$\eta_0^i(x) = C_\rho^{i1}\beta_0^\rho + \int_{x^1}^{x} \dot{\eta}_0(t)dt .$$

Since $\lim_{q=\infty} \eta_q(X^1(b_q)) = \eta_0(x^1)$, it follows from (6.6) that the sequence $\{\eta_q(x)\}$ converges uniformly to $\eta_0(x)$ on $e^1 \leqq x \leqq e^2$ and the theorem is proved.

THEOREM 6.4. *Let $\{C_q\}$ be the sequence of extended arcs for which $\lim_{q=\infty} K(C_q, C_0) = 0$ and define the variation γ_q as above. The sequence of variations $\{\gamma_q\}$ converges in subsequence to a variation γ_0 which is admissible on $x^1 \leqq x \leqq x^2$.*

Let γ_0 consist of the parameters β_0 and the functions $\eta_0(x)$, $\alpha_0(x)$ of the preceding three theorems. That γ_0 is a variation follows directly from the definition of a variation and the properties of β_0, $\eta_0(x)$, and $\alpha_0(x)$. The variation γ_0 will be admissible if it is differentially admissible and satisfies the endpoint equations in § 1. Let M_δ be a subset of $x^1 \leqq x \leqq x^2$ on which $\{a_q(x)\}$ converges uniformly to $a_0(x)$ and whose complement relative to $x^1 \leqq x \leqq x^2$ has measure less than δ, $\delta > 0$. By Taylor's theorem,

$$\dot{y}_q - \dot{y}_0 = P_{y^j}\{y_q^j - y_0^j\} + P_{a^h}\{a_q^h - a_0^h\} + R_q ,$$

the arguments of P_{y^j}, P_{a^h} being (x, y_0, a_0) and

$$|R_q| \leqq \varepsilon_q\{|y_q - y_0| + |a_q - a_0|\}$$

on M where $\varepsilon_q \to 0$ as $q \to \infty$. Then

$$\lim_{q=\infty} \int_{M\delta} \dot{\eta}_q(x)dx = \lim_{q=\infty} \int_{M\delta} \{P_{y^j}\eta_q^j + P_{a^h}\alpha_q^h\}dx + \lim_{q=\infty} \int_{M\delta} \frac{R_q}{k_q}dx .$$

Since the last integral on the right is bounded and $\varepsilon_q \to 0$ as $q \to \infty$, it follows from Theorems 6.2 and 6.3 that

$$\int_{M\delta} \dot{\eta}_0(x)dx = \int_{M\delta} \{P_{y^j}\eta_0^j + P_{a^h}\alpha_0^h\}dx$$

and γ_0 is differentially admissible. The endpoint conditions on an admissible arc yield

$$y_q^i(x^s) - y_0^i(x_0^s) = Y^{is}(b_q) - Y^{is}(b_0) .$$

Expressing the left side as $y_q(x^s) - y_0(x^s) + y_0(x^s) - y_0(x_0^s)$ and dividing by k_q we get

$$\eta_q^i(x^s) + \dot{y}_0^i(x_0'^s)X_\rho^s(b_0')\beta_q^\rho = Y_\rho^{is}(b_0')\beta_q^\rho$$

where

$$x_0'^s = x_0^s + \theta_1(x^s - x_0^s),\ 0 < \theta_1 < 1$$
$$b_0' = b_0 + \theta_2(b_q - b_0),\ 0 < \theta_2 < 1 .$$

When $q \to \infty$,

$$\eta_0^i(x_0^s) = \{Y_\rho^{is} - \dot{y}_0^i X_\rho^s\}\beta_0^\rho = C_\rho^{is}\beta_0^\rho$$

and γ_0 is admissible.

7. **Proof of the sufficiency theorem.** Two theorems involving $I^*(C_q)$ and $E_H^*(C_q)$ will be proved, then they will be used to obtain a proof of the sufficiency theorem of § 2.

THEOREM 7.1. *Let C_0 be an admissible arc on $x^1 \leqq x \leqq x^2$ satisfying condition S. If for any integer q there is an admissible arc $C_q \neq C_0$ in the $1/q$-neighborhood of C_0 such that $I(C_q) \leqq I(C_0)$ then*

$$\lim_{q=\infty} \frac{I^*(C_q) - I^*(C_0)}{k_q^2} = \frac{1}{2}I_2(\gamma_0) + \frac{1}{2}\int_{x^1}^{x^2} H_{a^h a^k}\alpha_0^h\alpha_0^k dx .$$

Applying Taylor's theorem to the right side of equation (3.1) for $I^*(C) - I^*(C_0)$ and dividing by k_q^2 we get equations (7.1) to (7.4)

$$\text{(7.1)} \qquad \frac{g(b_q) - g(b_0)}{k_q^2} = \frac{1}{k_q}g_\rho\beta_q^\rho + \frac{1}{2}g_{\rho\sigma}\beta_q^\sigma + R_{1q}$$

where $|R_{1q}| < \varepsilon_{1q} |\beta_q|^2$ and $\lim_{q=\infty} \varepsilon_{1q} = 0$. The derivatives are evaluated at $b = b_0$.

$$
(7.2)\quad \begin{aligned}
\frac{z^i(x^s)Y^{is}(b_q) - z^i(x_0^s)Y^{is}(b_0)}{k_q^2} &= \frac{1}{k_q}[\dot{z}^i Y^{is} X^s + z^i Y^{is}_\rho]_{s=1}^{s=2} \beta_q^\rho \\
&+ \frac{1}{2}[\ddot{z}^i Y^{is} X_\rho^s X_\sigma^s + \dot{z}^i\{Y_\sigma^{is} X_\rho^s + Y_\rho^{is} X_\sigma^s\} \\
&\qquad + \dot{z}^i Y^{is} X_{\rho\sigma}^s + z^i Y_{\rho\sigma}^{is}]_{s=1}^{s=2} \beta_q^\rho \beta_q^\sigma + R_{2q}
\end{aligned}
$$

where $|R_{2q}| < \varepsilon_{2q} |\beta_q|^2$ and $\lim_{q=\infty} \varepsilon_{2q} = 0$. Again the derivatives are evaluated at $b = b_0$.

$$
(7.3)\quad \begin{aligned}
\frac{1}{k_q^2}\int_{x^1}^{x^2} \{\dot{z}^i(y_q^i - y_0^i) &+ \{H(x, y_q, a_0, z) - H(x, y_0, a_0, z)\} \\
&+ (a^h - a_0^h) H_{a^h}(x, y_q, a_0, z)\} dx \\
&= \int_{x^1}^{x^2} \left\{\frac{1}{2} H_{y^i y^j} \eta_q^i \eta_q^j + H_{y^i a^h} \eta_q^i \alpha_q^h\right\} dx + \int_{x^1}^{x^2} R_{3q} dx
\end{aligned}
$$

where $|R_{3q}| < \varepsilon_{3q} |\eta_q|^2$ and $\lim_{q=\infty} \varepsilon_{3q} = 0$. The derivatives $H_{y^i y^j}$, $H_{y^i a^h}$ are evaluated along C_0.

$$
(7.4)\quad \begin{aligned}
\frac{1}{k_q^2}\int_{x^1(b_0)}^{x^1(b)} &\{\dot{z}^i y_0^i + H(x, y_0, a_0, z)\} dx \\
&= \frac{1}{k_q}\{\dot{z}^i y_0^i + H(x, y_0, a_0, z)\} X_\rho^1 \beta_q^\rho \\
&+ \frac{1}{2}\{\ddot{z}^i y_0^i + H_x + H_{a^h} \dot{a}_0^h + H_{z^i} \dot{z}^i\} X_\rho^1 X_\sigma^1 \beta_q^\rho \beta_q^\sigma \\
&+ \frac{1}{2}\{\dot{z}^i y_0^i + H\} X_{\rho\sigma}^1 \beta_q^\rho \beta_q^\sigma + R_{4q}
\end{aligned}
$$

where $|R_{4q}| < \varepsilon_{4q} |\beta_q|^2$ and $\lim_{q=\infty} \varepsilon_{4q} = 0$. All the terms on the right are evaluated along C_0 at $x = X^1(b_0)$. A result similar to this also holds for the integral remaining in the expression for $(I^*(C_q) - I^*(C_0))/k_q^2$ with R_{5q} as the error in place of R_{4q}. The definition of R_{3q} and the boundedness of $|\eta_q|^2$ yield the fact that $\lim_{q=\infty} \int_{x^1}^{x^2} R_{3q} dx = 0$. Substituting equations (7.1) to (7.4) into equation (3.1), applying condition S and a theorem of Hestenes [7, Lemma 6.3] we get the desired result.

THEOREM 7.2. *Let C_0 be an admissible arc satisfying condition S. Let $\{C_q\}$ be admissible arcs related to C_0 as described in the last theorem and chosen so that the corresponding variation γ_q defined previously converge to a variation γ_0 as described. Then*

$$
\liminf_{q=\infty} \frac{E_H^*(C_q)}{k_q^2} + \frac{1}{2}\int_{x^1(b_0)}^{x^2(b_0)} H_{a^h a^k} \alpha_0^h \alpha_0^k dx \geqq 0 .
$$

For large q, $E_H(C_q) > 0$ for $C_q \neq C_0$. Applying Taylor's theorem to $E_H(C_q)$ it follows that

$$\frac{E_H^*(C_q)}{k_q^2} \geqq -\frac{1}{2}\int_M H_{a^h a^k}(x, y_q, a_0, z)\alpha_q^h \alpha_q^k dx + \int_M R_{6q} dx \tag{7.6}$$

where M is a subset of $x^1 \leqq x \leqq x^2$ on which $\{a_q(x)\}$ converges uniformly to $a_0(x)$. Since $|R_{6q}| < \varepsilon_{6q}|\alpha_q|^2$ and $\lim_{q=\infty}\varepsilon_{6q} = 0$ it follows from the boundedness of $\int_{x^1}^{x^2}|\alpha_q|^2 dx$ that $\lim_{q=\infty}\int_M R_{6q}dx = 0$. Now

$$\begin{aligned} &-\frac{1}{2}\int_M H_{a^h a^k}(x, y_q, a_0, z)\alpha_q^h \alpha_q^k dx \\ &= -\frac{1}{2}\int_M H_{a^h a^k}(x, y_0, a_0, z)\alpha_0^h \alpha_0^k dx \\ &\quad -\frac{1}{2}\int_M \{H_{a^h a^k}(x, y_q, a_0, z) - H_{a^h a^k}(x, y_0, a_0, z)\}\alpha_q^h \alpha_q^k dx \\ &\quad -\frac{1}{2}\int_M H_{a^h a^k}(x, y_0, a_0, z)\{\alpha_q^h \alpha_q^k - \alpha_0^h \alpha_0^k\} dx\ . \end{aligned} \tag{7.7}$$

From the continuity of $H_{a^h a^k}$ and the boundedness of $\int_{x^1}^{x^2}|\alpha_q|^2 dx$ we get

$$\lim_{q=\infty}\int_M \{H_{a^h a^k}(x, y_q, a_0, z) - H_{a^h a^k}(x, y_0, a_0, z)\}\alpha_q^h \alpha_q^k dx = 0\ .$$

The last integral in equation (7.7) can be written as

$$\begin{aligned} \int_M H_{a^h a^k}\alpha_q^h \alpha_q^k dx &= \int_M H_{a^h a^k}\{\alpha_q^h - \alpha_0^h\}\{\alpha_q^k - \alpha_0^k\} dx \\ &\quad + \int_M H_{a^h a^k}\{\alpha_q^h \alpha_0^k + \alpha_0^h \alpha_q^k\} dx - \int_M H_{a^h a^k}\alpha_0^h \alpha_0^k dx\ . \end{aligned}$$

Since $\{\alpha_q(x)\}$ converges weakly to $\alpha_0(x)$ on M,

$$\begin{aligned} \liminf_{q=\infty}\int_M -\frac{1}{2}H_{a^h a^k}\alpha_q^h \alpha_q^k dx &= -\frac{1}{2}\int_M H_{a^h a^k}\alpha_0^h \alpha_0^k dx \\ &\quad + \liminf_{q=\infty}\int_M -\frac{1}{2}H_{a^h a^k}\{\alpha_q^h - \alpha_0^h\}\{\alpha_q^k - \alpha_0^k\} dx\ . \end{aligned} \tag{7.8}$$

Therefore, from (7.6), (7.7) and (7.8),

$$\begin{aligned} &\liminf_{q=\infty}\frac{E_H^*(C_q)}{k_2^q} + \frac{1}{2}\int_M H_{a^h a^k}\alpha_0^h \alpha_0^k dx \\ &\geqq \liminf_{q=\infty}\int_M -\frac{1}{2}H_{a^h a^k}\{\alpha_q^h - \alpha_0^h\}\{\alpha_q^k - \alpha_0^k\} dx\ . \end{aligned} \tag{7.9}$$

Since C_0 satisfies condition II_N with multipliers $z^i(x)$ it also satisfies the strengthened condition of Clebsch,

$$H_{a^h a^k}\pi^h\pi^k \leqq 0$$

in a neighborhood of C_0 for all $(\pi) \neq (0)$. Hence the last integral in (7.9) is nonnegative and the theorem is proved for every subset M on which $\{a_q(x)\}$ converges uniformly to $a_0(x)$. Let M_1 be the complement of M on $x^1 \leqq x \leqq x^2$. Then

$$\int_{x^1}^{x^2} H_{a^h a^k}\alpha_0^h\alpha_0^k dx = \int_M H_{a^h a^k}\alpha_0^h\alpha_0^k dx + \int_{M_1} H_{a^h a^k}\alpha_0^h\alpha_0^k dx \,.$$

Since the integrand $H_{a^h a^k}\alpha_0^h\alpha_0^k$ is integrable on $x^1 \leqq x \leqq x^2$, the last integral of the preceding equation must go to zero as the measure of M_1 tends to zero. Thus the theorem is proved over $x^1 \leqq x \leqq x^2$. We now turn to the proof of Theorem 2.1. Suppose it is false. For any integer q there is an admissible arc $C_q \neq C_0$ in the $1/q$-neighborhood of C_0 such that $I(C_q) \leqq I(C_0)$. From equation (3.2) and Theorem 7.1,

$$0 \geqq I_2(\gamma_0) + \frac{1}{2}\int_{x^1}^{x^2} H_{a^h a^k}\alpha_0^h\alpha_0^k dx + \liminf_{q=\infty} \frac{E_H^*(C_q)}{k_q^2} \tag{7.10}$$

which implies, by virtue of Theorem 7.2, that $I_2(\gamma_0) \leqq 0$. Statement (e) of condition S requires that γ_0 must be null. Consequently $I_2(\gamma_0) = 0$ and

$$\int_{x^1}^{x^2} H_{a^h a^k}\alpha_0^h\alpha_0^k dx = 0 \,.$$

By Theorem 2.2 and the inequality (7.10),

$$0 \geqq \liminf_{q=\infty} \frac{E_H^*(C_q)}{k_q^2} = h \liminf_{q=\infty} \int^{x^2} \frac{|\alpha_q|^2}{l_q(x)} dx$$

which is impossible because of equation (6.1). Hence $\gamma_0 \neq 0$ and the assumption that $I(C_q) \leqq I(C_0)$ is false. This proves the sufficiency theorem.

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