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DUALITY AND TYPES OF COMPLETENESS IN LOCALLY CONVEX SPACES

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The purpose of this paper is to extend and sharpen a result of Grothendieck concerning dual properties of complete locally convex topological vector spaces. Among other things, this leads to a rough dual characterization of sequential completeness and to the definition of a new type of completeness, which is studied briefly.

A dual characterization of completeness has been obtained in various forms. The first such was obtained by Grothendieck [3], who showed that the completion $\tilde{E}$ of E is the set of all linear functionals ℓ on the dual E' of E whose restriction to each equicontinuous subset Q of E' is continuous in the the topology induced on Q by the weak-* topology. Ptak [8] and Collins [1] have proven essentially equivalent results to the effect that ℓ is in $\tilde{E}$ if and only if its null-space is relatively closed in every Q. Both of these approaches raise the following question: the open and closed sets required for ℓ in the various Q's are given by a relatively small subset X of E, and we should expect some relationship to exist between this subset and ℓ. Luxemburg [7] has exhibited a partial answer (ℓ is in the closure in $\tilde{E}$ of the linear span of X) using the Grothehdieck approach. It is one of our main purposes to improve this result; in fact, using the approach of [8] and [1] we will show (Theorem 2.4) that if X is suitably normalized, then ℓ is in the closure in $\tilde{E}$ of X, and we will be able to identify with precision those parts of X which are "close" to ℓ. In addition we will generalize the dual notion of completeness to include weaker types, one of which appears to be new.

The material is divided up as follows: after a brief resume of our notation and terminology in § 1, we define the notion of (α, β)-closure on the dual space in § 2 and prove most of our fundamental results. In § 3 we use these concepts to define (α, β)-completeness and derive the existence of completions. In § 4 we identify the various types of (α, β)-completeness by their properties on the original space and in § 5 we present some suggestions for a dual theory of net convergence.

1. Notation and terminology.

Throughout, E will be a Hausdorff locally convex topological vector space (lcs) over the real numbers R, with (topological) dual space E' and completion $\tilde{E}$. When we use topological terms with regard to E', this will refer to the weak-*

topology (in most circumstances any topology compatible with the duality $\langle E, E'\rangle$ will do). If $A \subseteqq E$,

$$A^0 = \{u \in E: |u(x)| \leqq 1, \text{ all } x \in A\}.$$

(if $A \subseteqq E'$, $A^0 \subseteqq E$ is formed similarly.) $Q \subseteqq E'$ is equicontinuous if Q^0 is a neighborhood of 0 in E. For Q equicontinuous we let p_Q be the pseudo-norm whose unit ball is Q^0, and if p is an arbitrary pseudo-norm on E, we set

$$\begin{aligned} Q_p &= \{u \in E': |u(x)| \leqq p(x), \text{ all } x \in A\} \\ &= \{x \in E : p(x) \leqq 1\}^0. \end{aligned}$$

$\tilde{p}$ is the unique extension of p to $\tilde{E}$. A *hyperspace* is a subspace of codimension 1.

Let $A \subseteqq E$. A is absolutely convex (ac) if it is convex and circled, i.e., if $r_1A + r_2A \subseteqq A$ whenever $|r_1| + |r_2| \leqq 1$, $r_1, r_2 \in \boldsymbol{R}$. $\Gamma(A)$ is the absolutely convex hull of A, $\mathfrak{L}(A)$ the linear span of A, card A the cardinality of A, and clA (or $cl_E A$ when the space is to emphasized) is the closure of A. If $B \subseteqq E$, $A \sim B$ is the set theoretic difference $(= A \sim (A \cap B))$. If $f: E \to \boldsymbol{R}$, then $f^\perp = \{x \in E : f(x) = 0\}$.

We will be quite free with our use of notation. If $\mathscr{A}$ is a set of subsets of E, $A \subseteqq E$, $a \in E$, we will write $\mathfrak{L}(\mathscr{A}, A, a)$ for $\mathfrak{L}(\cup \mathscr{A} \cup A \cup \{a\})$. Note also that if $x \in E$, $x^\perp \subseteqq E'$.

2. $(\aleph, \beta)$-closure. We will be interested in the order properties of cardinal numbers for notational purposes. As a convenience we will add to the class of cardinals the symbols "$\aleph_0 -$", where $\alpha \leqq \aleph_0 -$ if and only if $\alpha < \aleph_0$, and "V", where $\alpha \leqq V$ for all cardinals α. The reader may well wonder why we do not include "$\alpha -$" for any cardinal α. This is because we are able to show the equivalence of $\aleph_0 -$ and V to ordinary cardinal numbers for our purposes, while this may not be true for $\alpha -$, $\alpha > \aleph_0$ and could cause difficulties in the proof of Corollary 3.3. Throughout we will use α, β, γ, and δ to represent elements of this extended class.

NOTATION. Let E be a lcs, $x \in E$, and $\varepsilon \geqq 0$. A *slice* in E' is a set of the form

$$\text{Sl}(x, \varepsilon) = \{u \in E': |u(x)| \leqq \varepsilon\}.$$

Note that if $\varepsilon > 0$, then $\text{Sl}(x, e) = (\Gamma(x/\varepsilon))^0$ and that $\text{Sl}(x, 0) = x^\perp$.

DEFINITION. Let E be a lcs, $X \subseteqq E$, $M \subseteqq E'$, and Q an equicontinuous subset of E'. Then we say that *M is an intersection of slices of X on Q* if for all $x \in X$, there is an $\varepsilon_x \geqq 0$ such that

$$M \cap Q = \bigcap_{x \in X} \mathrm{Sl}(x, \varepsilon_x) \cap Q .$$

A subset M of E' is *(α, β)-closed* if there is a subset X of E with card $X \leqq \alpha$ such that for every equicontinuous $Q \subseteqq E'$ there is a set $Y \subseteqq X$ with card $Y \leqq \beta$ such that M is an intersection of slices of Y on Q. we will summarize this by writing

$$X \xrightarrow{\beta} M \quad \text{or} \quad X \longrightarrow M .$$

REMARKS. If M is (α, β)-closed, it is necessarily ac, and if $\alpha \leqq \gamma$, $\beta \leqq \delta$, then M is (γ, δ)-closed. Also, if M is an intersection of slices of X on Q, then M is an intersection of slices on Q', for any $Q' \subseteqq Q$.

NOTATION. For fixed $M \subseteqq E'$ and for $X \in E$ any Q any equicontinuous subset of E', we define

$$\varepsilon_{x, Q} = \inf \{\varepsilon \geqq 0 : \mathrm{Sl}(x, \varepsilon) \supseteqq M \cap Q\} .$$

Note that since Q is weakly bounded, $\varepsilon_{x,Q} < \infty$ for all x and Q. If $Q \subseteqq Q'$ then $\varepsilon_{x,Q} \leqq \varepsilon_{x,Q'}$. Finally, if M is an intersection of slices of X on Q, then

$$M \cap Q = \bigcap_{x \in X} \mathrm{Sl}(x, \varepsilon_{x,Q}) \cap Q .$$

We begin our study by relating the above concepts to the $\mathfrak{T}^f$ topology on E' (see [6, Section 21, 8.-10., pp. 269ff]). $\mathfrak{T}^f$ is also often called the $e - w^*$ topology.), the weakest topology on E' agreeing with the weak-* topology on equicontinuous subsets of E'.

PROPOSITION 2.1. An ac subset M of E' is $\mathfrak{T}^f$-closed if and only if it is (V, V)-closed.

Proof. Clearly if M is (V, V)-closed, then $M \cap Q$ is a weakly closed subset of Q for every equicontinuous Q, so M is $\mathfrak{T}^f$-closed.

Now suppose M is $\mathfrak{T}^f$-closed and let Q be a closed ac equicontinuous subset of E'. If $u \in E' \sim M$, then since $M \cap Q$ is closed and convex in E', there is an $x \in E$ such that

$$\sup \{v(x) : v \in Q \cap M\} = \varepsilon < u(x) ,$$

by [6, § 20, 7.(5), p. 246]. Since $Q \cap M$ is ac we have

$$\sup \{-v(x) : v \in Q \cap M\} = \varepsilon ,$$

and therefore $\varepsilon = \varepsilon_{x, Q}$ and $u \notin \mathrm{Sl}(x, \varepsilon_{x, Q})$. Hence we have

$$Q \cap M = \bigcap_{x \in E} \mathrm{Sl}(x, \varepsilon_{x, Q}) .$$

Since every equicontinuous Q is contained in a closed ac equicontinuous subset, we are done.

Note that to get $M \subset Q$, we need intersect the slices with Q only if Q is not ac and closed. While this is true for $X = E$, it is of course not true for X in general.

COROLLARY 2.2. *A lcs E is B-complete [B_r-complete] if and only if every (V, V)-closed subspace [(V, V)-closed dense subspace] of E' is closed.*

(For definitions, see [8].)

Proof. By [8, (3.3), p. 49 and (4.1), p. 54].

COROLLARY. 2.3. *M is a (V, V)-closed hyperspace if and only if $M = x^{\perp}$ for some $x \in \tilde{E}$.*

Proof. By [6, Section 21, 9.(1), p. 271].

Henceforth we will make the blanket assumption that M is a hyperspace in E', E a lcs. We now state our fundamental theorem, the proof of which will occupy the rest of this section. In the process, we will develop several subsidiary results which will aid us in later work.

THEOREM 2.4. *Suppose $X \subseteqq E$ and $X \xrightarrow{\beta} M$, and let $u \in E' \sim M$. Set*

$$Y = \{x/u(x) \colon x \in X,\ u(x) \neq 0\} .$$

Then $Y \xrightarrow{\beta} M$ and $M = x^{\perp}$ for some $x \in cl_{\tilde{E}}\ Y$.

We begin by proving a series of lemmas.

LEMMA 2.5. *Let $M = f^{\perp}$ where f is a linear functional on E' which is bounded on equicontinuous subsets of E', and let $u \in E' \sim M$. Then if Q is any ac equicontinuous subset of E' containing u, we have*

$$Q \subseteqq 6\Gamma((r_f + 1)Q \cap M,\ r_f u)$$

where

$$r_f = \sup \{ |f(v)| \colon v \in Q\}/|f(u)| \} ,$$

and if $\mathscr{Q}$ is a fundamental system of equicontinuous subset of E' in the sense that every equicontinuous subset of E' is contained in some member of $\mathscr{Q}$, then so is

$$\mathscr{Q}' = \{\Gamma(Q \cap M, ru): Q \in \mathscr{Q}, r > 0\} .$$

In particular, the above results hold if M is $(\boldsymbol{V}, \boldsymbol{V})$-closed.

Proof. If $v \in Q$, then $|f(v)/r_f f(u)| \leqq 1$ and since $2t/(t-1) = 2 + 2/(t-1)$ is strictly decreasing for $t < 1$, we can find a $t \in [-1, 1/3]$ such that

$$2t/(t-1) = f(v)/r_f f(u) .$$

Then $f[2tr_f u + (1-t)v] = 0$ so

$$2tr_f u + (1-t)v = m \in f^{\perp} = M ,$$

and since

$$|2r_f t| + |1-t| \leqq 2r_f + 2 , \qquad t \in [-1, 1/3] ,$$

we have $m \in (2r_f + 2)Q$. We write

$$v = [2/(1-t)][(1/2)m - (t/2)(2r_f u)] ,$$

and since for $t \in [-1, 1/3]$, $1/2 + |t|/2 \leqq 1$ and $|2/(1-t)| \leqq 2(3/2) = 3$, we have

$$v \in 3\Gamma((2r_f + 2)Q \cap M, 2r_f u) = 6\Gamma((r_f + 1)Q \cap M, r_f u) .$$

The remaining statements follow easily, the last resulting from $M = x^{\perp}$ for some $x \in \tilde{E}$ by Corollary 2.3 and the fact that any $x \in \tilde{E}$ is continuous on any equicontinuous Q [6, Section, 21, 4. (5), p. 263].

The principal use of this lemma will be to conclude that whenever M is $(\boldsymbol{V}, \boldsymbol{V})$-closed and $u \in E' \sim M$, then there is fundamental system of equicontinuous Q such that each $Q = \Gamma(Q \cap M, ru)$ for some $r > 0$.

COROLLARY 2.6. *Let $X \xrightarrow{\beta} M$, suppose that $u \in E' \sim M$, and let $Y = \{x \in X : u(x) \neq 0\}$. Then $Y \xrightarrow{\beta} M$.*

Proof. If Q is equicontinuous then $Q \subseteqq Q' = \Gamma(Q' \cap M, ru)$, $r > 0$. Any $v \in Q'$ is of the form $v = sm + tru$, $|s| + |t| \leqq 1$, $m \in Q' \cap M$ and for any $x \in X \sim Y$,

$$|v(x)| = |s|\,|m(x)| \leqq \varepsilon_{x,Q'} ,$$

so $v \in \mathrm{Sl}(x, \varepsilon_{x,Q'})$ and hence $Q' \subseteqq \mathrm{Sl}(x, \varepsilon_{x,Q'})$, so if M is an intersection of slices of $Z \subseteqq X$ an Q', it is on intersection of slices of $Z \cap Y$ on

Q', and hence on Q by an earlier remark.

Our main lemma will be obtained by generalizing the proof of Corollary 2.6, but first we need a more precise notion of when two hyperspaces are "near" each other. For $x \in E$ and $Q \subseteqq E'$ equicontinuous, we define

$$b_{x,Q} = \sup \{ | u(x) | : u \in Q\} < \infty .$$

For a given M, the ratio $\Delta_{x,Q} = \varepsilon_{x,Q}/b_{x,Q}$ can be considered as a measure of the difference between $x^\perp$ and M on Q (conventionally, we set $\Delta_{x,Q} = 1$ when $b_{x,Q} = 0$); in fact $0 \leqq \Delta_{x,Q} \leqq 1$, $\Delta_{x,Q} = 1$ implies that $Q \subseteqq x^\perp$ and hence that $x^\perp$ and M are unrelated on Q, and $\Delta_{x,Q} = 0$ implies that $M \cap Q = x^\perp \cap Q$, as we will digress to prove in the following lemma.

LEMMA 2.7. *If Q is an equicontinuous subset of E', $x \in E$, then $\Delta_{x,Q} = 0$ implies that $x^\perp \cap Q = M \cap Q$.*

Proof. First assume that Q is ac. Since $\Delta_{x,Q} = 0$ we have $\varepsilon_{x,Q} = 0$ and $Q \nsubseteqq x^\perp$, so choose $u_1 \in Q \sim x^\perp$. If $x^\perp \cap Q \neq M \cap Q$, then since $x^\perp \cap Q \supseteqq M \cap Q$, $u_1 \notin M$ and we may choose a $u_2 \in (x^\perp \cap Q) \sim M$. $E' = \mathfrak{L}(M, u_2)$ so $u_1 = m + ru_2$, $m \in M$ and

$$\begin{aligned} m &= (1 + | r |)(u_1/(1 + | r |) - ru_2/(1 + | r |)) \\ &\in (1 + | r |)M \cap Q \subseteqq (1 + | r |)x^\perp \cap Q , \end{aligned}$$

and therefore

$$0 = m(x) = u_1(x) - ru_2(x) = u_1(x) ,$$

so $u_1 \in x^\perp$, a contradiction. If Q is arbitrary, let $Q' = \Gamma Q$. Clearly $\varepsilon_{x,Q'} = \varepsilon_{x,Q}$, $b_{x,Q'} = b_{x,Q}$, so $\Delta_{x,Q'} = \Delta_{x,Q} = 0$ and $x^\perp \cap Q' = M \cap Q'$. Therefore $x^\perp \cap Q = M \cap Q$.

Note also that if $x^\perp = y^\perp$, then $\Delta_{x,Q} = \Delta_{y,Q}$. The notion of $\Delta_{x,Q}$ as a measure of difference will be brought out more fully as this section progresses.

LEMMA 2.8. *Let Q be an ac equicontinuous subset of E' and suppose that M is an intersection of slices of $X \subseteqq E$ on Q. Then if $Y \subseteqq X$ and $inf \{\Delta_{y,Q} : y \notin Y\} > 0$, M is an intersection of slices of Y on Q. In particular, $Q \nsubseteqq M$ implies that*

$$\inf \{\Delta_{x,Q} : x \in X\} = 0 .$$

Proof. Let $0 < \varepsilon \leqq 1$. Then

$$\left(\frac{1}{\varepsilon} Q\right) \cap M = \frac{1}{\varepsilon} \left(\bigcap_{x \in X} \mathrm{Sl}(x, \varepsilon_{x,Q}) \cap Q\right) = \bigcap_{x \in X} \mathrm{Sl}(x, \varepsilon_{x,Q}/\varepsilon) \cap \frac{1}{\varepsilon} Q .$$

$Q \subseteqq (1/\varepsilon)Q$ since $\varepsilon \leqq 1$, so

$$Q \cap M = \bigcap_{x \in X} \mathrm{Sl}(x, \varepsilon_{x,Q}/\varepsilon) \cap Q ,$$

and if we set $X_\varepsilon = \{x \in X : \Delta_{x,Q} < \varepsilon\}$, we have

$$Q \cap M = \textstyle\bigcap_{x \in X_\varepsilon} \mathrm{Sl}(x, \varepsilon_{x,Q}/\varepsilon) \cap Q ,$$

since if $x \in X \sim X_\varepsilon$, then $\varepsilon_{x,Q}/\varepsilon \geqq b_{x,Q}$ so $\mathrm{Sl}(x, \varepsilon_{x,Q}/\varepsilon) \supseteqq Q$ and is thus superfluous in the intersection. By definition of $\varepsilon_{x,Q}$,

$$Q \cap M \subseteqq \bigcap_{x \in X_\varepsilon} \mathrm{Sl}(x, \varepsilon_{x,Q}) \cap Q \subseteqq \bigcap_{x \in X_\varepsilon} \mathrm{Sl}(x, \varepsilon_{x,Q}/\varepsilon) \cap Q = Q \cap M .$$

Therefore M is an intersection of slices of X_ε on Q. For $Y \subseteqq X$, set $\varepsilon = \inf\{\Delta_{y,Q}: y \notin Y\}$. If $\varepsilon > 0$, then the first statement of the lemma follows from the above and the fact that $Y \supseteqq X_\varepsilon$. If $\inf\{\Delta_{x,Q}: x \in X\} > 0$, then

$$Q \cap M = \bigcap_{x \in \varnothing} \mathrm{Sl}(x, \varepsilon_{x,Q}) \cap Q = E' \cap Q = Q ,$$

so $Q \subseteqq M$, a contradiction.

Our final lemma is a series of computations which further demonstrate the relation between $\Delta_{x,Q}$ and the "difference" between $x^\perp$ and M.

LEMMA 2.9. *Let $M \subseteqq E'$, $v \in E' \sim M$, and let $x \in E$ be such that $u(x) = 1$. Suppose that $Q = \Gamma(Q \cap M, ru)$, $r > 0$, is an equicontinuous subset of E'. Then we have*

(2.10) *If $\Delta_{x,Q} < 1$, then $\varepsilon_{x,Q} \leqq r\Delta_{x,Q}/(1 - \Delta_{x,Q})$. In particular, if $\Delta_{x,Q} \leqq \varepsilon/(r + \varepsilon)$, $\varepsilon \geqq 0$, then $\varepsilon_{x,Q} \leqq \varepsilon$.*

(2.11) *If $M = \tilde{x}^\perp$ for some $\tilde{x} \in \tilde{E}$ with $u(\tilde{x}) = 1$, then $\tilde{p}_Q(x - \tilde{x}) = \varepsilon_{x,Q}$.*

Proof. If v is any element of Q,

$$v = sm + tru , \qquad |s| + |t| \leqq 1,\ m \in M \cap Q ,$$

we have

$$\begin{aligned} |v(x)| &\leqq |s|\,|m(x)| + |t|\,r\,|u(x)| \\ &\leqq |s|\varepsilon_{x,Q} + |t|\,r \\ &\leqq \varepsilon_{x,Q} + r, \end{aligned}$$

so $b_{x,Q} \leqq \varepsilon_{x,Q} + r$ and

$$\varepsilon_{x,Q} = \Delta_{x,Q} b_{x,Q} \leqq \Delta_{x,Q} r + \Delta_{x,Q}\varepsilon_{x,Q} ,$$

from which (2.10) follows immediately.

To prove (2.11), let $v \in Q$ be as above. Then $|v(x - \tilde{x})| = |s||m(x)|$ so

$$\tilde{p}_Q(x - \tilde{x}) = \sup_{v \in Q} |v(x - \tilde{x})| = \sup |s||m(x)| = \varepsilon_{x,Q} .$$

We will say that $X \subseteqq E$ is *normalized by* $u \in E'$ if $u(x) = 1$ for all $x \in X$. We now proceed to the proof of Theorem 2.4.

Proof of Theorem 2.4. We assume that X is normalized by u, i.e., that X is the Y of the statement of the theorem. We may do this by Corollary 2.6 and the fact that for $r \neq 0$, $\mathrm{Sl}(x/r, \varepsilon) = \mathrm{Sl}(x, |r| \varepsilon)$. Choose $\tilde{x} \in \tilde{E}$ such that $M = \tilde{x}^\perp$ and $u(\tilde{x}) = 1$. Let $\mathscr{Q}$ be the set of all equicontinuous subsets Q of E' with the property that $Q = \Gamma(Q \cap M, ru)$ for some $r > 0$, and let $\mathscr{D} = \{\langle Q, \varepsilon\rangle : Q \in \mathscr{Q}, \varepsilon > 0\}$ with ordering given by $\langle Q, \varepsilon\rangle \leqq \langle Q', \varepsilon'\rangle$ if $Q \subseteqq Q'$ and $\varepsilon' \leqq \varepsilon$. By Lemma 2.8, for every $\langle Q, \varepsilon\rangle \varepsilon \mathscr{D}$ there is an $x = x_{Q,\varepsilon} \in X$ with

$$\Delta_{x,Q} \leqq \varepsilon/(r + \varepsilon) \qquad (Q = \Gamma(Q \cap M, ru)) ,$$

and hence

$$\tilde{p}_Q(x_{Q,\varepsilon} - \tilde{x}) = \varepsilon_{x,Q} \leqq \varepsilon$$

by (2.10) and (2.11). If $\langle Q', \varepsilon'\rangle \in \mathscr{D}$ and $\langle Q', \varepsilon'\rangle \geqq \langle Q, \varepsilon\rangle$, then $Q' \supseteqq Q$ so $p_Q \leqq p_{Q'}$ and hence

$$\begin{aligned} \tilde{p}_Q(x_{Q',\varepsilon'} - \tilde{x}) &\leqq \tilde{p}_{Q'}(x_{Q',\varepsilon'} - \tilde{x}) \\ &\leqq \varepsilon' \leqq \varepsilon . \end{aligned}$$

Since by Lemma 2.5 $\{p_Q : Q \in \mathscr{Q}\}$ is a basic set of pseudonorms, $\{x_{Q,\varepsilon} : \langle Q, \varepsilon\rangle \in \mathscr{D}\}$ is a net in X converging to $\tilde{x}$.

As Corollary, we can show that we obtain all (α, β)-closed hyperspaces by restricting β to $1 \leqq \beta \leqq \aleph_0$. To be precise, we have

COROLLARY 2.12. *Let M be a hyperspace in E'. Then M is (α, V)-closed if and only if M is $(\alpha, \aleph_0)$-closed.*

Proof. Necessity is obvious. Part of the proof of sufficiency is contained in the following lemma, which is a partial converse of Lemma 2.8.

LEMMA 2.13. *Let Q be an equicontinuous subset of E' with $Q = \Gamma(Q \cap M, ru)$, $u \notin M$, $r > 0$, and suppose that for some $X \subseteqq E$, we have $\inf\{\Delta_{x,Q} : x \in X\} = 0$. Then M is an intersection of slices of x on Q.*

Proof. Assume that X is normalized by u. Then by (2.10), $\inf\{\varepsilon_{x,Q}: x \in X\} = 0$. If

$$v \in \bigcap_{x \in X} \mathrm{Sl}(x, \varepsilon_{x,Q}) \cap Q ,$$

then $v = tru + sm$, $|s| + |t| \leqq 1$, $m \in M \cap Q$, so we have

$$\begin{aligned} \varepsilon_{x,Q} &\geqq |tru(x) + sm(x)| \\ &\geqq |t| r - |s| |m(x)| \\ &\geqq |t| r - |s| \varepsilon_{x,Q} , \end{aligned}$$

so $|t| \leqq (1 + |s|)\varepsilon_{x,Q}/r$ for all $x \in X$ and hence $t = 0$ and $v \in M$. This gives us one of the required inclusions, the other being obvious.

Each $Q' \subseteqq E'$ is contained in some $Q = \Gamma(Q \cap M, ru)$, and for each $n > 0$, there is an $x_n \in X$ such that $\Delta_{x_n,Q} \leqq 1/n$ by Lemma 2.8. Hence by the above lemma, M is an intersection of slices of $\{x_n\}$ on Q, and hence the same is true of Q', which completes the proof of the Corollary.

It is convenient to note here that the β's considered may be still further restricted.

PROPOSITION 2.14. Let Q be an ac equicontinuous subset of E', $Q \not\subseteqq M$, and suppose that for $X = \{x_1, \cdots, x_n\}$ a subset of E, M is an intersection of slices of X on Q. Then there is an i with $\Delta_{x_i,Q} = 0$.

Proof. Let $\varepsilon_i = \varepsilon_{x_i,Q}$, $b_i = b_{x_i,Q}$ for all i. Suppose that the Proposition is false, i.e., $\varepsilon_i = 0$ only if $b_i = 0$. Let $u \in Q \sim M$ and let

$$r = \min\{1, \{\varepsilon_i/|u(x_i)|: \varepsilon_i \neq 0, u(x_i) \neq 0\}\} .$$

Then $0 < r \leqq 1$ so $ru \in Q$ and $|ru(x_i)| \leqq \varepsilon_i$ for all i with $\varepsilon_i \neq 0$, $u(x_i) \neq 0$. But this inequality certainly holds if $u(x_i) = 0$, and if $\varepsilon_i = 0$, $b_i = 0$ and hence $ru(x_i) = 0 = \varepsilon_i$ since $ru \in Q$. Therefore $|ru(x_i)| \leqq \varepsilon_i$ for all i, so $ru \in M \cap Q$ and $u \in M$, a contradiction.

COROLLARY 2.15. *A hyperspace in E' is $(\aleph_0 -, \aleph_0 -)$-closed if and only if it is closed (i. e., (1, 1)-closed).*

Proof. Suppose $\{x_1, \cdots, x_n\} \to M$. If Q is ac and $Q \not\subseteqq M$, then there is an i such that $\varepsilon_{x_i,Q} = \Delta_{x_i,Q} = 0$. If $Q' \supseteqq Q$, then $\varepsilon_{x_i,Q'} \geqq \varepsilon_{x_i,Q}$, $b_{x_i,Q'} \geqq b_{x_i,Q} > 0$, so in fact there must be an i such that $e_{x_i,Q} = \Delta_{x_i,Q} = 0$ for *all* ac $Q \not\subseteqq M$. Hence by Lemma 2.7,

$$x_i^{\perp} \cap Q = M \cap Q$$

for all ac $Q \not\subseteqq M$ and therefore $x_i^{\perp} = M$, which gives us the result.

COROLLARY 2.16. *A hyperspace in E' is $(\alpha, \aleph_0 -)$-closed if and only if it is $(\alpha, 1)$-closed.*

Proof. Sufficiency is obvious and necessity follows immediately from Proposition 2.14 and Lemma 2.7.

Hence we have shown that we do not really need the symbols $\aleph_0 -$ and V as we promised we would at the beginning of this section (we can always assume that $\alpha, \beta \leqq \text{card } E$). However we will continue to use V as a convenient notation.

3. (α, β)-completeness, Completions.

DEFINITION. An lc space E is *(α, β)-complete* if and only if every (α, β)-closed hyperspace in E' is closed.

If follows immediately from the results of § 2 that we need only consider (α, β)-completeness for $\alpha \geqq \aleph_0, \beta = 1$ or $\aleph_0$. We will devote this section to the study of (α, β)-completions using the methods of § 2 and will defer until § 4 characterizations of (α, β)-completeness wholly in terms of the original space E.

Our basic result is the following:

THEOREM 3.1. *Let $\tilde{x} \in \tilde{E}, u \in E' \sim \tilde{x}^\perp$, and let $Q = \Gamma(\tilde{x}^\perp \cap Q, ru)$, $r > 0$, be an equicontinuous subset of E'. Suppose that for some $X \subseteqq E, \tilde{x}^\perp (= \mathrm{Sl}\,(\tilde{x}, 0))$ is an intersection of slices of X on Q. Then for all $\varepsilon \geqq 0$, $\mathrm{Sl}\,(\tilde{x}, \varepsilon)$ is an intersection of slices of X on Q.*

Proof. The proof is clear when contemplated geometrically. $E' \sim \mathrm{Sl}(\tilde{x}, \varepsilon) = A_\varepsilon \cup B_\varepsilon$ where A_ε and B_ε are half-spaces. If $v \in Q \sim \mathrm{Sl}(\tilde{x}, \varepsilon)$, say $v \in A_\varepsilon$, then, $A_\varepsilon \cap Q$ is similar to $A_0 \cap Q$, v corresponding to $\bar{v} \in A_0 \cap Q$. $\bar{v} \notin$ some $\mathrm{Sl}\,(x, \varepsilon_{x,Q})$ and running the similarity backwards we get $v \notin \mathrm{Sl}\,(x, \nu_x) \supseteqq \mathrm{Sl}\,(\tilde{x}, \varepsilon) \cap Q$ for suitable ν_x. The proof is now merely a matter of computation, which we proceed to perform.

Assume $Q \not\subseteqq \mathrm{Sl}\,(x, \varepsilon_{x,Q})$ for all $x \in X$, that X is normalized by u, that $u(\tilde{x}) = 1$, and that $\varepsilon < r$ (if $r \leqq \varepsilon$, then $ru \in \mathrm{Sl}\,(x, \varepsilon) \supseteqq Q \cap M$ so $Q \subseteqq \mathrm{Sl}\,(x, \varepsilon)$ and the result is trivial. For each $x \in X$, set $\nu_x = (r - \varepsilon)\varepsilon_{x,Q}/r + \varepsilon$. If $v \in Q \cap \mathrm{Sl}\,(\tilde{x}, \varepsilon)$, then $v = tm + sru$, $|s| + |t| \leqq 1$, $m \in Q \cap M$ and $\varepsilon \geqq |v(\tilde{x})| = |s|r$. Therefore $|s| \leqq \varepsilon/r$ and

$$\begin{aligned} |v(x)| &\leqq |t|\,|m(x)| + |s|\,r \\ &\leqq (1 - |s|)\varepsilon_{x,Q} + |s|\,r \\ &= |s|\,(r - \varepsilon_{x,Q}) + \varepsilon_{x,Q}\,. \end{aligned}$$

Since $Q \not\subseteqq \mathrm{Sl}\,(x, \varepsilon_{x,Q})$, $ru \notin \mathrm{Sl}\,(x, \varepsilon_{x,Q})$ so $r > \varepsilon_{x,Q}$ and the right hand quantity above is largest when $|s|$ is largest, i.e., when $|s| = \varepsilon/r$, so

$$\begin{aligned} |v(x)| &\leqq \tfrac{\varepsilon}{r}(r - \varepsilon_{x,Q}) + \varepsilon_{x,Q} \\ &= (r - \varepsilon)\varepsilon_{x,Q}/r + \varepsilon = \nu_x , \end{aligned}$$

and therefore $\mathrm{Sl}\,(\tilde{x},\varepsilon) \subseteqq Q \cap \mathrm{Sl}(x, \nu_x)$.

Now suppose that $v \in Q \sim \mathrm{Sl}\,(\tilde{x}, \varepsilon)$. Then $|v(\tilde{x})| > \varepsilon$, say $v(\tilde{x}) > \varepsilon$. $v = sm + tru$ and since $Q \cap M$ is ac, we may assume that $s \geqq 0$, $s + |t| = 1$. $v(\tilde{x}) = tr > \varepsilon$ so $v = sm + (1 - s)ru$ and $(1 - s)r > \varepsilon$, i.e., $sr < r - \varepsilon$. Set

$$\begin{aligned} \bar{v} &= \frac{r}{r - \varepsilon}(v - \varepsilon u) \\ &= \frac{1}{r - \varepsilon}(rsm + [(1 - s)r - \varepsilon]ru) . \end{aligned}$$

$(1 - s)r > \varepsilon$ and $rs + [(1 - s)r - \varepsilon] = r - \varepsilon$ so $\bar{v} \in Q$.

$$\bar{v}(\tilde{x}) = r(v(\tilde{x}) - \varepsilon)/(r - \varepsilon) > 0$$

so $\bar{v} \notin M$ and there is an $x \in X$ such that $\bar{v} \notin \mathrm{Sl}\,(x, \varepsilon_{x,Q})$, i.e., $|\bar{v}(x)| > \varepsilon_{x,Q}$.

$$\begin{aligned} \bar{v}(x) &= \frac{1}{r - \varepsilon}(rsm(x) + [(1 - s)r - \varepsilon]r) \\ &> \frac{rsm(x)}{r - \varepsilon} \end{aligned}$$

so if $\bar{v}(x) < 0$, then $m(x) < 0$ and

$$m(x) < \frac{rsm(x)}{r - \varepsilon} < \bar{v}(x) < -\varepsilon_{x,Q}$$

so $m \notin \mathrm{Sl}\,(x, \varepsilon_{x,Q})$, a contradiction. Hence $\bar{v}(x) > \varepsilon_{x,Q}$ and

$$\begin{aligned} \nu_x &= (r - \varepsilon)\varepsilon_{x,Q}/r + \varepsilon \\ &< (r - \varepsilon)\bar{v}(x)/r + \varepsilon u(x) \\ &= v(x) , \end{aligned}$$

and $v \notin \mathrm{Sl}\,(x, \nu_x)$. The case $v(\tilde{x}) < -\varepsilon$ follows from the above by symmetry, so we are done.

By Lemma 2.5 and oúr usual argument we obtain

COROLLARY 3.2. *Let* $X \xrightarrow{\beta} \tilde{x}^{\perp} \subseteqq E'$, $\tilde{x} \in E$. *Then for all* $\varepsilon \geqq 0$, $X \xrightarrow{\beta} \mathrm{Sl}\,(\tilde{x}, \varepsilon)$.

Notice that we do *not* claim that if $\tilde{x}^{\perp}$ is intersection of slices of $Y(\subseteqq X)$ on Q for *arbitrary* Q, then $\mathrm{Sl}(\tilde{x}, \varepsilon)$ is an intersection of slices from Y on Q. For the latter we must use a Y' which is perhaps larger than Y and which is such that $\tilde{x}^{\perp}$ is an intersection of slices of Y' on Q' for some $Q' = \Gamma(Q' \cap \tilde{x}^{\perp}, ru) \supseteqq Q$.

COROLLARY 3.3. *Let $\bar{E}$ be the set of all $\tilde{x} \in \tilde{E}$ for which $\tilde{x}^{\perp}$ is (α, β)-closed on E, and suppose that M is (α', β')-closed on $\bar{E}$, i.e., for some $X \subseteqq \bar{E}$, card $X \leqq \alpha'$ and $X \xrightarrow{\beta'} M$. Then M is (α'', β'')-closed on E where $\alpha'' = \max(\alpha, \alpha')$ and $\beta'' = \max(\beta, \beta')$.*

Proof. We may assume that $\alpha, \alpha' \geqq \aleph_0$ and that β and β' are either 1 or $\aleph_0$. For each $x \in X$, there is a subset X_x of E such that $X_x \xrightarrow{\beta} x$ and card $X_x \leqq \alpha$. Let $\bar{X} = \bigcup_{x \in X} X_x \subseteqq E$. Then we have card $\bar{X} \leqq$ card $X \cdot \sup_x(\text{card } X_x) \leqq \alpha' \cdot \alpha = \alpha''$. For each equicontinuous $Q \subseteqq E'$ we may choose $Y \subseteqq X$ with card $Y \leqq \alpha'$ such that M is an intersection of slices of Y on Q. By Corollary 3.2, for each $y \in Y$ there is a $Y_y \subseteqq X_y$ with card $Y_y \leqq \beta$ such that $\mathrm{Sl}(y, \varepsilon_{y,Q})$ is an intersection of slices of $\bar{Y}_y$ on Q. Then clearly M is an intersection of slices of $\bar{Y} = \bigcup_{y \in Y} Y_y$ on Q and card $\bar{Y} \leqq \beta \cdot \beta' = \beta''$, so we are done.

DEFINITION. The *(α, β)-completion* $E^{\sim\alpha,\beta}$ of E is the set of all $\tilde{x} \in \tilde{E}$ with $\tilde{x}^{\perp}$ (α, β)-closed on E.

Note. $0^{\perp} = E'$ is not a hyperspace, but is trivially (α, β)-closed and hence in $E^{\sim\alpha,\beta}$.

COROLLARY 3.4. *$E^{\sim\alpha,\beta}$ is (α, β)-complete and is the smallest (α, β)-complete subspace of $\tilde{E}$ containing E.*

Proof. If $\tilde{x}^{\perp}$ is (α, β)-closed over $E^{\sim\alpha,\beta}$, then $\tilde{x} \in E^{\sim\alpha,\beta}$ by Corollary 3.3 so $E^{\sim\alpha,\beta}$ is (α, β)-complete. Clearly $E^{\sim\alpha,\beta}$ is the minimal (α, β)-complete set containing E, so we are done if we can show that $E^{\sim\alpha,\beta}$ is a subspace. Given the results of § 4 this is easily obtained, but we feel that it is instructive to continue in the spirit of § 2.

Since $E^{\sim\alpha,\beta}$ obviously contains rx whenever it contains x ($(rx)^{\perp} = x^{\perp}$ or E'), we are done if we can prove that whenever $\tilde{x}, \tilde{y} \in \tilde{E}$, $\tilde{x}^{\perp}$, $\tilde{y}^{\perp}(\alpha, \beta)$-closed, we have $M = (\tilde{x} + \tilde{y})^{\perp}$ is (α, β)-closed. We may assume that $\tilde{x} \neq 0 \neq \tilde{y}$ and that $\tilde{x} \neq r\tilde{y}$ for all $r \in \boldsymbol{R}$. Let $X \xrightarrow{\beta} \tilde{x}^{\perp}$, $Y \xrightarrow{\beta} \tilde{y}^{\perp}$ and choose $u \in \tilde{x}^{\perp} \sim \tilde{y}^{\perp}$, $v \in \tilde{y}^{\perp} \sim \tilde{x}^{\perp}$ so that $u(\tilde{y}) = v(\tilde{x}) = 1$. Clearly $M = \mathfrak{L}(u - v, \tilde{x}^{\perp} \cap \tilde{y}^{\perp})$. Finally, we assume that X and Y are normalized by v and u respectively. Let Q be ac and equicontinuous with u,

$v \in Q$ and choose $X' \subseteqq X$, $Y' \subseteqq Y$ so that card X', card $Y' \leqq \beta$ and $\tilde{x}^{\perp}[\tilde{y}^{\perp}]$ is an intersection of slices of X' $[Y']$ on Q. We will show that $(\tilde{x} + \tilde{y})^{\perp}$ is an intersection of slices of $X' + Y'$ on Q.

Every element w of E' is of the form $ru - sv + m$, $m \in \tilde{x}^{\perp} \cap \tilde{y}^{\perp}$, and $w \in M$ if and only if $r = s$. Then for $x \in X'$, $y \in Y'$,

$$w(x + y) = (r - s) + ru(x) - sv(y) + m(x + y)$$

$(u(y) = v(x) = 1)$. Set

$$\mu = \sup \{ |r|, |s| : ru = sv + m \in Q \text{ for some } m \in \tilde{x}^{\perp} \cap \tilde{y}^{\perp} \} .$$

$\tilde{y}$ is weakly continuous on Q, whose weak closure is compact, and $(ru - sv + m)(\tilde{y}) = r$ (and similarly for $\tilde{x}$), so $\mu < \infty$. Now suppose that $w_0 = r_0 u - s_0 v + m_0 \in Q$, $r_0 \neq s_0$, i.e., $w_0 \notin M$. We may choose $x \in X'$, $y \in Y'$ so that

$$\varepsilon_{x,Q},\ \varepsilon_{y,Q} < | r_0 - s_0 |/8(1 + 2\mu)$$

by (2.10) and the fact that if $Q \subseteqq Q'$, $\varepsilon_{x,Q} \leqq \varepsilon_{x,Q'}$. $u \in \tilde{x}^{\perp} \cap Q$ and $v \in \tilde{y}^{\perp} \cap Q$, and

$$m_0 = w_0 - r_0 u + s_0 v \in (1 + |r_0| + |s_0|)Q \subseteqq (1 + 2\mu)Q$$

so

$$\begin{aligned} | w_0(x + y) | &= | (r_0 - s_0) + r_0 u(x) + s_0 v(y) + m_0(x + y) | \\ &\geqq | r_0 - s_0 | - | r_0 | \, | u(x) | - | s_0 | \, | v(y) | \\ &\quad - | m_0(x) | - | m_0(y) | \\ &\geqq | r_0 - s_0 | - \mu\varepsilon_{x,Q} - \mu\varepsilon_{y,Q} - \varepsilon_{x,(1+2\mu)Q} \\ &\quad - \varepsilon_{y,(1+2\mu)Q} \\ &> | r_0 - s_0 | - 2(| r_0 - s_0 |/8) - (1 + 2\mu)\varepsilon_{x,Q} \\ &\quad - (1 + 2\mu)\varepsilon_{y,Q} \\ &> | r_0 - s_0 | - 4(| r_0 - s_0 |/8) \\ &= | r_0 - s_0 |/2 . \end{aligned}$$

If $w \in Q \cap M$, $w = ru - rv + m$, then similarly we have

$$\begin{aligned} | w(x + y) | &\leqq | r | \, | u(x) | + | r | \, | v(y) | + | m(x) | + | m(y) | \\ &\leqq 4(| r_0 - s_0 |/8) = | r_0 - s_0 |/2 , \end{aligned}$$

so $w_0 \notin \mathrm{Sl}\,(x + y, | r_0 - s_0 |/2)$ and $Q \cap M \subseteqq \mathrm{Sl}\,(x + y, | r_0 - s_0 |/2)$, from which the result follows.

Note that we cannot conclude from $X \to \tilde{x}^{\perp}$ and $Y \to \tilde{y}^{\perp}$ that $X + Y \to (\tilde{x} + \tilde{y})^{\perp}$. X and Y must first be suitably normalized.

Our final result in this section follows immediately from Corollary 3.3.

COROLLARY 3.5. *Let* $\alpha'' = \max(\alpha, \alpha')$. *Then*

$$(E^{\sim\alpha,\beta})^{\sim\alpha',\beta} = (E^{\sim\alpha',\beta})^{\sim\alpha,\beta} = E^{\sim\alpha'',\beta} .$$

There is much we do not know about various relations of this type. For instance, we do not known if any of the equalities in

$$(E^{\sim\alpha,1})^{\sim\alpha,\aleph_0} = (E^{\sim\alpha,\aleph_0})^{\sim\alpha,1} = E^{\sim\alpha,\aleph_0}$$

is true in general.

4. Characterizations in *E*. In this section we will determine necessary and sufficient conditions for E to be (α, β)-complete which involve only E itself. There are two cases, $(\alpha, \aleph_0)$-completeness and $(\alpha, 1)$-completeness, which will be studied in turn. First we will show that the former is equivalent to α-completeness, as defined below.

DEFINITION. A lcs E is *α-complete* if whenever $X \subseteqq E$ and card $X \leqq \alpha$, then $cl_{\tilde{E}} X \subseteqq E$.

A few remarks are in order concerning the relationship between $\aleph_0$-completeness and sequential completeness. Clearly the former implies the latter, but the converse is false, as is demonstrated by the space $H[\mathfrak{T}_s]$ of [4] which is sequentially complete (in fact, quasi-complete) and separable but not complete. Also $\aleph_0$-completeness does not imply quasi-completeness, for let S be the topological space of [2, Ex. 4N, p 64], i.e., an uncountable set all of whose points are open except for a single exceptional point whose neighborhoods are complements of countable sets. The space of all continuous $\boldsymbol{R}$-valued functions on S is then easily seen to be $\aleph_0$-complete but not quasi-complete in the topology induced by the product topology on $\boldsymbol{R}^S$.

Our first equivalence theorem is

THEOREM 4.1. *The following conditions on a lcs E are equivalent*:

(a) *E is (α, V)-complete*

(b) *E is $(\alpha, \aleph_0)$-complete*

(c) *E is α-complete* .

Proof. (b) $\Rightarrow$ (a) follows immediately from Corollary 2.12 but we will obtain a second proof by showing (b) $\Rightarrow$ (c) $\Rightarrow$ (a) ((a) $\Rightarrow$ (b) trivially) and simultaneously develop some results which will prove useful in later work.

(b) $\Rightarrow$ (c) follows immediately from:

LEMMA 4.2. *Let* $\{x_\lambda\}$ *be a net in* E, $\lim_\lambda x_\lambda = x \in \tilde{E}$, *let* p *be a continuous pseudo-norm on* E, *and let* λ_n *be chosen for each* $n > 0$ *so that* $p(x_\lambda - x_\mu) \leqq 1/n$ *if* $\lambda, \mu > 1_n$. *Then*

$$x^\perp \cap Q_p = \bigcap_{n=1}^{\infty} \mathrm{Sl}\,(x_{\lambda_n}, 1/n) \cap Q_p \;.$$

Proof. If $u \in x^\perp \cap Q_p$, then $u(x) = 0$ and $|u| \leqq p$. u is continuous on E so

$$|u(x_{\lambda_n})| = |u(x - x_{\lambda_n})| = \lim_\lambda |u(x_\lambda - x_{\lambda_n})| \\ \leqq \lim_\lambda p(x_\lambda - x_{\lambda_n}) \leqq 1/n$$

so $u \in \mathrm{Sl}\,(x_{\lambda_n}, 1/n)$ for all n. Conversely, if $u \in \mathrm{Sl}\,(x_{\lambda_n}, 1/n)$ for all n, fix n and choose $\lambda \geqq \lambda_n$ so that $|u(x) - u(x_{\lambda_n})| \leqq 1/n$. Then

$$|u(x)| \leqq |u(x - x_\lambda)| + |u(x_\lambda - x_{\lambda_n})| + |u(x_{\lambda_n})| \\ \leqq 1/n + p(x_\lambda - x_{\lambda_n}) + 1/n \leqq 3/n.$$

Since n is arbitrary, $u(x) = 0$ and $u \in x^\perp \cap Q_p$, proving the Lemma.

(c) $\Rightarrow$ (a) is a trivial Corollary of Theorem 2.4, but it can also be given a very simple direct proof as follows; We first show that if $X \subseteqq E$, $X \rightarrow x^\perp$, $x \in \tilde{E}$, then $x \in cl_{\tilde{E}}\mathfrak{L}(X)$. If $x \notin cl_{\tilde{E}}\mathfrak{L}(X)$ then there is a $u \in E'$ such that $u(\mathfrak{L}(X)) = \{0\}$ and $u(x) \neq 0$. But then $Q = \Gamma\{u\}$ is equicontinuous, $x^\perp \cap Q = \{0\} \neq Q$, and $Q \subseteqq \mathrm{Sl}\,(y, 0)$ for all $y \in X$, contradicting $X \rightarrow x^\perp$. Now assume (c) and let M be (α, V)-closed, $X \subseteqq E, X \rightarrow M$. By Corollary 2.3 and the above remarks $M = x^\perp$ for some $x \in cl_{\tilde{E}}\mathfrak{L}(X)$. If X is finite then $\mathfrak{L}(X)$ is closed in $\tilde{E}$, being being finite dimensional, so $x \in E$; if X is infinite the set Y of all finite linear combinations of elements of X with rational coefficients has the same cardinality as X and since Y is dense in X, $x \in cl_{\tilde{E}} Y \leqq E$ by hypothesis. In either case, $M = x^\perp$ is closed and we are we are done.

From Lemma 4.2 we obtain

COROLLARY 4.3. *Let* $X \subseteqq E$ *and suppose that* $x \in cl_{\tilde{E}}X$. *Then* $X \rightarrow x^\perp$ *and* $x^\perp$ *is* (card X, $\aleph_0$)*-closed.*

By [7, Th 4.3, p. 310] and the proof of (c) $\Rightarrow$ (a) we have

COROLLARY. 4.4. $X \rightarrow M = x^\perp$ *for some* $x \in E$ *if and only if for all equicontinuous* Q *in* E', x, *considered as a function on* Q, *is continuous in the topology of uniform convergence on finite subsets of* X.

We now turn our attention to $(\alpha, 1)$-completeness. We will need notions of completeness and convergence which as far as we know are original. Moreover, they have applications to other areas of topological vector spaces (see [5]).

DEFINITION. Let E be a lcs, $\mathscr{P}$ a set of continuous pseudo-norms on E, and $\{x_\lambda\}$ a net in E. Then $\{x_\lambda\}$ is *0-convergent to x on $\mathscr{P}$* if for any $p \in \mathscr{P}$, there is a λ_0 such that if $\lambda \geqq \lambda_0$ then $p(x - x_\lambda) = 0$. If $\mathscr{P}$ is the set of all continuous pseudo-norms on E, we say that $\{x_\lambda\}$ is *0-convergent to x*. *0-Cauchy on $\mathscr{P}$* and *0-Cauchy* are defined in an analogous manner.

We collect some useful facts in the Proposition below. The proofs are trivial.

PROPOSITION 4.4. Let E, $\mathscr{P}$, and $\{x_\lambda\}$ be as in the definition above. Then

(a) If $\{x_\lambda\}$ 0-converges to x on $\mathscr{P}$, then $\{x_\lambda\}$ is 0-Cauchy on $\mathscr{P}$ and $\{x_\lambda\}$ converges to x, and conversely, if $\{x_\lambda\}$ is 0-Cauchy on $\mathscr{P}$ and converges to x, then $\{x_\lambda\}$ is 0-convergent to x on $\mathscr{P}$.

(b) If $\mathscr{P}$ defines the topology of E, then $\{x_\lambda\}$ is 0-Cauchy [0-convergent to x] if and only if it is 0-Cauchy on $\mathscr{P}$ [0-convergent to x on $\mathscr{P}$].

DEFINITION. Let E, $\mathscr{P}$ be as above. Then E is *α-0-complete* [*on $\mathscr{P}$*] if and only if every net $\{x_\lambda\}$ in E with card $\{x_\lambda\} \leqq \alpha$ which is 0-Cauchy [on $\mathscr{P}$] is 0-convergent [on $\mathscr{P}$]. If E is V-0-complete, we say that E is *0-complete*.

An immediate consequence of Proposition 4.4 is:

PROPOSITION 4.5.

(a) Every complete lcs is 0-complete.

(b) If $\mathscr{P}$ defines the topology of E then E is α-0-complete if and only if E is α-0-complete on $\mathscr{P}$.

Proposition 4.5 (b) states that 0-completeness is independent of the set of pseudo-norms defining a topology, a fact which is useful in many proofs.

We may now characterize $(\alpha, 1)$-completeness.

THEOREM 4.6. *The following conditions on a lcs E are epuivalent:*

(a) *E is $(\alpha, 1)$-complete.*

(b) *E is* $(\alpha, \aleph_0 -)$*-complete.*
(c) *E is* α*-0-complete.*

Proof. The equivalence of (*a*) and (*b*) follows from Corollary 2.16.

(a) $\Rightarrow$ (c). Let $\{x_\lambda\}$ be 0-Cauchy, hence 0-convergent to some $x \in \tilde{E}$. For any equicontinuous $Q \subseteqq E'$ there is a λ_0 such that $\lambda, \mu \geqq \lambda_0$ implies that $p_Q(x_\lambda - x_\mu) = 0$. Thus we may set $\lambda_n = \lambda_0$ for all n in Lemma 4.2, and we obtain

$$\begin{aligned} x^\perp \cap Q_{(p_Q)} &= x^\perp \cap Q^{00} \\ &= \bigcap_{n=1}^{\infty} \mathrm{Sl}\,(x_{1_0}, 1/n) \cap Q^{00} \\ &= \mathrm{Sl}\,(x_{\lambda_0}, 0) \cap Q^{00}\,, \end{aligned}$$

and hence $x^\perp \cap Q = \mathrm{Sl}\,(x_{\lambda_0}, 0) \cap Q$ and $x^\perp$ is (card $\{x_\lambda\}$, 1)-closed.

(c) $\Rightarrow$ (a). Let $M \subseteqq E'$ be an $(\alpha, 1)$-closed hyperspace, $X \to M$ for $X \subseteqq E$. Let $u \in E' \sim M$, let X be normalized by u, and let $\mathscr{Q}$ be the set of all equicontinuous $Q \subseteqq E'$ such that $Q = \Gamma(Q \cap M, ru)$ for some $r > 0$. Let $x \in \tilde{E}$ be such that $x^\perp = M$ and $u(x) = 1$. If $Q \in \mathscr{Q}$, there is an $x_Q \in X$ such that $Q \cap M = \mathrm{Sl}\,(x_Q, \varepsilon_{x_Q,Q})$, and by Proposition 2.14, $\varepsilon_{x_Q,Q} = 0$. Hence by (2.11), $\tilde{p}_Q(x - x_Q) = 0$ and if $Q' \in \mathscr{Q}$, $Q' \supseteqq Q$, then

$$\tilde{p}_Q(x - x_{Q'}) \leqq \tilde{p}_{Q'}(x - x_{Q'}) = 0\,.$$

Therefore ordering $\mathscr{Q}$ by inclusion makes $\{x_Q\}$ a net in E which is 0-convergent to x on $\{p_Q : Q \in \mathscr{Q}\}$, and hence is 0-convergent by Lemma 2.5 and Proposition 4.4 (b). Moreover card $\{x_Q\} \leqq$ card $X \leqq \alpha$ so $x \in E$ by assumption.

In the case of $(V, 1)$-completeness, we have some additional properties.

COROLLARY 4.7. *The following conditions of a lcs E are equivalent*:

(a) *E is 0-complete*
(b) *E is* $(V, 1)$*-complete*
(c) *E is* $(V, \aleph_0 -)$*-complete*
(d) *If M is a hyperspace in E′ such that for every equicontinuous subset Q of E′ there is an* $x \in E$ *such that* $M \cap Q = x^\perp \cap Q$, *then M is closed.*
(e) *If M is a hyperspace in E′ such that for every equicontinuous subset Q of E′,* $M \cap Q = [cl(\mathfrak{L}(M \cap Q))] \cap Q$, *then M is closed.*

Proof. We have already shown the equivalence of (a) through (c) and their equivalence with (d) follows from Propositions 2.7 and

2.14. Clearly (e) $\Rightarrow$ (d) so we will be done if we can show that (d) $\Rightarrow$ (e). To this end, let M be a hyperspace in E', Q an ac equicontinuous subset of E', $F = cl(\mathfrak{L}(M \cap Q))$, and $M \cap Q = F \cap Q$. We wish to find an $x \in E$ such that $x^\perp \cap Q = M \cap Q$. If $Q \subseteqq M$ we may simply take $x = 0$, so assume there is a $u \in Q \sim M = Q \sim F$. Define $f : \mathfrak{L}(F, u) \rightarrow \boldsymbol{R}$ by $f(v + ru) = r$ for $v \in F$. f is continuous (its null-space is closed) and thus may be extended to a continuous $\bar{f}$ on E'. Choose $x \in E$ such that $\bar{f}(v) = v(x)$ for $v \in E'$. Clearly $x^\perp \cap Q \supseteqq F \cap Q = M \cap Q$ and if $v \in x^\perp \cap Q$, $v = w + ru$ for some $w \in M$, $r \in \boldsymbol{R}(E' = \mathfrak{L}(M, u))$ so

$$w/(1 + |r|) = ru/(1 + |r|) - v/(1 + |r|) \in Q, M .$$

Since $Q \cap M = Q \cap F$, $w \in F$ so $0 = v(x) + ru(x) = ru(x)$. $u(x) \neq 0$ and hence $r = 0$ and $v \in M$, completing the proof.

We complete our characterizations of (α, β)-completeness on E itself by noting that every lcs is $(\aleph_0 -, \aleph_0 -)$-complete by Corollary 2.15.

It is interesting to note that we have proven all of the implications between (α, β)-and (γ, δ)-completeness for various α, β, γ, δ. This can be demonstrated by showing the independence of $(\alpha, 1)$-and $(\gamma, \aleph_0)$-completeness for $\alpha > \gamma \geqq \aleph_0$. Since any noncomplete normed linear space is clearly $(\boldsymbol{V}, 1)$-complete but not $(\aleph_0, \boldsymbol{V})$-complete, we need only find a $(\gamma, \aleph_0)$-complete space which isn't $(\alpha, 1)$-complete. Let Λ be an index set of cardinality α and let E be the subspace of $\Pi_{\lambda \in \Lambda}\boldsymbol{R}(= \boldsymbol{R}^\alpha)$ consisting of all elements all but γ of whose coordinates are 0. Letting $\langle x_\lambda \rangle$ be the element of E whose λ th coordinate is x_λ, we set $p_\lambda(\langle x_\lambda \rangle) = |x_\lambda|$. Then $\{p_\lambda\}$ defines the topology of E and clearly the $(\alpha, 1)$-completion on $\{p_\lambda\}$ of E is $\boldsymbol{R}^\alpha$. However if $X \subseteqq E$ has cardinality no greater than γ, the number of coordinates in which some element of X is nonzero has cardinality no greater than $\gamma^2 = \gamma$, so E is $(\gamma, \boldsymbol{V})$-complete.

5. **Nets of hyperspaces.** We end our study with a brief outline of a dual theory of net convergence. The theory is of necessity sketchy and incomplete, but we will present some possibilities for further investigation. We begin in the spirit of the preceding sections and will ultimately translate our ideas into notions concerning only the dual space.

Let $\{x_\lambda\}$ be a net in E and let M be a $(\boldsymbol{V}, \boldsymbol{V})$-closed hyperspace in E'. We say that $\{x_\lambda^\perp\}$ *converges to* M if $\lim_\lambda \Delta_{x_\lambda, Q} = 0$ for all equicontinuous $Q \neq M$ in E' (compare with Lemma 2.8). By Lemma 2.13, if $\{x_\lambda^\perp\}$ converges to M, then $\{x_\lambda\} \rightarrow M$ and by Lemma 2.9, we have immediately

PROPOSITION 5.1. Let $x \in \tilde{E}$ and let $\{x_\lambda\} \subseteqq E$ be normalized by u for some $u \in E'$ with $u(x) = 1$. Then $\lim_\lambda x_\lambda = x$ if and only if $\{x_\lambda^\perp\}$ converges to $x^\perp$.

Notice that it is necessary to assume that M is (V, V)-closed (or at least that $M = f^\perp$ where f is bounded on equicontinuous subsets) in order to prove that $\{x_\lambda\} \rightarrow M$, even though this hypothesis is not necessary for Lemma 2.13. The reason is that the Lemma concerns only those Q of the form $Q = \Gamma(Q \cap M, ru)$, and we know that this class is sufficiently large only when Lemma 2.5 is applicable. This in fact is one of the major problems of this study and one to which we will return for further comment at the end of this section.

We now consider our notion entirely in terms of E'. Let $\mathscr{H}[\overline{\mathscr{H}}]$ be the set of all closed $[(V, V)$-closed$]$ hyperspaces in E'. For any $H, M \in \overline{\mathscr{H}}$ and Q an equicontinuous subset of E', we define a number $Q(H, M)$ as follows:

First we assume that Q is ac and closed. Then there is a hyperplane of support of Q, H', which is parallel to H, and a maximal r such that $0 \leqq r \leqq 1$ and rH' is a hyperplane of support to $Q \cap M$. We define $Q(H, M) = r$. For arbitrary Q set

$$Q(H, M) = [cl\Gamma(Q)](H, M) .$$

It is easy to see that if $H = x^\perp$, then $Q(H, M) = \Delta_{x,Q}$. If we define $\lim_\lambda H_\lambda = M$ to mean $\lim_\lambda Q(H_\lambda, M) = 0$ for all $Q \cap M$, then by Proposition 5.1 we have

PROPOSITION 5.2. Let $\{H_\lambda\}$ be a net in $\mathscr{H}$ and $M \in \overline{\mathscr{H}}$. Then $\lim_\lambda H_\lambda = M$ if and only if there are $x_\lambda \in E$ with $x_\lambda^\perp = H_\lambda$ for each λ and an $x \in \tilde{E}$ with $x^\perp = M$ such that $\lim_\lambda x_\lambda = x$.

Unfortunately the above characterization introduces more problems than it solves. It would be desirable to avoid the assumption that $M \in \overline{\mathscr{H}}$ in Proposition 5.2 but as we mentioned above, we do not know if this can be done. Moreover we have been able to obtain no results at all on a Cauchy-type theory, with at least part of the problem certainly due to the lack of knowledge mentioned above. We are further bothered by the fact that $Q(H, M)$ is not symmetric in H and M, as the reader can demonstrate to himself by letting Q be a square in $\boldsymbol{R}^2$ with H parallel to a pair of sides and M not parallel to either pair of sides. Nonetheless we feel that further study of these concepts or modifications thereof could prove quite interesting.

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