PACIFIC JOURNAL OF MATHEMATICS
Vol. 18, No. 1, 1966

FRATTINI SUBGROUPS AND Φ-CENTRAL GROUPS

HOMER BECHTELL

Φ-central groups are introduced as a step in the direction of determining sufficiency conditions for a group to be the Frattini subgroup of some finite p-group and the related extension problem. The notion of Φ-centrality arises by uniting the concept of an E-group with the generalized central series of Kaloujnine. An E-group is defined as a finite group G such that $\Phi(N) \leqq \Phi(G)$ for each subgroup $N \leqq G$. If $\mathscr{H}$ is a group of automorphisms of a group N, N has an $\mathscr{H}$-central series $N = N_0 > N_1 > \cdots > N_r = 1$ if $x^{-1}x^{\alpha} \in N_j$ for all $x \in N_{j-1}$, all $\alpha \in \mathscr{H}$, x^{α} the image of x under the automorphism $\alpha \in \mathscr{H}$, $j = 0, 1, \cdots, r-1$.

Denote the automorphism group induced on $\Phi(G)$ by transformation of elements of an E-group G by $\mathscr{H}$. Then $\Phi(\mathscr{H}) = \mathscr{I}(\Phi(G))$, $\mathscr{I}(\Phi(G))$ the inner automorphism group of $\Phi(G)$. Furthermore if G is nilpotent, then each subgroup $N \leqq \Phi(G)$, N invariant under $\mathscr{H}$, possess an $\mathscr{H}$-central series. A class of nilpotent groups N is defined as Φ-central provided that N possesses at least one nilpotent group of automorphisms $\mathscr{H} \neq 1$ such that $\Phi(\mathscr{H}) = \mathscr{I}(N)$ and N possesses an $\mathscr{H}$-central series. Several theorems develop results about Φ-central groups and the associated $\mathscr{H}$-central series analogous to those between nilpotent groups and their associated central series. Then it is shown that in a p-group, Φ-central with respect to a p-group of automorphism $\mathscr{H}$, a nonabelian subgroup invariant under $\mathscr{H}$ cannot have a cyclic center. The paper concludes with the permissible types of nonabelian groups of order p^4 that can be Φ-central with respect to a nontrivial group of p-automorphisms.

Only finite groups will be considered and the notation and the definitions will follow that of the standard references, e.g. [6]. Additionally needed definitions and results will be as follows: The group G is the *reduced partial product* (or reduced product) of its subgroups A and B if A is normal in $G = AB$ and B contains no subgroup K such that $G = AK$. For a reduced product, $A \cap B \leqq \Phi(B)$, (see [2]). If N is a normal subgroup of G contained in $\Phi(G)$, then $\Phi(G/N) \cong \Phi(G)/N$, (see [5]). An elementary group, i.e., an E-group having the identity for the Frattini subgroup, splits over each of its normal subgroups, (see [1]).

1. For a group G, $\Phi(G) = \Phi$, $G/\Phi = F$ is *Φ-free* i.e., $\Phi(F)$ is the identity. The elements of G by transformation of Φ induce auto-

morphisms $\mathscr{H}$ on Φ. Denoting the centralizer of Φ in G by M, $G/M \cong \mathscr{H} \leqq \mathscr{A}(\Phi)$, $\mathscr{A}(\Phi)$ the automorphism group of Φ. Then if $\mathscr{I}(\Phi)$ denotes the inner automorphisms of Φ, one has $\mathscr{I}(\Phi) \leqq \Phi(\mathscr{H})$ and by a result of Gaschütz [5, Satz 11], $\mathscr{I}(\Phi)$ normal in $\mathscr{A}(\Phi)$ implies that $\mathscr{I}(\Phi) \leqq \Phi(\mathscr{A}(\Phi))$.

Supposing first that $M \not\leqq \Phi$, there exists a reduced product $G = MK$ such that $M \cap K \leqq \Phi(K)$ and $M\Phi(K)/M \cong \Phi(G/M) \cong \Phi(\mathscr{H})$. Moreover $M\Phi/\Phi \cong A \leqq F$. Thus A is normal in F and the elements in A correspond to the identity transformation on Φ. Thus $F/A \cong G/M\Phi$ corresponds to a subgroup of outer automorphisms of Φ, namely $F/A \cong \mathscr{H}/\mathscr{I}(\Phi)$. Since F is Φ-free, there exists a reduced product $F = AB$ such that $A \cap B \leqq \Phi(B)$ and $F/A \cong B/A \cap B$. By combining these latter statements, $\mathscr{H}/\mathscr{I}(\Phi) \cong B/A \cap B$. Moreover $\Phi(B/A \cap B) \cong \Phi(B)/A \cap B \cong \Phi(\mathscr{H}/\mathscr{I}(\Phi)) = \Phi(\mathscr{H})/\mathscr{I}(\Phi)$, i.e., $\Phi(B)/A \cap B \cong \Phi(\mathscr{H})/\mathscr{I}(\Phi)$. However note that if $\Phi(K) \leqq \Phi(G)$, then $M\Phi(K)/M \leqq M\Phi(G)/M \cong \mathscr{I}(\Phi) \leqq \Phi(\mathscr{H})$. Thus $\mathscr{I}(\Phi) = \Phi(\mathscr{H})$.

Now suppose that $M \leqq \Phi$. Then $\Phi(G/M) \cong \Phi(G)/M \cong \Phi(\mathscr{H})$. Since $M = Z$, Z the center of Φ, and $\Phi/Z \cong \mathscr{I}(\Phi)$ again it follows that $\mathscr{I}(\Phi) = \Phi(\mathscr{H})$.

LEMMA 1. *A necessary condition that a group N be the Frattini subgroup of an E-group G is that $\mathscr{A}(N)$ contains a subgroup $\mathscr{H}$ such that $\Phi(\mathscr{H}) = \mathscr{I}(\Phi)$.*

COROLLARY 1.1. *A necessary and sufficient condition that the centralizer of Φ in an E-group G be the center of Φ is that $G/\Phi \cong \mathscr{H}/\mathscr{I}(\Phi)$.*

Using the notation of the above, $G/M\Phi \cong \mathscr{H}/\mathscr{I}(\Phi) \cong T \leqq F$. However $G/\Phi \cong F$ elementary implies $F = ST$, $S \cap T = 1$, S normal in F and $F/S = T$. Then:

THEOREM 1. *Necessary conditions that a nilpotent group N be the Frattini subgroup of an E-group G is that $\mathscr{A}(N)$ contains a subgroup $\mathscr{H}$ such that*

(1) *$\Phi(\mathscr{H}) = \mathscr{I}(N)$, and*

(2) *there exists an extension of N to a group M such that $M/N \cong \mathscr{H}/\mathscr{I}(N)$.*

A sufficiency condition may well be lacking since M/N elementary only implies that $\Phi(M) \leqq N$; equality is not implied.

Let K denote a normal subgroup of an E-group G such that $\Phi < K \leqq G$ and that M is the G-centralizer of K. If $M \not\leqq \Phi$ but $M\Phi < K$ properly, $M\Phi/\Phi \not\cong \mathscr{I}(K)$. On the other hand $K < M\Phi$ implies $M\Phi/M \cong$

$\Phi/M\cap\Phi\cong\Phi/K\cap M\cong(K\cap M)\Phi/K\cap M\cong\mathscr{I}(K)$. Thus $\Phi(G/M)\cong\Phi(\mathscr{H})=\mathscr{I}(K)$, $\mathscr{H}$ the group of automorphisms of K induced by transformation of elements in G. Summarizing:

THEOREM 2. *If K is a subgroup normal in an E-group G, $\Phi < K \leqq G$, and M is the G-centralizer of K, then $\Phi(\mathscr{H}) = \mathscr{I}(K)$, $\mathscr{H}$ the group of automorphisms of K induced by transformation of elements of G, if and only if $K \leqq M\Phi$, i.e., $K = \Phi Z$, Z the center of K.*

On the other hand if K is a subgroup of Φ, the following decomposition of Φ is obtained:

THEOREM 3. *If a subgroup K of Φ normal in an E-group G has an automorphism group $\mathscr{H}$ induced by transformation of elements of G with $\Phi(\mathscr{H}) = \mathscr{I}(K)$, then $\Phi = KB$, B the centralizer of K in Φ, and $K\cap B \neq 1$ unless $K = 1$. $\mathscr{K}$ denotes the automorphism group of B induced by transformation of elements of G, then $\Phi(\mathscr{K}) = \mathscr{I}(B)$.*

Proof. Denote the G-centralizer of K by M. Then $G/M \cong \mathscr{H}$ and $MK/M \cong K/Z \cong \mathscr{I}(K)$, Z the center of K. Since the homomorphic image of an E-group is an E-group, then $\mathscr{H}$ is an E-group and $\mathscr{H}/\mathscr{I}(K)$ is an elementary group. Hence G/KM is an elementary group which implies that $\Phi \leqq KM$ since G is an E-group. $B = M\cap\Phi$ is normal in G and it follows that $\Phi = KB$. Since K is nilpotent, the center of K exists properly unless G is an elementary group.

Symmetrically K is contained in the G-centralizer J of B. Then as above JB/J is mapped into $\Phi(\mathscr{K})$ and since G/JB is elementary, the mapping is onto, i.e., $JB/J \cong \Phi(\mathscr{K}) \cong B/J\cap B \cong \mathscr{I}(B)$.

REMARK 1. Note that in Theorem 3, each subgroup K contained in the center of Φ and normal in G satisfies the condition $\Phi(\mathscr{H}) = \mathscr{I}(K)$ and so $\mathscr{H}$ is an elementary group.

For normal subgroups N of a nilpotent group G, transformation by elements of G on N induce a group of automorphisms $\mathscr{H}$ for which a series of subgroups exist, $N = N_0 > N_1 > \cdots > N_r = 1$, such that $x^{-1}x^a \in N_i$, $a \in \mathscr{H}$, $x \in N_{i-1}$. Following Kaloujnine [8], N is said to have an *$\mathscr{H}$-central series.* In general E-groups do not have this property on the normal subgroups except in the trivial case of $\mathscr{H}$ the identity mapping. If N is nilpotent and $\mathscr{I}(N) \leqq \mathscr{H}$ then the series can be refined to a series for which $|N_{i-1}/N_i|$ is a prime integer.

A group N does not necessarily have an $\mathscr{H}$-central series for each subgroup $\mathscr{H} \leqq \mathscr{A}(N)$ even if N is nilpotent. For example if N is the quaternion group and $\mathscr{H}$ is $\mathscr{A}(N)$, N has only one proper characteristic subgroup.

Combining Lemma 1 with the above one has:

THEOREM 4. *A necessary condition that a group N be the Frattini subgroup of a nilpotent group G is that $\mathscr{A}(N)$ contains a nilpotent subgroup $\mathscr{H}$ such that*

(1) *$\Phi(\mathscr{H}) = \mathscr{I}(N)$ and*

(2) *N possesses an $\mathscr{H}$-central series.*

The dihedral group N of order eight has an $\mathscr{H}$-central series for $\mathscr{H} = \mathscr{A}(N)$, however $|\Phi(\mathscr{H})| = 2$ and $|\mathscr{I}(N)| = 4$. There are Abelian groups which trivially satisfy (1) but not (2). So both conditions are necessary.

A Φ-central group N will be defined as a nilpotent group possessing at least one nilpotent group of automorphisms $\mathscr{H} \neq 1$ such that

(1) $\Phi(\mathscr{H}) = \mathscr{I}(N)$ and

(2) N possesses an $\mathscr{H}$-central series.

Φ-central groups have the following properties:

THEOREM 5.

(1) *If N is Φ-central with respect to an automorphism group $\mathscr{H}$, M a subgroup of N invariant under $\mathscr{H}$, and S a subgroup of N/M invariant under $\mathscr{H}^*$, $\mathscr{H}^*$ the group of automorphisms induced on N/M by $\mathscr{H}$, then there exists a subgroup K of N containing M, invariant under $\mathscr{H}$, with $K/M \cong S$. Moreover $\mathscr{H}^* \cong \mathscr{H}/\mathscr{M}$, $\mathscr{M}$ the set of all $a \in \mathscr{H}$ such that $x^{-1}x^a \in M$.*

(2) *If N is Φ-central with respect to an automorphism group $\mathscr{H}$ and M is a member of the $\mathscr{H}$-central series, then N/M is Φ-central with respect to $\mathscr{H}^*$, $\mathscr{H}^*$ the group of automorphisms induced on N/M by $\mathscr{H}$.*

Proof. The proof of (1) relies on the fact that the groups considered are nilpotent and $\mathscr{I}(N) \leqq \mathscr{H}$. The only additional comment necessary for (2) is that under a homomorphic mapping of a nilpotent group the Frattini subgroup goes onto the Frattini subgroup of the image (see [2]).

THEOREM 6. *Let N be a group Φ-central under an automorphism group $\mathscr{H}$. If M is a subgroup of N invariant under $\mathscr{H}$ then*

(1) *M possesses an $\mathscr{H}$-central series,*

(2) *M possesses a proper subgroup of fixed points under $\mathscr{H}$, and*

(3) *M can be included as a member of an $\mathscr{H}$-central series of N.*

Proof. As Kaloujnine [8] has introduced, a *descending $\mathscr{H}$-central chain* can be defined by $N = N_0 \geqq N_1 \geqq \cdots \geqq N_j \geqq \cdots$ for $N_j = [N_{j-1}, \mathscr{H}]$, $[N_{j-1}, \mathscr{H}]$ the set of $x^{-1}x^a$ for all $x \in N_{j-1}$, $a \in \mathscr{H}$. A series occurs if for some integer r, $N_r = 1$. Analogous to the corresponding proofs for nilpotent groups, a group possessing an $\mathscr{H}$-central series, possesses a descending $\mathscr{H}$-central series, M possesses a proper subgroup of fixed points under, $\mathscr{H}$ (the set corresponds to a generalized center of N relative to $\mathscr{H}$) and M can be included as a member of an $\mathscr{H}$-central series of N. However M may not necessarily be a Φ-central group.

Even though the notion of Φ-centrality is derived from the properties of the Frattini subgroup of a nilpotent group, it is not a sufficient condition for group extension purposes e.g., consider the extension of cyclic group of order three to the symmetric group on three symbols.

Since $\Phi(K)$ for a nilpotent group K is the direct product of the Frattini subgroup of the Sylow p-subgroups of K (see Gaschütz [5, Satz 6]), then the determination of the nilpotent groups N which can be the Frattini subgroup of some nilpotent group G reduces to the consideration of G as a p-group. The next section discusses several properties of Φ-central p-groups.

2. *Only p-groups and their p-groups of automorphisms will be considered.*

LEMMA 2. (Blackburn [3].) *If M is a group invariant under a group of automorphisms $\mathscr{H}$ and N is a subgroup of M of order p^2 invariant under $\mathscr{H}$, then $\mathscr{H}$ possesses a subgroup $\mathscr{M}$ of index at most p under which N is a fixed-point set.*

Proof. $\mathscr{H}$ is homomorphic to a p-group of $\mathscr{A}(N)$ and $|\mathscr{A}(N)| = p(p-1)$ since $\mathscr{H}$ is a p-group. The kernel has index at most p.

LEMMA 3. *A group N, Φ-central under the automorphism group $\mathscr{H}$, can contain no nonabelian subgroup M of order p^3 and invariant under $\mathscr{H}$.*

Proof. If M is invariant under $\mathscr{H}$, then M contains a subgroup K of order p^2 invariant under $\mathscr{H}$ by Theorem 6. By Lemma 2, $\mathscr{H}$ possesses a subgroup $\mathscr{M}$ of index at most p under which K is a fixed-point set. Since $\mathscr{H}$ contains $\mathscr{I}(N)$, $K \leqq Z(N)$, $Z(N)$ the center of N. Consequently $K \leqq Z(M)$, M must be Abelian, and so a contradiction.

COROLLARY 3.1. (Hobby [7].) *No nonabelian p-group of order p^3 can be the Frattini subgroup of a p-group.*

Proof. Denote the induced group of automorphisms on $\Phi(G)$ by the elements of a p-group G by $\mathscr{H}$. Then $\Phi(G)$ is Φ-central under $\mathscr{H}$.

COROLLARY 3.2. *Each Frattini subgroup of order greater than* p^3 *of a* p*-group* G *contains an Abelian subgroup* N *of order* p^3 *and normal in* G.

LEMMA 4. *Let* N *be a group* Φ*-central under an automorphism group* $\mathscr{H}$. *A noncyclic Abelian subgroup* M *of* N, *invariant under* $\mathscr{H}$ *and having order* p^3 *contains an elementary Abelian subgroup* K *of order* p^2, *invariant under* $\mathscr{H}$ *and a fixed-point set for* $\mathscr{I}(N)$.

Proof. If M is invariant under $\mathscr{H}$ and elementary Abelian, M contains an elementary Abelian subgroup K of order p^2 and invariant under $\mathscr{H}$ by Theorem 6. On the other hand if M is invariant under $\mathscr{H}$ and of the form $\{x, y \mid x^{p^2} = x^p = 1\}$, the characteristic subgroup $K = \{x^p, y\}$ in M has order p^2 and is invariant under $\mathscr{H}$. In either case K is invariant under a subgroup $\mathscr{M}$ of index at most p by Lemma 2. The result follows since $\Phi(\mathscr{H}) = \mathscr{I}(N) \leqq \mathscr{M}$.

COROLLARY 4.1. *A noncyclic Abelian normal subgroup* M *of a* p*-group* G, $|M| = p^3$, *and* $M \leqq \Phi(G)$, *contains an elementary Abelian subgroup* N *of order* p^2, *normal in* G, *and contained in the center of* $\Phi(G)$.

THEOREM 7. *Let* N *denote a group* Φ*-central under an automorphism group* $\mathscr{H}$. *Each nonabelian subgroup* M *of* N, *invariant under* $\mathscr{H}$, *contains an elementary Abelian subgroup* K *of order* p^2 *which is invariant under* $\mathscr{H}$ *and is a fixed-point set under* $\mathscr{I}(N)$.

Proof. Suppose M is a nonabelian subgroup of least order for which the theorem is not valid. By Lemma 3, $|M| \geqq p^4$. Since $\Phi(M) \neq 1$, denote by P the cyclic subgroup of order p, consisting of fixed-points under $\mathscr{H}$ and contained in $\Phi(M)$. One such subgroup always exists by Theorem 6. Then $M/P \leqq N/P$, both are invariant under $\mathscr{H}^*$, and N/P is Φ-central under $\mathscr{H}^*$, $\mathscr{H}^*$ the induced automorphisms on N/P by $\mathscr{H}$.

If M/P is Abelian, then M/P not cyclic implies that the elements of order p in M/P form a characteristic subgroup K/P, invariant under $\mathscr{H}^*$, which is elementary Abelian and $|K/P| \geqq p^2$. Thus K/P contains a subgroup L/P of order p^2 and invariant under $\mathscr{H}^*$. This implies that L is a noncyclic commutative subgroup invariant under $\mathscr{H}$ by Lemma 3.

For M/P nonabelian, M/P contains an elementary Abelian subgroup

L/P of order p^2 invariant under $\mathscr{H}^*$ by the induction hypothesis. Again Lemma 3 implies that L of order p^3 is a noncyclic commutative subgroup invariant under $\mathscr{H}$.

By Lemma 4, K exists for L and hence for M in both cases.

COROLLARY 7.1. *A nonabelian subgroup invariant under $\mathscr{H}$ of a group N, Φ-central under an automorphism group $\mathscr{H}$, cannot have a cyclic center.*

COROLLARY 7.2. *A nonabelian normal (characteristic) subgroup of a p-group G that is contained in $\Phi(G)$ cannot have a cyclic center.*

REMARK 2. Corollary 7.2 is stronger than the results of Hobby [7, Theorem 1, Remark 1] and includes a theorem of Burnside [2] that no nonabelian group whose center is cyclic can be the derived group of a p-group. Together with Lemma 5, the results, as necessary conditions, prove useful in determining whether or not a p-group could be the Frattini subgroup of a given p-group.

LEMMA 5. *Let N denote a group Φ-central under an automorphism group $\mathscr{H}$. An Abelian subgroup $M \leqq N$ of type $(2, 1)$ and invariant under $\mathscr{H}$, is contained in the center of N.*

Proof. The result holds for N Abelian so consider the case of N nonabelian. If $M = \{x, y \mid x^{p^2} = y^p = 1\}$, then as in Lemma 4, $\{x^p, y\}$ is invariant under $\mathscr{H}$ and is contained in the center of N. Since M contains only p cyclical subgroups of order p^2 and $x^a \neq x^j$ for an integer j and $a \in \mathscr{H}$, it follows that x^a has at most p images under $\mathscr{H}$. Therefore the subgroup $\mathscr{M}$ of $\mathscr{H}$ having x as a fixed point has index at most p in $\mathscr{H}$. Since $\Phi(\mathscr{H}) = \mathscr{I}(N) \leqq \mathscr{M}$, then x is fixed by $\mathscr{I}(N)$ i.e., x is in the center of N.

COROLLARY 5.1. *An Abelian subgroup M of type $(2, 1)$, normal in a p-group G, and contained in $\Phi(G)$ is contained in the center of $\Phi(G)$.*

THEOREM 8. *The following two types of nonabelian groups of order p^4 cannot be Φ-central groups with respect to a nontrivial p-group of automorphisms $\mathscr{H}$:*

(1) $A = \{x, y, z \mid x^{p^2} = y^p = z^p = 1, [x, z] = y, [x, y] = [y, z] = 1\}$.

(2) $B = \{x, y \mid x^{p^2} = y^{p^2} = 1, [x, y] = x^p\}$.

Proof. Consider (1) and note that each element of order p^2 is of the form $z^a x^b y^c$ for $b \neq 0 \pmod{p^2}$. Then $(z^a x^b y^c)^p = x^{pb} y^{pc+ab(1+2+\cdots+(p-1))} = x^{pb} y^{abp(p-1)/2} = x^{pb}$ for $(b, p) = 1$. Thus $\{x^p\} = A^p$ is characteristic in A of order p. If A was Φ-central with respect to an automorphism group $\mathscr{H}$ then A/A^p would be Φ-central with respect to the automorphism group $\mathscr{H}^*$ induced on A/A^p by $\mathscr{H}$. This contradicts Lemma 3 if $\mathscr{H}$ is nontrivial and so A cannot be Φ-central with respect to a nontrivial p-group of automorphisms $\mathscr{H}$.

Each maximal subgroup in (2) is Abelian, of order p^3, and type $(2, 1)$. If B was Φ-central under a nontrivial p-group of automorphisms $\mathscr{H}$ then one of these maximal subgroups, say M, is invariant under $\mathscr{H}$. By Lemma 5, M is contained in the center of B and thus B is Abelian. So B cannot be Φ-central with respect to a nontrivial p-group of automorphisms $\mathscr{H}$.

COROLLARY 8.1. *The types* (1) *and* (2) *of p-groups of Theorem* 8 *cannot be Frattini subgroups of p-groups.*

REMARK 3. The remaining two types of nonabelian p-groups are of the forms

(3) $\{x, y, z \mid x^{p^2} = y^p = z^p = 1, [x, z] = x^p, [y, x] = [y, z] = 1\}$ and

(4) $\{x, y, z, w \mid x^p = y^p = z^p = w^p = 1, [z, w] = x, [y, w] = [x, w] = 1\}$.

Without attempting a classification it is sufficient to show the existence of p-groups G having $\Phi(G)$ of form (3) or (4). For $p > 5$, the group $G = \{x, y, z, w \mid x^{p^2} = y^{p^2} = z^p = w^p = 1, \quad [y^p, z] = [y^p, x] = [x, w] = 1, [y^p, w] = [x, z] = [z, w] = x^p, [z, y] = y^p, [x, y] = z, [w, y] = x\}$, $|G| = p^6$, and $\Phi(G)$ is of the form (3). Then for $p = 5$, $G = \{u, v, w, x, y, z \mid u^p = v^p = w^p = x^p = y^p = z^p = 1, [v, w] = [v, x] = [v, z] = [x, y] = 1, [v, y] = [x, w] = [w, y] = u, [w, z] = v, [x, z] = w, [y, z] = x\}$, $|G| = p^6$, and $\Phi(G)$ is of type (4).

Groups G of order p^6 other than those given in Remark 3 exist having nonabelian $\Phi(G)$. However for all such cases $\Phi(G)$ contains a characteristic subgroup N of order p^2 such that G/N is not of form (3) nor (4) i.e., G cannot be the Frattini subgroup of any p-group. Remark 3 provides a ready source of examples of p-groups which are Φ-central, or in particular are Frattini subgroups of some p-group. This offsets the conjecture that such a source consisted of p-groups of relatively "large" order. The examples raise the following question: If the group F is the Frattini subgroup of a group G, does there always exist a group G^* such that $\Phi(G^*) \cong F$ and the centralizer of $\Phi(G^*)$ in G^* is the center of $\Phi(G^*)$?

BIBLIOGRAPHY

1. H. Bechtell, *Elementary groups*, Trans. Amer. Math. Soc. (to appear).
2. ———, *Reduced partial products*, Amer. Math. Monthly, Notes Section (to appear).
3. N. Blackburn, *On prime-power groups*, Proc. Camb. Phil. Soc. **53** (1957), 19-27.
4. W. Burnside, *On some properties of groups whose orders are powers of primes*, Proc. Lond. Math. Soc. (2) **11** (1912), 225-245.
5. W. Gaschütz, *Über die Φ-Untergruppen endlicher Gruppen*, Math. Z. **58** (1953), 160-170.
6. M. Hall, *The theory of groups*, Macmillan, New York, 1959.
7. C. Hobby, *The Frattini subgroup of a p-group*, Pacific J. Math. **10** (1960), 209-212.
8. L. Kaloujnine, *Über gewisse Beziehungen zwischen einer Gruppen und ihren Automorphismen*, Berliner Mathematische Tagung, 1953, 164-172.

Received February 13, 1965.

BUCKNELL UNIVERSITY
LEWISBURG, PENNSYLVANIA