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ASYMPTOTIC VALUES OF A HOLOMORPHIC FUNCTION WITH RESPECT TO ITS MAXIMUM TERM

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Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be holomorphic with radius of convergence $R(0 < R \leqq \infty)$, and let $\mu(r)$ denote the maximum term and $\nu(r)$ the central index of $f(z)$. By definition for $r > 0, \mu(r) = \max\{|a_n| r^n \mid n = 0, 1, 2, \cdots\}$ and $\nu(r) = \max\{n \mid \mu(r) = |a_n| r^n\}$ so that $\mu(r) = |a_{\nu(r)}| r^{\nu(r)}$. In previous papers we have investigated the limiting values of the quotient $\mu(r)/M(r)$ as $r \to R$. Here, as usual, $M(r)$ denotes the maximum modulus of $f(z)$. Recently Clunie and Hayman have disproved a conjecture of Erdös that if $\mu(r)/M(r)$ tends to a limit, the limit must be zero.

In this paper we consider a more general problem. There are two complex functions $\mu(z)$ and $m(z)$ which can be regarded as complex extensions of $\mu(r)$ in a natural way. We are led to investigate the limiting values of $f(z)/\mu(z)$ and $f(z)/m(z)$ along curves tending to $|z| = R$, and we call these μ and m asymptotic values. We prove that for a class of functions which are either of very slow growth, or have gap power series, there are no μ or m asymptotic values. On the other hand, for the admissible functions of Hayman, ∞ is a μ and m asymptotic value along the positive real axis, while 0 is a μ and m asymptotic value along any other path in an angle excluding the positive real axis.

Definitions. First we extend μ to a complex function by the formula

$$\mu(re^{i\theta}) = \mu(r)e^{i\nu(r)\theta} ,$$

for $r > 0$ and $0 \leqq \theta < 2\pi$. Then $\mu(r) = |\mu(re^{i\theta})|$ and $\mu(z) = |a_{\nu(|z|)}| z^{\nu(|z|)}$. We also define a "complex maximum term" $m(z)$ given by

$$m(re^{i\theta}) = \mu(r) \exp\{i\nu(r)\theta + i \arg a_{\nu(r)}\}$$

for $r > 0$ and $0 \leqq \theta < 2\pi$. Then $m(z) = a_{\nu(|z|)} z^{\nu(|z|)}$ and as before $\mu(r) = |m(re^{i\theta})|$. Note that $\mu(z)$ and $m(z)$ are continuous in each annulus where $\nu(|z|)$ is continuous, but in general have discontinuities where $\nu(|z|)$ is discontinuous.

Let $\gamma(t)$ be a (continuous) curve such that $|\gamma(t)| \to R$ as $t \to \infty$. If $f(\gamma(t))/\mu(\gamma(t))$ $(f(\gamma(t))/m(\gamma(t)))$ tends to a limit $\omega\,(0 \leqq |\omega| \leqq \infty)$ as $t \to \infty$ we say that ω is a *μ asymptotic value* (*m asymptotic value*) of $f(z)$ and $\gamma(t)$ is a corresponding *μ asymptotic path* (*m asymptotic* *path*). Further let $\gamma(t)$ be a μ(or m) asymptotic path written in polar coordinates $(r(t), \theta(t))$. Then $\gamma(t)$ is *nonessential* if and only if there exists $\varepsilon > 0$ such that for all curves of the form $(r(t), \phi(t))$ such that $|\phi(t) - \theta(t)| < \varepsilon$ for sufficiently large t we have that $(r(t), \theta(t))$ is a μ(or m) asymptotic path with the same μ(or m) asymptotic value as $(r(t), \theta(t))$. Otherwise $\gamma(t)$ is *essential*. Note that f has μ(or m) asymptotic value ∞ (0) if and only if $|f(z)|/\mu(|z|) \to \infty$ (0) along a curve $\gamma(t)$. Also if $a_n \geqq 0$ for sufficiently large n then the μ and m asymptotic values are the same, as are the μ and m asymptotic paths.

Let $\{\rho(n)\}$ be the sequence of jump points of $\nu(r)$, counting multiplicity, and assume throughout this paper that $\mu(r) \to \infty$ as $r \to R$ (so that $\rho(n) \to R$ as $n \to \infty$). This last assumption avoids triviality and implies that if $a_n \geqq 0$ for $n > n_0$ then $\mu(z) = m(z)$ for $|z|$ sufficiently near R. We denote by $\{n_k\}$ the range of $\nu(r)$ (so that $\nu(\rho(n_k)) = n_k$) and we define:

$$L = \limsup_{k\to\infty} \rho(n_{k+1})/\rho(n_k) .$$

$$S = \limsup_{k\to\infty} (n_{k+1} - n_k) .$$

$$\left.\begin{matrix} A \\ \mathrm{a} \end{matrix}\right\} = \begin{matrix} \limsup \\ k\to\infty \inf \end{matrix} \frac{n_{k+1} - n_k}{n_k - n_{k-1}} .$$

$$\left.\begin{matrix} \Phi \\ \phi \end{matrix}\right\} = \begin{matrix} \limsup \\ k\to\infty \inf \end{matrix} (n_{k+1} - n_k) \log\left(\frac{\rho(n_{k+1})}{\rho(n_k)}\right) .$$

THEOREM 1. *If $L > 1$ and $S < \infty$, then $f(z)$ has no μ or m asymptotic values. (The hypothesis $L > 1$ implies $f(z)$ is a transcendental entire function.)*

2. Statement of theorems.

THEOREM 2. *Suppose $0 < \phi = \Phi < \infty$ and $1 \leqq a = A < \infty$, and that $f(z)$ has the form*

$$f(z) = 1 + \sum_{k=1}^{\infty} \frac{z^{n_k}}{\rho(1)\cdots\rho(n_k)} .$$

Then $f(z)$ has no μ asymptotic values.

Next suppose that $f(z)$ is real for real z and $f(r) > 0$ for $R_0 < r < R$. Let

$$a(r) = \frac{rf'(r)}{f(r)} \quad \text{and} \quad b(r) = ra'(r) .$$

Following Hayman [9] we call $f(z)$ *admissible* if $b(r) \to +\infty$ as $r \to R$

and there exists a function $\delta(r)$ defined for $R_0 < r < R$ and satisfying $0 < \delta(r) < \pi$ such that

$$f(re^{i\theta}) \sim f(r)e^{i\theta a(r)-(1/2)\theta^2 b(r)} \tag{2.1}$$

as $r \to R$ uniformly for $|\theta| \leqq \delta(r)$; while uniformly for $\delta(r) \leqq |\theta| \leqq \pi$,

$$f(re^{i\theta}) = O\left(\frac{f(r)}{\sqrt{b(r)}}\right) \quad \text{as} \quad r \to R\,. \tag{2.2}$$

THEOREM 3. (i) *For admissible functions the positive real axis is an essential μ asymptotic path with μ asymptotic value ∞. Any path in an angle outside the positive real axis is a nonessential μ asymptotic path with μ asymptotic value* 0.

(ii) *Let $f(z)$ be an admissible entire function satisfying the condition*

$$a(r) - \nu(r) = O\left(\frac{b(r)}{\log b(r)}\right)^{1/2} \quad as \quad r \to \infty\ , \tag{2.3}$$

and let $0 < c < \infty$. Then c is a μ asymptotic value of $f(z)$ along the curve whose equation in polar coordinates is $(r, \phi(r))$ where

$$\phi(r) = \{b(r)^{-1} \log 2\pi c^{-2} b(r)\}^{1/2}\,.$$

(iii) *Let $g(z) = f(e^{i\psi}z)$ where $0 < \psi < 2\pi$ and $f(z)$ is an admissible entire function for which* (2.3) *is satisfied and $\nu(r)$ assumes every integer as a value. If $0 < c < \infty$, then c is a m asymptotic value of $g(z)$ along the curve $(r, \phi(r) - \psi)$ but $g(z)$ has no μ asymptotic values other than* 0 *and* ∞.

In §7 we give some examples of functions illustrating our theorems.

3. LEMMA 1. (cf. [6], [7]). *Let $1 \leqq r \leqq (\rho(n_{k+1})/\rho(n_k))$. Then*

$$\frac{M(\rho(n_k)r)}{\mu(\rho(n_k)r)} \geqq \frac{\pi}{4}(1 + r^{n_{k-1}-n_k})\,.$$

Proof. Let $\mu(\rho(n_k)) = |a_{n_k}|\rho(n_k)^{n_k} = |a_{n_{k-1}}|\rho(n_k)^{n_{k-1}}$. Then

$$a_{n_k}\{\rho(n_k)re^{i\theta}\}^{n_k} + a_{n_{k-1}}\{\rho(n_k)re^{i\theta}\}^{n_{k-1}}$$
$$= \frac{1}{2\pi i}\int_{|\xi|=\rho(n_k)r} \frac{f(\xi)}{\xi}\left\{\left(\frac{\rho(n_k)re^{i\theta}}{\xi}\right)^{n_k} + \left(\frac{\rho(n_k)re^{i\theta}}{\xi}\right)^{n_{k-1}}\right\} d\xi\,.$$

Hence

$$|a_{n_k}\{\rho(n_k)re^{i\theta}\}^{n_k} + a_{n_{k-1}}\{\rho(n_k)re^{i\theta}\}^{n_{k-1}}|$$

$$\leqq \frac{M(\rho(n_k)r)}{2\pi}\int_0^{2\pi}|1+e^{i\phi}|\,d\phi = \frac{4M(\rho(n_k)r)}{\pi}\ .$$

Choose $\theta = -(n_k - n_{k-1})^{-1}(\arg a_{n_k} - \arg a_{n_{k-1}})$. Then

$$\mu(\rho)(n_k)r)(1 + r^{n_{k-1}-n_k}) \leqq \frac{4M(\rho(n_k)r)}{\pi}\ .$$

and the lemma follows.

LEMMA 2. (cf. [9; pp. 71, 83]). *Let $K > 1$ and $0 < c < \infty$. If $f(z)$ is admissible we may assume that $\delta(r)$ satisfies*

$$\left\{\frac{\log 2\pi c^{-2}b(r)}{b(r)}\right\}^{1/2} \leqq \delta(r) \leqq \left\{\frac{K\log b(r)}{b(r)}\right\}^{1/2}$$

for $r(c) < r < R$.

Proof. The first inequality must always be satisfied. Indeed admissibility implies

$$\frac{|f(re^{i\delta(r)})|}{f(r)} \sim \exp\left\{-\frac{1}{2}b(r)\delta(r)^2\right\} = O(b(r)^{-1/2})\ .$$

Hence $\exp\{1/2\, b(r)\delta(r)^2\} \geqq c^{-1}(2\pi b(r))^{1/2}$ for $r(c) < r < R$, and this is equivalent to the first inequality.

For the second inequality suppose $f(z)$ is admissible with a function $\delta_1(r)$. Let $\delta(r) = \min\{\delta_1(r), (K\log b(r)/b(r))^{1/2}\}$. We show that $f(z)$ is admissible with $\delta(r)$. Let $\delta(r) \leqq |\theta| \leqq \delta_1(r)$. Then by (2.1)

$$\frac{b(r)^{1/2}|f(re^{i\theta})|}{f(r)} \sim b(r)^{1/2}\exp\left\{-\frac{1}{2}b(r)\theta^2\right\} \leqq b(r)^{1/2-K/2} = O(1)\ .$$

This is equivalent to (2.2). Thus we may replace $\delta_1(r)$ by $\delta(r)$ without destroying the truth of (2.2).

4. **Proof of Theorem 1.** Without loss of generality, we may assume $f(0) = 1$. Let $1 < \alpha < \beta < L_1 < L$, and $\alpha < (\pi/(4-\pi))^{1/S}$. There exists a sequence $\{k_p\} = \{k(p)\}$ of integers such that $\rho(n_{k(p)}+1)/\rho(n_{k(p)}) > L_1$. Then if $\phi_p(w) = f(\rho(n_{k(p)})w)/\mu(\rho(n_{k(p)})w)$ for $w \in \Omega_1 = \{w \mid 1 < |w| < L_1\}$, have, writing $n_{k_p} = n$, (cf. [6])

$$|f(\rho(n)w)| \leqq 1 + \sum_{k=1}^{\infty}\frac{\rho(n)^k|w|^k}{\rho(1)\cdots\rho(k)}\ ,$$

and

$$\mu(\rho(n)|w|) = \frac{\rho(n)^n|w|^n}{\rho(1)\cdots\rho(n)}\ .$$

Hence

$$(4.1)\quad \begin{aligned} |\phi_p(w)| &\leqq \frac{\rho(1)\cdots\rho(n)}{\rho(n)^n\,|w|^n}\left\{1+\sum_{k=1}^{\infty}\frac{\rho(n)^k\,|w|^k}{\rho(1)\cdots\rho(k)}\right\} \\ &= 1+\sum_{j=1}^{\infty}\frac{\rho(n)^j\,|w|^j}{\rho(n+1)\cdots\rho(n+j)}+\sum_{j=-n}^{0}\frac{\rho(n+j+1)\cdots\rho(n)}{\rho(n)^{-j}\,|w|^{-j}} \\ &\leqq 1+\sum_{j=1}^{\infty}\left(\frac{|w|}{L_1}\right)^j+\sum_{j=1}^{\infty}|w|^{-j}\,. \end{aligned}$$

Therefore $\{\phi_p(w)\}$ is uniformly bounded on compact subsets of Ω_1 and so it is a normal family. Thus there is a subsequence of $\{\phi_p(w)\}$ which converges uniformly on compact subsets of Ω_1. We may therefore assume that $\{k(p)\}$ has been so chosen that $\{\phi_p(w)\}$ itself converges uniformly on compact subsets of Ω_1 to a holomorphic function $G(w)$.

We shall show that $G(w)$ is nonconstant, for suppose $G(w)\equiv C$ on Ω_1 The constant term in the Laurent expansion of $\phi_p(w)$ about the origin is 1, and so for $1<r<L_1$ we would have

$$C=\frac{1}{2\pi i}\int_{|w|=r}\frac{G(w)}{w}\,dw=\lim_{p\to\infty}\frac{1}{2\pi i}\int_{|w|=r}\frac{\phi_p(w)}{w}\,dw=1\,.$$

Thus $G(w)\equiv 1$ on Ω_1. But by the lemma

$$M(r,G)=\lim_{p\to\infty}M(r,\phi_p)\geqq\frac{\pi}{4}(1+r^{-S})\,,\quad\text{for}\quad 1<r<L_1\,.$$

In particular for $r=\alpha$ we have $M(\alpha,G)>1$. Hence $G(w)$ must be nonconstant.

Let $\Omega=\{w\,|\,\alpha\leqq|w|\leqq\beta\}$ and suppose that $f(z)$ has a asymptotic value ω. Then there exists a curve $\gamma(t)$ with $|\gamma(t)|\to\infty$ as $t\to\infty$ such that $f(\gamma(t))/\mu(\gamma(t))\to\omega$ as $t\to\infty$.

There exists an unbounded set I with the following property: for each $t\in I$ there is a unique integer p such that

$$\rho(n_{k(p)})\leqq|\gamma(t)|<\rho(n_{k(p)}+1)\,.$$

Write $\gamma(t)=\rho(n_{k(p)})\gamma_p(t)$; then $1\leqq|\gamma_p(t)|\leqq L+o(1)$, so $\{\gamma_p(t)\}$ is bounded. We now consider the set T of limit points of $\{\gamma_p(t)\}$ as $t\to\infty$, $t\in I$, which lie in Ω and prove they are an uncountable set on which $G(w)$ is constant. In fact, let Σ be the intersection of Ω with the positive real axis, and define $\chi:\Sigma\to T$ as follows. For each $x\in\Sigma$, there exists $t_p\in I$ such that $|\gamma(t_p)|=\rho(n_{k(p)})x$; then $|\gamma_p(t_p)|=x$. Choose a limit point v of $\{\gamma_p(t_p)\}$, and define $\chi(x)=v$. Then χ is one-one since $|\chi(x)|=x$. Thus T is uncountable, since Σ is.

Furthermore $G(w)$ is constant on T, for suppose $\gamma_p(t_s)\to b\in T$ for a sequence $\{t_s\}$ with $t_s\in I$. By virtue of uniform convergence $\phi_p(\gamma_p(t_s))\to G(b)$. But we are assuming ω is a μ asymptotic value and so $G(b) = \omega$. Hence G is constant on T. This is a contradiction; therefore G has no μ asymptotic values.

For m asymptotic values we define

$$\psi_p(w) = \frac{f(\rho(n_{k(p)})w)}{m(\rho(n_{k(p)})w)} \qquad \text{for} \quad w \in \Omega ,$$

and we still have (4.1) holding with ϕ_p replaced by ψ_p. Thus $\{\psi_p(w)\}$ is a normal family and the rest of the proof goes through in exactly the same manner as for μ asymptotic values.

5. **Proof of theorem 2.** Since $f(z)$ has positive coefficients we need only consider μ asymptotic values. We again suppose that $f(z)$ has μ asymptotic value ω. Let $\gamma(t)$ be a μ asymptotic path corresponding to ω. For a given t take m to be the unique integer for which $\rho(n_m) \leqq |\gamma(t)| < \rho(n_{m+1})$ and define $\gamma_m(t) = C + iD$ where

$$\gamma(t) = \rho(n_m) \exp\left(\frac{\phi C}{n_{m+1} - n_m} + iD\right)$$

and $0 \leqq D < 2\pi$. It is easy to see that $0 \leqq Re\, \gamma_m(t) \leqq 1 + o(1)$ so that $\{\gamma_m(t)\}$ is bounded.

Now write $P_m(w) = f(z)/\mu(z)$ where $z = \rho(n_m) \exp(\phi w/(n_{m+1} - n_m))$. Then [8] $P_m(w)$ tends uniformly on $\Lambda = \{w \mid 0 \leqq Re\, w \leqq \beta\}$, $1/2 < \beta < 1$, to a nonconstant analytic function $Q(w)$ as $m \to \infty$. For completeness we sketch a proof of this. We have

$$1 \leqq \exp\left\{\frac{\phi Re\, w}{n_{m+1} - n_m}\right\} = \left|\exp\left(\frac{\phi w}{n_{m+1} - n_m}\right)\right| .$$

For sufficiently large m

$$0 \leqq \phi Re\, w \leqq \phi\beta < (n_{m+1} - n_m) \log \frac{\rho(n_{m+1})}{\rho(n_m)} ,$$

and so

$$\rho(n_m) \leqq \rho(n_m) \left|\exp\left(\frac{\phi w}{n_{m+1} - n_m}\right)\right| = |z| < \rho(n_{m+1}) .$$

Hence

$$\nu(|z|) = n_m,\ \mu(z) = \frac{z^{n_m}}{\rho(1) \cdots \rho(n_m)} .$$

Write

$$\sigma^{(j)}(m) = \begin{cases} \rho(n_{m+1})^{n_{m+1} - n_m} \cdots \rho(n_{m+j})^{n_{m+j} - n_{m+j-1}} & , \ j > 0 \\ 1 & , \ j = 0 \\ \{\rho(n_m)^{n_m - n_{m-1}} \cdots \rho(n_{m+j+1})^{n_{m+j+1} - n_{m+j}}\}^{-1} & , \ j < 0 . \end{cases}$$

Then

$$\frac{f(z)}{\mu(z)} = \sum_{j=-m}^{\infty} \frac{\rho(n_m)^{n_{m+j}-n_m}}{\sigma^{(j)}(m)} \exp\left\{\frac{n_{m+j}-n_m}{n_{m+1}-n_m}\phi w\right\}. \tag{5.1}$$

Since $1 \leqq a \leqq A < \infty$, and $\phi > 0$, there exist numbers A_1, A_2, ϕ_1 so that

$$\left.\begin{aligned} &0 < A_1 < \frac{n_{m+1}-n_m}{n_m - n_{m-1}} < A_2 < \infty\,, \\ &0 < \phi_1 < (n_{m+1}-n_m)\log\frac{\rho(n_{m+1})}{\rho(n_m)} \end{aligned}\right\} \quad \text{for } m = 1, 2, \cdots.$$

Let $j \geqq 2$. Then

$$\begin{aligned} &(n_{m+j}-n_m)\log\rho(n_{m+1}) - \log\sigma^{(j)}(m) \\ &\qquad = -\sum_{q=2}^{j}(n_{m+j}-n_{m+q-1})\log\frac{\rho(n_{m+q})}{\rho(n_{m+q-1})} \\ &\qquad \leqq -\sum_{q=2}^{j}(n_{m+q}-n_{m+q-1})\log\frac{\rho(n_{m+q})}{\rho(n_{m+q-1})} \\ &\qquad \leqq -(j-1)\phi_1\,. \end{aligned}$$

Similarly we have for $-j = k \geqq 2$, $(n_{m+j}-n_m)\log\rho(n_m) - \log\sigma^{(j)}(m) \leqq (j+1)\phi_1/A_2$. Hence

$$\left|\frac{z^{n_{m+j}-n_m}}{\sigma^{(j)}(m)}\right| \leqq \begin{cases} e^{-(j-1)\phi_1}\,, & j \geqq 2 \\ 1\,, & -2 < j < 2 \\ e^{(j+1)\phi_1/A_2}\,, & j \leqq -2\,. \end{cases}$$

Hence by the Weierstrass M-test the series (5.1) converges uniformly in both m and w. Hence we have

$$\lim_{m\to\infty}\frac{f(z)}{\mu(z)} = \sum_{-\infty}^{\infty}\lim_{m\to\infty}\left\{\frac{\rho(n_m)^{n_{m+j}-n_m}}{\sigma^{(j)}(m)}\exp\left(\frac{n_{m+j}-n_m}{n_{m+1}-n_m}\phi w\right)\right\}. \tag{5.2}$$

Further for $j > 0$

$$\begin{aligned} &(n_{m+j}-n_m)\log\rho(n_m) - \log\sigma^{(j)}(m) \\ &\qquad = -\sum_{q=1}^{j}\sum_{p=q}^{j}\left(\prod_{s=q}^{p-1}\frac{n_{m+s+1}-n_{m+s}}{n_{m+s}-n_{m+s-1}}\log\left(\frac{\rho(n_{m+q})}{\rho(n_m)}\right)^{n_{m+q}-n_{m+q-1}}\right) \end{aligned}$$

and so

$$\begin{aligned} &\lim_{m\to\infty}\{(n_{m+j}-n_m)\log\rho(n_m) - \log\sigma^{(j)}(m) \\ &\qquad = \begin{cases} -\dfrac{1}{2}j(j+1)\phi & \text{if } A = a = 1 \\ -\dfrac{(a^{j+1}-a(j+1)+j)\phi}{(a-1)^2} & \text{if } 1 < a = A < \infty\,. \end{cases} \end{aligned} \tag{5.3}$$

A similar argument shows that (5.3) is valid when $j < 0$. Hence we have from (5.2)

(5.4)

$$\lim_{m\to\infty}\frac{f(z)}{\mu(z)}=\begin{cases}\sum_{-\infty}^{\infty}\exp\left\{-\frac{\phi j}{2}(j+1-2w)\right\} & \text{when } A=a=1\,,\\ \sum_{-\infty}^{\infty}\exp\left\{-\frac{\phi}{(a-1)^2}(a^{j+1}-(j+1)a+j-(a-1)(a^j-1)w)\right\}\,, & \text{when } A=a>1.\end{cases}$$

It can be easily verified that the two expressions on the right of (5.4) are not constant.

Just as in Theorem 1, we now can prove that the set T of limit points of $\{\gamma_m(t)\}$ is uncountable, and that $Q(w)$ is constant on T, contradicting the fact that $Q(w)$ is nonconstant on Λ. Hence $f(z)$ has no μ asymptotic values.

6. **Proof of Theorem 3.** (i) We may assume by Lemma 2 that $\delta(r) = o(1)$. Furthermore according to [9] $a_n > 0$ for $n > n_0$, and so we need only consider μ asymptotic values. We have [9; pp. 68–69]

$$\frac{f(re^{i\theta})}{\mu(re^{i\theta})} \sim \sqrt{2\pi b(r)}\exp\left\{i(a(r)-\nu(r))\theta - \frac{1}{2}\theta^2 b(r)\right\} \tag{6.1}$$

uniformly for $|\theta| \leqq \delta(r)$, and

$$\frac{f(re^{i\theta})}{\mu(re^{i\theta})} = o(1) \text{ uniformly for } \delta(r) \leqq |\theta| \leqq \pi\,. \tag{6.2}$$

It is immediate from (6.2) that any path in an angle outside the real axis has μ asymptotic value 0 and is nonessential. From (6.1) we have

$$\frac{f(r)}{\mu(r)} \sim \sqrt{2\pi b(r)}\,,$$

and so the positive real axis has μ asymptotic value ∞. To prove that it is essential it suffices to show that there exists a curve $(r, \phi(r))$ (in polar coordinates) such that for each $\varepsilon > 0$ there exists $r(\varepsilon)$ for which $r > r(\varepsilon)$ implies $|\phi(r)| < \varepsilon$, and $(r, \phi(r))$ does not have μ asymptotic value ∞. We take

$$\phi(r) = \{b(r)^{-1}\log(2\pi c^{-2}b(r))\}^{1/2}\,,$$

where $0 < c < \infty$. Then by Lemma 2, $|\phi(r)| \leqq \delta(r)$ for $r > r(c)$ and

$$\left|\frac{f(re^{i\phi(r)})}{\mu(re^{i\phi(r)})}\right| \sim \sqrt{2\pi b(r)}\, e^{-(1/2)\phi(r)^2 b(r)} = c\,,$$

so that $(r, \phi(r))$ cannot have μ asymptotic value ∞.

(ii) If (2.3) is satisfied for $f(z)$ then $(a(r) - \nu(r))\phi(r) = o(1)$, and so

$$\frac{f(re^{i\phi(r)})}{\mu(re^{i\phi(r)})} \sim \sqrt{2\pi b(r)}\, e^{-(1/2)\phi(r)^2 b(r)} = c\,.$$

(iii) We have

$$\frac{g(re^{i(\phi(r)-\psi)})}{m(re^{i(\phi(r)-\psi)}, g)} = \frac{f(re^{i\phi(r)})}{\mu(re^{i\phi(r)}, f)} \sim c\,,$$

so that c is a m asymptotic value of $g(z)$ along $(r, \phi(r) - \psi)$. However

$$\frac{g(re^{i(\theta-\psi)})}{\mu(re^{i(\theta-\psi)}, g)} \sim e^{i\psi\nu(r)} \sqrt{2\pi b(r)}\, e^{i(a(r)-\nu(r))\theta - 1/2 b(r)\theta^2}$$

uniformly for $0 \leqq |\theta| \leqq \delta(r)$. Since $\nu(r)$ assumes every integer as a value, g can have no μ asymptotic values other than 0 and ∞.

7. Examples. (i) Theorem 1 shows that $\sum_{n=0}^{\infty} \lambda^{-(1/2)n(n+1)} e^{i\alpha_n} z^n$, where $1 < \lambda < \infty$ and $0 \leqq \alpha_n < 2\pi$, has no μ or m asymptotic values. Here $\rho(n) = \lambda^n$ and $L = \lambda$. Similarly it follows from Theorem 2 that if $0 < \alpha < \infty$ the functions

$$\sum_{k=0}^{\infty} \frac{z^{k^2}}{(k^2!)^\alpha}\,, \quad \sum_{k=0}^{\infty} \frac{z^{k^2}}{\Gamma(1 + \alpha k^2)}\,,$$

and $\sum_{k=0}^{\infty} z^{k^2}/k^{2\alpha k^2}$ have no μ or m asymptotic values. For each of these functions $\phi = 4\alpha$.

(ii) The function e^z is admissible with $a(r) = b(r) = r$ and $\nu(r) = [r]$, so Theorem 3 (i), (ii) apply to it. More generally the Mittag-Leffler function

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1 + \alpha n)} \quad (0 < \alpha < 2)$$

is admissible with $a(r) = \alpha^{-1} r^{\alpha^{-1}} + o(1)$, $b(r) = \alpha^{-2} r^{\alpha^{-1}} + o(1)$, and $\nu(r) = \alpha^{-1} r^{\alpha^{-1}} + O(1)$, so that $a(r) - \nu(r) = O(1)$. These facts follow from

$$E_\alpha(z) - e^{z^{1/\alpha}} = O(1/z) \text{ for } |\arg z| \leqq \frac{1}{2}\alpha\pi \qquad [1;\ \text{p. } 175]\,.$$

If $f(z)$ is admissible so is $e^{f(z)}$ [9].

(iii) Let $L_\beta(z) = \sum_{n=0}^{\infty} \{z/(\log(n + \beta))\}^n$ where $\beta > 1$. It is known [3; p. 346] that *$L_\beta(z)$ tends to zero on every ray except the real axis, where it tends to* ∞. Hence $e^{-L_\beta(z)}$ has μ and m asymptotic value 0 along every ray from 0 to ∞.

(iv) Theorem 3 (ii), (iii) show that every positive real number is an m asymptotic value of some function. If b is any complex number we can construct a function having b as a μ asymptotic value. We take for example

$$f(z, b) = e^{iz^2} + \frac{b}{z\sqrt{2\pi}}(e^{z^2} - 1) .$$

For $r > (|b|)/(\sqrt{2\pi})$ we have $\mu(r,f) = \mu(r, e^{iz^2}) \sim (e^{r^2})/(r\sqrt{2\pi})$. It follows easily that $\lim_{r\to\infty} f(r)/\mu(r) = b$.

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Added in proof. An application of these results may be found in the authors' paper "*Asymptotic values of holomorphic functions of irregular growth,*" Bull. Amer. Math. Soc. 71 (1965), 747–749.

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