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FIXED-POINT THEOREMS FOR FAMILIES OF CONTRACTION MAPPINGS

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Let X be a nonempty, bounded, closed and convex subset of a Banach space B. A mapping $f: X \to X$ is called a *contraction mapping* if $\| f(x) - f(y) \| \leqq \| x - y \|$ for all $x, y \in X$. Let $\mathfrak{F}$ be a nonempty commutative family of contraction mappings of X into itself. The following results are obtained.

(i) Suppose there is a compact subset M of X and a mapping $f_1 \in \mathfrak{F}$ such that for each $x \in X$ the closure of the set $\{f_1^n(x): n = 1, 2, \cdots\}$ contains a point of M (where f_1^n denotes the n^{th} iterate, under composition, of f_1). Then there is a point $x \in M$ such that $f(x) = x$ for each $f \in \mathfrak{F}$.

(ii) If X is weakly compact and the norm of B strictly convex, and if for each $f \in \mathfrak{F}$ the f-closure of X is nonempty, then there is a point $x \in X$ which is fixed under each $f \in \mathfrak{F}$. A third theorem, for finite families, is given where the hypotheses are in terms of weak compactness and a concept of Brodskii and Milman called normal structure.

Fixed-point theorems for families of continuous linear (or affine) transformations have been obtained by Kakutani [6], Markov [8], Day [2], and others. Recently De Marr [3] proved the following fixed-point theorem: If X is a nonempty, compact, convex subset of a Banach space B and if $\mathfrak{F}$ is a nonempty family of commuting contraction mappings of X into itself, then the family $\mathfrak{F}$ has a common fixed point in X. In Theorem 1 of this paper hypotheses of a type considered by Göhde in [5] are used to obtain a generalization of De Marr's result.

Throughout this paper we shall denote the *diameter* of a subset $A \subseteqq B$ by $\delta(A)$, i.e.,

$$\delta(A) = \sup\{\| x - y \|: x, y \in A\} .$$

THEOREM 1. *Let X be a nonempty, bounded, closed, convex subset of a Banach space B; let M be a compact subset of X. Let $\mathfrak{F}$ be a nonempty commutative family of contraction mappings of X into itself with the property that for some $f_1 \in \mathfrak{F}$ and for each $x \in X$ the closure of the set $\{f_1^n(x): n = 1, 2, \cdots\}$ contains a point of M. Then there is a point $x \in M$ such that $f(x) = x$ for each $f \in \mathfrak{F}$.*

Proof. Let K be a nonempty closed convex subset of X such that $f(K) \subseteqq K$ for each $f \in \mathfrak{F}$. Select a point $x \in K$. Since $f(K) \subseteqq K$, we have $\{f_1^n(x)\} \subseteqq K$. Hence it follows that

$$K \cap M \supseteqq \overline{\{f_1^n(x)\}} \cap M \neq \varnothing .$$

Thus we may apply Zorn's Lemma to obtain subset X^* of X which is minimal with respect to being nonempty, closed, convex and mapped into itself by each $f \in \mathfrak{F}$. Let $M^* = X^* \cap M$; from the above remarks we know $M^* \neq \varnothing$. By a theorem of Göhde [5, p. 54], f_1 has a nonempty fixed-point set H in M^*. Since H is the set of all fixed-points of f_1, it is closed. Let $x \in H$ and $y = f(x)$. Then we have

$$f_1(y) = f_1[f(x)] = f[f_1(x)] = f(x) = y$$

since the set $\mathfrak{F}$ is commutative and x is a fixed-point of f_1. Hence $y \in H$ and $f(H) \subseteqq H$ for each $f \in \mathfrak{F}$. We are therefore able to find a subset H^* of H which is minimal with respect to being nonempty, closed and mapped into itself by each $f \in \mathfrak{F}$.

Let $g \in \mathfrak{F}$. Since H^* is compact and g continuous, $g(H^*)$ is closed. For each $f \in \mathfrak{F}$, $f[g(H^*)] = g[f(H^*)] \subseteqq g(H^*)$. Thus if $g(H^*)$ is a proper subset of H^* for some $g \in \mathfrak{F}$, then the minimality of H^* is contradicted. Hence H^* is mapped *onto* itself by each member of $\mathfrak{F}$. Let W denote the convex closure of H^*. Since H^* is compact, so is W. If $\delta(W) > 0$ it follows (see De Marr [3; Lemma 1]) that there is a point $x \in W$ such that

$$\sup\{\| x - z \| : z \in W\} = r < \delta(W) .$$

We shall show that this leads to a contradiction and thereby conclude that $\delta(W) = 0$. Thus, let

$$C_1 = \{w \in W\colon \| w - z \| \leqq r \quad \text{for all} \quad z \in H^*\} ,$$
$$C_2 = \{w \in X^*\colon \| w - z \| \leqq r \quad \text{for all} \quad z \in H^*\} .$$

Clearly $C_1 = C_2 \cap W$. Since H^* is mapped onto itself by each member of $\mathfrak{F}$, it is easily seen $f(C_2) \subseteqq C_2$ for each $f \in \mathfrak{F}$. Since C_2 is a nonempty closed convex subset of X^*, the minimality of X^* implies $C_2 = X^*$. Therefore $C_1 = W$. But since $\delta(H^*) = \delta(W)$ there are points $x,\ y \in H^*$ such that $\| x - y \| > r$. However $H^* \subseteqq W = C_1$ implies $\| x - y \| \leqq r$. This contradiction shows $\delta(W) = 0$ and H^*(hence X^*) consists of a single point which must be fixed under each mapping in $\mathfrak{F}$.

That De Marr's theorem follows from the above is evident.

The following definition may be found in [4].

DEFINITION. Let X be a nonempty subset of a Banach space B and let $f\colon X \to X$ be a contraction. The *f-closure* of X, denoted by X^f, is the set of points $y \in B$ such that for some $x \in X$ a subsequence of $\{f^n(x)\}$ converges to y.

THEOREM 2. *Suppose X is a nonempty, weakly compact, convex subset of a Banach space B whose norm is strictly convex. Suppose $\mathfrak{F}$ is a nonempty commutative family of contraction mappings of X into itself such that for each $f \in \mathfrak{F}$, $X^f \neq \emptyset$. Then there is an $x \in X$ such that $f(x) = x$ for each $f \in \mathfrak{F}$.*

Proof. It follows from a result of Edelstein [4; p. 441, II] that each member of $\mathfrak{F}$ has a nonempty fixed point set in X. (Although the mappings in [4] are defined on the entire Banach space the same results can be obtained when the domain is restricted as in this theorem.) Because the norm of B is strictly convex, and the mappings considered are contractions, it is easily seen that each of these fixed-point sets is convex (and closed). As closed convex subsets of the weakly compact set X, they are themselves weakly compact. Thus we need only show that these fixed-point sets have the finite intersection property to conclude that there is a point common to all of them.

We make the inductive assumption that each n members of $\mathfrak{F}$ have a common fixed-point in X. Let $f_1, f_2, \cdots, f_{n+1} \in \mathfrak{F}$. Let M be the set of common fixed points of $f_1, \cdots, f_n$. Then M is weakly compact and if $y \in M$, $f_i[f_{n+1}(y)] = f_{n+1}[f_i(y)] = f_{n+1}(y)$ for each $i = 1, 2, \cdots, n$. Hence $f_{n+1}(y) \in M$ and $f_{n+1}(M) \subseteqq M$. Let y be a point of X fixed under f_{n+1}. The strict convexity of the norm together with the weak compactness of M enable us to obtain a unique point $x \in M$ nearest to y. Since f_{n+1} is a contraction it then follows that $f_{n+1}(x) = x$. Thus x is a common fixed point of $f_1, \cdots, f_{n+1}$. The proof is now complete.

The concept defined below was first introduced by Brodskii and Milman in [1].

DEFINITION. A bounded convex set K in a Banach space B is said to have *normal structure* if for each convex subset H of K which contains more than one point there is a point $x \in H$ which is not a diametral point of H, (i.e. sup $\{\| x - y \|: y \in H\} < \delta(H)$).

By replacing strict convexity of the norm by normal structure and removing the requirement that $X^f \neq \emptyset$ we obtain the following theorem for weakly compact sets X. Unfortunately, we have only been able to establish this theorem for finite families (or, of course, finitely generated families) of commuting contractions.

THEOREM 3. *Suppose X is a nonempty, weakly compact, convex subset of a Banach space B and suppose that X has normal structure. If $\mathfrak{F}$ is a finite family of commuting contraction mappings of X into itself then there is an $x \in X$ such that $f(x) = x$ for each $f \in \mathfrak{F}$.*

That this theorem holds if $\mathfrak{F}$ consists of a single mapping follows from [7]. However, we take this opportunity to establish a slightly more general result which also serves our purpose.

THEOREM 4. *Let X be a bounded, closed, convex subset of a Banach space B and suppose that X has normal structure. Let M be a weakly compact subset of X. Assume f is a contraction mapping of X into itself with the property that for each $x \in X$, the closure of $\{f^n(x): n = 1, 2, \cdots\}$ contains a point of M. Then there is an $x \in M$ such that $f(x) = x$.*

Proof of Theorem 4. Since closed and convex subsets of X are weakly closed and since M is weakly compact, Zorn's lemma gives us a subset X^* of X which is minimal with respect to being nonempty, closed, convex, mapped into itself by f, and having points in common with M. By normal structure, if $\delta(X^*) > 0$ then there is a point $x \in X^*$ such that

$$\sup\{\| x - z \|: z \in X^*\} = r < \delta(X^*) .$$

Assume, then, that $\delta(X^*) > 0$. Let

$$C = \{z \in X^*: \| z - y \| \leqq r \quad \text{for each} \quad y \in X^*\} .$$

Then C is nonempty. Let K denote the convex closure of $f(X^*)$. Since $K \subseteqq X^*$, then $f(K) \subseteqq f(X^*)$. The closure of $f(X^*)$ is contained in K and the hypotheses on f imply that this set intersects M. Hence $M \cap K \neq \emptyset$. By the minimality of X^* we conclude that $K = X^*$. Let

$$C_1 = \{z \in X^*: \| z - y \| \leqq r \quad \text{for all} \quad y \in f(X^*)\} .$$

Clearly $C \subseteqq C_1$. But if $z \in C_1$ then any spherical ball of radius r centered at z must contain $f(X^*)$, and hence it must contain $K = X^*$. Consequently $C_1 \subseteqq C$, and therefore $C_1 = C$.

Let $z \in C$ and $y \in f(X^*)$. Then $y = f(x)$ for some $x \in X^*$ and we have

$$\| f(z) - y \| = \| f(z) - f(x) \| \leqq \| z - x \| \leqq r .$$

Therefore $f(C) \subseteqq C$. This implies, by the minimality of X^*, that $C = X^*$. But $\delta(C) \leqq r < \delta(X^*)$. This contradiction shows that $\delta(X^*) = 0$. Therefore X consists of a single point which must be fixed under f.

We now return to Theorem 3.

Proof of Theorem 3. Suppose $\mathfrak{F} = \{f_1, f_2, \cdots, f_n\}$. Since X is weakly compact we can find a subset X^* of X minimal with respect to being nonempty, closed, convex and mapped into itself by each element of $\mathfrak{F}$. Let W denote the set of points of X^* fixed under $f_1 f_2 \cdots f_n$. By Theorem 4, $W \neq \emptyset$. Furthermore $f_i(W) = W$ for $i = 1, 2, \cdots, n$. Let H be the convex closure of W. By normal structure H contains a point x such that

$$\sup \{\| x - z \| : z \in H\} = r < \delta(H)$$

provided $\delta(H) > 0$. As before, we assume $\delta(H) > 0$ and obtain a contradiction. Let

$$C = \{x \in X^* : \| x - z \| \leqq r \text{ for all } z \in H\} .$$

Then C is a nonempty closed convex subset of X^* and, moreover,

$$C = \{x \in X^* : \| x - z \| \leqq r \quad \text{for each} \quad z \in W\} .$$

Thus $f_i(C) \subseteqq C$ and $C = X^*$, which is impossible since $\delta(C \cap H) \leqq r < \delta(H)$. Hence $\delta(H) = 0$, so H consists of the desired fixed point.

Several questions remain unanswered, the most notable perhaps being:

(1) Is Theorem 2 true with strict convexity deleted?

(2) Is Theorem 3 true with the hypothesis of normal structure deleted?

The answers to these questions are not even known in the case that $\mathfrak{F}$ consists of a single mapping (cf. [4], [7]).

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