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FUNCTIONS ANALYTIC IN A FINITE DISK AND HAVING ASYMPTOTICALLY PRESCRIBED CHARACTERISTIC

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Let $f(z)$ be analytic in the region $|z| < R\ (R \leqq +\infty)$. Then in the interval $0 \leqq r < R$, Nevanlinna's characteristic

$$T(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \overset{+}{\log} |f(re^{i\theta})|\, d\theta$$

is known to be nonnegative, nondecreasing and convex in $\log r$; however, it is not known whether these properties characterize completely $T(r, f)$.

Recently, A. Edrei and W. H. J. Fuchs have investigated one aspect of this question; they have shown that if $\Lambda(r)$ is an arbitrary convex function of $\log r$ defined for $r_0 \leqq r < +\infty$ and such that $\log r = o(\Lambda(r))$ as $r \to +\infty$, then it is possible to find an entire function $f(z)$ such that

$$\text{(A)} \qquad T(r, f) \backsim \Lambda(r) \qquad (r \to +\infty),$$

except possibly for values of r belonging to an exceptional set of finite measure. In this note I establish an analogue of this result for the case of functions regular in a disk of finite radius R.

The proof of (A) in the case $R < +\infty$, as well as in the case $R = +\infty$, depends on the construction of certain infinite products which have applications to other problems of the same nature. To illustrate this fact, I use these products to find, very simply, examples of functions $F(z)$ which are bounded on $|z| < 1$ and such that the derivatives $F'(z)$ have unbounded characteristic.

The main result of this note is given by the following Theorem 1. The notion of *order* which appears in the statement of the theorem is the one introduced by R. Nevanlinna [9]: if $\Lambda(r)$ is a positive nondecreasing function defined in $0 \leqq r < R\ (< +\infty)$, the order λ of Λ(r) is

$$\lambda = \limsup_{r \to R-} \frac{\log \Lambda(r)}{\log\left(\dfrac{R}{R-r}\right)}\ .$$

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THEOREM 1. *Let $A(r)$ be a given convex function defined for $0 \leqq r < R\ (< +\infty)$ and satisfying*

$$(1) \qquad \lim_{r\to R-} \frac{A(r)}{-\log(R-r)} = +\infty .$$

Then there exists a function $f(z)$ regular in $|z| < R$ and such that

(i) *if $A(r)$ is of finite order,*

$$(2) \qquad T(r, f) \sim A(r) \qquad (r \to R-) ;$$

(ii) *if $A(r)$ has infinite order,* (2) *still holds provided r avoids an exceptional set E of intervals in $[0, R)$. The set E satisfies*

$$(3) \qquad \text{means } E(r, R) = 0\left(\frac{(R-r)^{\tau}}{A(r)}\right) \qquad (r \to R-) ,$$

where $E(r, R)$ denotes the intersection of E with (r, R) and $\tau(\geqq 1)$ is a given constant.

We have assumed convexity instead of logarithmic convexity because, for functions defined on a finite interval, these two notions are asymptotically equivalent.

In a paper as yet unpublished, J. Clunie has improved the results of [2] by eliminating the need for any exceptional sets. It seems that, with a few modifications, his ingenious argument would lead to an improvement of Theorem 1 which, in addition to removing the exceptional set E, would also replace the condition (1) by the simpler

$$\lim_{r\to R-} A(r) = +\infty .$$

The construction given here of a function $f(z)$ satisfying (2) may be of interest because of its relative simplicity, and also because with minor modifications, given in § 5 of this note, it yields a very simple solution of a problem of Bloch and Nevanlinna.

I would like to thank Professor Edrei for suggesting the problem of finding an analogue for the disk of the results in [2]. I am also indebted to Professor Edrei and Dr. G. T. Cargo for their helpful remarks about the Bloch-Nevanlinna problem, and to Dr. Clunie for allowing me to see the manuscript of his paper.

1. Preliminaries. It is clearly no restriction to assume $R = 1$ and to consider only functions $A(r)$ of the special form

$$(1.1) \qquad A(r) = \int_0^r \phi(t)dt \qquad (\phi(0) = 0) ,$$

where $\phi(t)$ is continuous, strictly increasing and unbounded. This is justified by the elementary remark that any $A(r)$ satisfying the conditions of Theorem 1 is asymptotically equivalent, as $r \to 1-$, to a function of the form (1.1) [cf. for example 12, p. 69].

We shall need:

LEMMA 1. *Let $A(r)$ and $G(r)$ be positive, continuous, increasing, unbounded functions defined for $r^* \leqq r < 1$ ($r^* \geqq 0$) and such that*

$$\lim_{r\to 1-} \frac{A(r)}{G(r)} = +\infty \,. \tag{1.2}$$

Then the function

$$\beta(r) = \inf_{r \leqq t < 1} \frac{A(t)}{G(t)} \tag{1.3}$$

has the following properties on the interval $r^* \leqq r < 1$:

(i) *it is positive, nondecreasing, continuous and unbounded;*

(ii) *the function*

$$B(r) = \frac{A(r)}{\beta(r)} \tag{1.4}$$

is unbounded and strictly increasing.

Proof. The properties (i) as well as the inequality

$$B(r) \leqq \frac{A(r)}{G(r)} \tag{1.5}$$

follow at once from the definition (1.3).

From (1.5) we deduce that $B(r)$ is unbounded. To verify that it is increasing, let r, r' satisfy

$$r^* \leqq r < r' < 1 \,.$$

By definition, for some t_1 such that $r \leqq t_1 < 1$ we have

$$G(t_1) = \frac{A(t_1)}{\beta(r)} \,. \tag{1.6}$$

If $t_1 < r'$, the relations (1.6) and (1.5) (with r replaced by r') imply

$$\frac{A(r)}{\beta(r)} \leqq G(t_1) < G(r') \leqq \frac{A(r')}{\beta(r')} \,,$$

and hence

$$B(r) < B(r') \,. \tag{1.7}$$

If $r' \leqq t_1$, then by the definition (1.3) and (1.6),

$$\beta(r') \leqq \frac{A(t_1)}{G(t_1)} = \beta(r) ,$$

which implies $\beta(r') = \beta(r)$. Thus (1.7) follows from the inequality $A(r) < A(r')$. This completes the proof of Lemma 1.

In the sequel, we shall use the symbol K to denote a positive constant depending on one or more parameters. Since most of the inequalities in §§ 2–5 are valid only for values of certain parameters $t, r, p, \cdots$ sufficiently close to some limit, it is convenient to indicate this fact by writing, immediately after the relevant inequality, $(t_0 \leqq t < 1)$, $(r_0 \leqq r < 1)$, $(p \geqq p_0)$, $\cdots$. The quantities $K, t_0, r_0, p_0, \cdots$ are not necessarily the same each time they occur.

We assume that the reader is familiar with the fundamental concepts of Nevanlinna's theory of meromorphic functions, and in particular with the symbols: $\overset{+}{\log}$, $n(r, f)$, $N(r, f)$.

2. Construction of a function $f(z)$ with $N(r, 1/f) \sim A(r)$. Let $A(r)$ be any given function of the form (1.1) such that the growth condition (1) is satisfied. Denote the order of $A(r)$ by λ, and choose a constant A such that, if λ is finite,

$$A > \lambda + 2 . \tag{2.1}$$

If $\lambda = +\infty$, we consider the arbitrary number $\tau(\geqq 1)$ which appears in (3) and require that A satisfy

$$A > 6\tau . \tag{2.2}$$

Then define a function $G(r)$ on $0 \leqq r < 1$ by

$$G(r) = \max\left\{\sqrt{A(r)}\ ,\quad A \log \frac{1}{1-r}\right\} .$$

By (1), $A(r)$ and $G(r)$ satisfy the hypotheses of Lemma 1, and hence there exists a continuous, nondecreasing, unbounded function $\beta(r)$ such that, on some interval $r^* \leqq r < 1$,

$$\beta(r) \leqq \sqrt{A(r)}\ , \tag{2.3}$$

$$\beta(r) \leqq \frac{A(r)}{-A \log (1-r)}\ , \tag{2.4}$$

and such that the function

$$B(r) = \frac{A(r)}{\beta(r)} \tag{2.5}$$

is continuous, increasing, and unbounded.

Let α be any constant such that $0 < \alpha \leqq 1$ and

$$\alpha B(r^*)\sqrt{\beta(r^*)} < 1 ,$$

and observe that the equations

$$k = \alpha B(r_k)\sqrt{\beta(r_k)} \qquad (k = 1, 2, \cdots) \tag{2.6}$$

define uniquely an increasing sequence $\{r_k\}$, with $\lim_{k\to\infty} r_k = 1$.

Next, put

$$s_k^k = \exp\left(\frac{k}{\alpha\sqrt{\beta(r_k)}}\right) = \exp\left(B(r_k)\right) \qquad (k \geqq 1)$$

and note that s_k^k increases to $+\infty$ with k, while the terms s_k form a monotone sequence converging to 1.

Denote by $[x]$ the greatest integer in x, and define new sequences $\{q_k\}$ and $\{Q_k\}$ by

$$q_k = [3ks_1s_2 \cdots s_k] \qquad (k \geqq 1) ,$$
$$Q_k = q_1 + q_2 + \cdots + q_k \qquad (k \geqq 1) .$$

The following relations are elementary consequences of the above definitions and will be taken for granted.

$$q_k > s_k^k = e^{B(r)} \qquad (k \geqq 1) , \tag{2.7}$$

$$q_{k+1} > q_k \qquad (k \geqq 1) , \tag{2.8}$$

$$\lim_{k\to\infty} \frac{q_{k+1}}{q_k} = 1 , \tag{2.9}$$

$$\lim_{k\to\infty} \frac{q_k}{Q_k} = 0 . \tag{2.10}$$

Finally, we define a sequence $\{t_k\}$ by the conditions

$$\phi(t_k) = Q_k \qquad (k = 1, 2, \cdots) , \tag{2.11}$$

it is clear that this sequence exists and is uniquely defined, with $0 < t_1 < t_2 < \cdots < 1$ and $\lim_{k\to\infty} t_k = 1$.

We consider now the formal product

$$\prod_{k=1}^{\infty}\left(1 + \left\{\frac{z}{t_k}\right\}^{q_k}\right) = f(z) \tag{2.12}$$

and establish some of its simple properties.

If ρ is any number in $t_1 \leqq \rho < 1$, we can define an integer $p = p(\rho)$ by

$$t_p \leqq \rho < t_{p+1} \,. \tag{2.13}$$

Then if $|z| = r < \rho$, we have in view of (2.8)

$$\sum_{t_k > \rho} \left|\frac{z}{t_k}\right|^{q_k} < \sum_{k=p}^{\infty} \left(\frac{r}{\rho}\right)^{q_k} \leqq \left(\frac{r}{\rho}\right)^{q_p} \frac{\rho}{\rho - r} \,. \tag{2.14}$$

These inequalities imply that the product (2.12) converges uniformly for $|z| \leqq \rho_0 (< \rho)$, and hence $f(z)$ is regular for $|z| < 1$.

It is clear from the representation (2.12) that the zeros of $f(z)$ satisfy

$$n\left(t, \frac{1}{f}\right) = \begin{cases} 0 & (0 \leqq t < t_1) \\ Q_k & (t_k \leqq t < t_{k+1}\,; \quad k \geqq 1) \,. \end{cases}$$

Using (2.9), (2.10) and (2.11), we have

$$n\left(t, \frac{1}{f}\right) \sim \phi(t) \qquad (t \to 1-) \,,$$

and hence

$$N\left(r, \frac{1}{f}\right) \sim \Lambda(r) \qquad (r \to 1-) \,. \tag{2.15}$$

We proceed to estimate the maximum modulus of $f(z)$. Let $t_1 \leqq r < \rho < 1$, $p = p(\rho)$. Then

$$\begin{aligned} \log M(r, f) &= \sum_{t_k \leqq r} \log \left(\frac{r}{t_k}\right)^{q_k} + \sum_{t_k \leqq r} \log \left(1 + \left\{\frac{t_k}{r}\right\}^{q_k}\right) \\ &\quad + \sum_{r < t_k \leqq \rho} \log \left(1 + \left\{\frac{r}{t_k}\right\}^{q_k}\right) + \sum_{t_k > \rho} \log \left(1 + \left\{\frac{r}{t_k}\right\}^{q_k}\right) \\ &\leqq N\left(r, \frac{1}{f}\right) + p \log 2 + \sum_{t_k > \rho} \log \left(1 + \left\{\frac{r}{t_k}\right\}^{q_k}\right) \,. \end{aligned} \tag{2.16}$$

Using $\log(1 + x) < x$ $(x > 0)$, (2.14) and Jensen's theorem, we obtain

$$\begin{aligned} N\left(r, \frac{1}{f}\right) &\leqq T(r, f) \\ &\leqq \log M(r, f) < N\left(r, \frac{1}{f}\right) + p + \left(\frac{r}{\rho}\right)^{q_p} \frac{\rho}{\rho - r} \,. \end{aligned} \tag{2.17}$$

3. Proof of Theorem 1 when $\Lambda(r)$ has finite order λ. Putting $\sigma = \lambda + 1/2$, the definition of order implies

$$\Lambda(r) < \frac{1}{(1 - r)^{\sigma}} \qquad (r_0 \leqq r < 1) \,. \tag{3.1}$$

Increasing, if necessary, the value of r_0, we may associate with each r in the interval $r_0 \leqq r < 1$ an integer p such that

$$q_{p-1} \leqq \frac{1}{(1-r)^{\lambda+2}} < q_p \,, \tag{3.2}$$

and then select any ρ in the interval

$$t_p \leqq \rho < t_{p+1} \,. \tag{3.3}$$

Since (3.3) coincides with (2.13), we see that the estimates following (2.13) will be valid if we can show that $r < \rho$.

By (3.2), (3.3), (2.11) and (1.1), we have

$$\frac{1}{(1-r)^{\lambda+2}} < q_p \leqq \phi(\rho) < \frac{2}{1-\rho}\int_{\rho}^{(1+\rho)/2} \phi(t)dt < \frac{2}{1-\rho} A\left(\frac{1+\rho}{2}\right),$$

so that, by (3.1)

$$\frac{1}{(1-r)^{\lambda+2}} < \frac{2^{\sigma+1}}{(1-\rho)^{\lambda+3/2}}$$

and hence

$$\frac{1-r}{1-\rho} > 2 \qquad (r_0 \leqq r < 1) \,,$$

$$r < \frac{1+r}{2} < \rho < 1 \,. \tag{3.4}$$

Using (3.2) and (3.4), we have

$$\begin{aligned} \left(\frac{r}{\rho}\right)^{q_p} \frac{\rho}{\rho - r} &< \exp\left(-\, q_p \int_r^{\rho} \frac{dt}{t}\right) \frac{2}{1-r} \\ &< \exp\left(-\, \frac{1}{(1-r)^{\lambda+2}} \frac{1-r}{2}\right) \frac{2}{1-r} = o(1) \\ &\qquad\qquad (r \to 1-) \,. \end{aligned} \tag{3.5}$$

Returning to (2.7), and using (2.9) and (3.2), we have

$$\begin{aligned} B(r_p) &< \log q_p = \log q_{p-1} + o(1) \\ &< (\lambda + 2) \log \frac{1}{1-r} + o(1) \qquad (r \to 1-) \,, \end{aligned}$$

so that by the definition (2.1)

$$B(r_p) < A \log \frac{1}{1-r} \qquad (r_0 \leqq r < 1) \,. \tag{3.6}$$

Hence (2.4) and (1.4) imply

$$A \log \frac{1}{1-r_p} \leqq \frac{\Lambda(r_p)}{\beta(r_p)} = B(r_p) < A \log \frac{1}{1-r} ,$$

and this shows that $r_p < r$. Using (2.6), (3.6) and (2.4), we have

$$\begin{aligned} p \leqq B(r_p)\sqrt{\beta(r_p)} &< A \log \frac{1}{1-r} \sqrt{\beta(r)} \\ &\leqq \sqrt{A\Lambda(r) \log \frac{1}{1-r}} \qquad (r_0 \leqq r < 1) , \end{aligned}$$

so that, in view of the growth condition (1),

$$p = o(\Lambda(r)) \qquad (r \to 1-) . \tag{3.7}$$

Theorem 1 then follows, for functions of finite order, on combining (3.5), (3.7) and (2.15) with (2.17). In fact, it is clear from (2.17) that the method yields the additional information

$$T(r, f) \sim N\left(r, \frac{1}{f}\right) \sim \log M(r, f) \sim \Lambda(r) \qquad (r \to 1-) .$$

4. **Remarks on the infinite order case.** Since the proof when $\lambda = +\infty$ proceeds in much the same way as the one given in Section 3 for finite orders, a sketch of the argument used will suffice.

The quantities ρ and p given in (2.17) are chosen as follows: For each r in $r_0 \leqq r < 1$, ρ is taken so that

$$\frac{\log \sqrt{\phi(\rho)}}{\sqrt{\phi(\rho)}} = \frac{\rho - r}{\rho} . \tag{4.1}$$

It is not hard to see that such a $\rho = \rho(r)$ exists and is unique. Then let p be the integer determined by

$$Q_p \leqq \phi(\rho) < Q_{p+1} . \tag{4.2}$$

The definitions of q_p and Q_p, together with (4.1) and (4.2), imply

$$\left(\frac{r}{\rho}\right)^{q_p} \frac{\rho}{\rho - r} = o(1) \qquad (r \to 1-) . \tag{4.3}$$

The exceptional set E needed when $\lambda = +\infty$ is defined by

$$E = \{r\colon\ p(r) > \alpha B(r)\sqrt{\beta(r)}, \quad r^* \leqq r < 1\} .$$

In view of (2.4)–(2.6), (2.17) and (4.3), $r \notin E$ implies

$$N\left(r, \frac{1}{f}\right) \leqq T(r, f) \leqq \log M(r, f) < N\left(r, \frac{1}{f}\right) + K\sqrt{\Lambda(r) \log \frac{1}{1-r}} ,$$

so that (2.15) and the growth condition (1) gives (2) for these r.

To complete the proof of Theorem 1, it remains only to show that E satisfies (3). This follows upon using (2.2), (2.3), (2.4) and (4.1) to estimate ρ, and then using this estimate and (1.1) to see that $r \in E$ implies

$$A\left(r + \frac{(1-r)^{\tau}}{A(r)}\right) > e^{\frac{1}{2}\sqrt{A(r)}} \qquad (r_0 \leqq r < 1) .$$

This relation together with Borel's growth lemma ([1], p. 19) then gives (3).

5. A solution of the Bloch-Nevanlinn problem. *If* $F(z)$ *is meromorphic in* $|z| < 1$, *does* $T(r, F) = 0(1)$ *imply* $T(r, F') = 0(1)$?

This problem was posed by Bloch and Nevanlinna [9, p. 138], and was first solved by 0. Frostman [4] who showed that the boundedness of $T(r, F)$ does not imply that of $T(r, F')$. Subsequently a number of further solutions have been given (cf. [3], [5], [6], [7], [8], [10], [11]).

Using the methods of §§ 2 and 3, we now construct a function $F(z)$ regular and bounded in the unit disk and such that $T(r, F')$ is unbounded. In view of the importance of the Dirichlet integral

$$D[F] = \iint_{|z|<1} |F'(z)|^2 d\omega$$

it might be of interest to point out that our example is such that, by choosing suitably one of the parameters involved, we can obtain $D[F]$ and $F(z)$ bounded and $T(r, F')$ unbounded.

Let $a(\geqq 2)$ be an integer, and put

$$Q_m = \sum_{k=1}^{m} a^k = \frac{a}{a-1}(a^m - 1) \qquad (m \geqq 1) , \tag{5.1}$$

$$t_m = 1 - \frac{\gamma}{Q_m} \qquad (m \geqq 1) , \tag{5.2}$$

where γ is a constant in $0 < \gamma < 1$. We shall verify that the product

$$f(z) = \prod_{m=1}^{\infty}\left(1 + \left\{\frac{z}{t_m}\right\}^{a^m}\right) \tag{5.3}$$

is analytic in the unit disk and satisfies

$$\alpha \log \frac{1}{1-r} < T(r, f) \leqq \log M(r, f) < \beta \log \frac{1}{1-r} \qquad (r_0 \leqq r < 1) , \tag{5.4}$$

where α and β are any constants such that

$$0 < \alpha < \frac{\gamma}{a}, \qquad \beta > \gamma + \frac{\log 2}{\log a}. \tag{5.5}$$

If γ and a are chosen so that $\gamma + (\log 2/\log a) < 1$, we can take $\beta < 1$. This implies that the function

$$F(z) = \int_0^z f(\zeta)d\zeta \tag{5.6}$$

is bounded on $|z| < 1$, since by (5.4)

$$|F(z)| \leqq \int_0^{|z|} M(r, f)dr = 0(1) \qquad (|z| \to 1-).$$

Further, if we take γ and a to permit $\beta < 1/2$, then $F(z)$ has bounded Dirichlet integral:

$$\int_0^{2\pi}\int_0^1 |F'(re^{i\theta})|^2 r\, dr\, d\theta < +\infty.$$

On the other hand, (5.4) shows that

$$T(r, F') > \alpha \log \frac{1}{1-r} \qquad (r_0 \leqq r < 1),$$

so that $F'(z)$ has unbounded characteristic on the unit disk.

The regularity of the product (5.3) is an immediate consequence of the definitions (5.1) and (5.2), which imply that the series $\sum |z/t_m|^{a^m}$ converges when $|z| < 1$.

To establish (5.4), let $n(t)$ denote the number of zeros of $f(z)$ in $|z| \leqq t$, and put $N(r) = \int_0^r (n(t)/t)dt$. By the definition of $n(t)$, if $t_m \leqq t < t_{m+1}$ then

$$n(t) = Q_m \leqq \frac{\gamma}{1-t} < Q_m + a^{m+1} = n(t)\left(1 + \frac{a^{m+1}}{Q_m}\right),$$

so that

$$n(t) \leqq \frac{\gamma}{1-t} < n(t)(a + o(1)) \qquad (t \to 1-). \tag{5.7}$$

Multiplying (5.7) by t^{-1} and integrating from t_1 to r yields

$$\frac{\gamma}{a + o(1)} \log \frac{1}{1-r} < N(r) < (\gamma + o(1)) \log \frac{1}{1-r} \quad (r \to 1-). \tag{5.8}$$

The first of these inequalities, together with Jensen's Theorem and (5.5), implies

$$\alpha \log \frac{1}{1-r} < T(r, f) \leqq \log M(r, f) \qquad (r_0 \leqq r < 1) .$$

The proof of the last inequality in (5.4) is similarly easy. For each r in $t_1 \leqq r < 1$ we choose the integer $p = p(r)$ given by

$$t_p \leqq r < t_{p+1} . \tag{5.9}$$

Estimating the maximum modulus of the product (5.3), we have, as in (2.16),

$$\begin{aligned} \log M(r, f) &\leqq \sum_{m=1}^{p} \log \left(1 + \left\{\frac{r}{t_m}\right\}^{a^m}\right) + \sum_{m=p+1}^{\infty} \left(\frac{r}{t_m}\right)^{a^m} \\ &< \sum_{m=1}^{p} \log \left(\frac{r}{t_m}\right)^{a^m} + p \log 2 + \sum_{m=p+1}^{\infty} \left(\frac{t_{p+1}}{t_m}\right)^{a^m} . \end{aligned} \tag{5.10}$$

From (5.2) and (5.9) it is clear that

$$a^p \leqq Q_p \leqq \frac{\gamma}{1-r} < \frac{1}{1-r} ,$$

and hence

$$p \log a < \log \frac{1}{1-r} .$$

Using this with (5.10) and putting $k = p + 1$, we obtain

$$\log M(r, f) < N(r) + \frac{\log 2}{\log a} \log \frac{1}{1-r} + \sum_{m=k}^{\infty} \left(\frac{t_k}{t_m}\right)^{a^m} ,$$

and this together with the second of the inequalities (5.8) implies

$$\log M(r, f) < \left(\gamma + \frac{\log 2}{\log a} + o(1)\right) \log \frac{1}{1-r} + \sum_{m=k}^{\infty} \left(\frac{t_k}{t_m}\right)^{a^m} \qquad (r \to 1-) . \tag{5.11}$$

To prove that $\sum_{m=k}^{\infty}(t_k/t_m)^{a^m}$ is suitably small, use (5.1) and (5.2) to see that

$$1 - a^{-n} < t_n < 1 - \frac{1}{2}\gamma a^{-n} \qquad (n \geqq n_0) . \tag{5.12}$$

Hence

$$\begin{aligned} \sum_{m=k}^{\infty} \left(\frac{t_k}{t_m}\right)^{a^m} &< K \sum_{m=k}^{\infty} t_k^{a^m} = K \sum_{m=k}^{\infty} (t_k^{a^k})^{a^{m-k}} \\ &< K \sum_{m=k}^{\infty} (t_k^{a^k})^{(m-k)} \qquad (k \geqq k_0) . \end{aligned}$$

By the second inequality in (5.12),

$$t_k^{a^k} < e^{-\gamma/2} \qquad (k \geqq k_0)$$

and hence

$$\sum_{m=k}^{\infty} \left(\frac{t_k}{t_m}\right)^{a^m} < K \sum_{n=0}^{\infty} e^{-(\gamma/2)n} = 0(1) \qquad (k \to \infty) \ . \tag{5.13}$$

Combining (5.11) and (5.13), the derivation of (5.4) is complete.

REFERENCES

1. A. Bloch, *Les fonctions holomorphes et méromorphes dans la cercle unité*, Mémorial des sciences mathématiques XX, Paris, 1926.
2. A. Edrei and W. H. J. Fuchs, *Entire and meromorphic functions with asymptotically prescribed characteristic*, Canad. J. Math., to appear.
3. H. Fried, *On analytic functions with bounded characteristic*, Bull. Amer. Math. Soc. **52** (1946), 694-699.
4. O. Frostman, *Sur les produits de Blaschke*, Kungl. fysiogr. sallsk. i Lund fohr. **12** (1942) no. 15, 169-182.
5. W. K. Hayman, *On Nevanlinna's second theorem and extensions*, Rend. Circ. Mat. Palermo (2) **2** (1953), 346-392.
6. P. B. Kennedy, *A property of bounded regular functions*, Proc. Royal Irish Acad. Sect. A **60** (1959), no. 2, 7-14.
7. ———, *A problem on bounded analytic functions*, Proc. Amer. Math. Soc. **15** (1964), 325-326.
8. A. J. Lohwater, G. Piranian and W. Rudin, *The derivative of a schlicht function*, Math. Scand. **3** (1955), 103-106.
9. R. Nevanlinna, *Le théorème de Picard-Borel et la théorie des functions méromorphes*, Paris, 1930.
10. W. Rudin, *On a problem of Bloch and Nevanlinna*, Proc. Amer. Math. Soc. **6** (1955), 202-204.
11. ———, *On the radial variation of analytic functions*, Duke Math. J. **22** (1955), 235-242.
12. A. Zygmund, *Trigonometrical series*, Warsaw, 1935.

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