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EMBEDDING THEOREMS FOR COMMUTATIVE BANACH ALGEBRAS

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One knows from the Gelfand theory that every commutative semisimple Banach algebra $\mathfrak{A}$ containing an identity is a separating subalgebra of the algebra of all complex continuous functions on the space of maximal ideals of $\mathfrak{A}$. We shall be concerned in this paper with conditions which when imposed on a separating Banach subalgebra $\mathfrak{A}$ of $C(\Omega)$, Ω a compact Hausdorff space, will guarantee that $\mathfrak{A} = C(\Omega)$. The conditions will take the form of restrictions on either the algebra or the space Ω. For example we prove that if $\mathfrak{A}$ is an ε-normal Banach subalgebra of $C(\Omega)$ then $\mathfrak{A} = C(\Omega)$ if an appropriate boundedness condition holds locally on Ω. If Ω is assumed to be an F space in the sense of Gillman and Henriksen this boundedness assumption is redundant. These results include a recent characterization of Sidon sets in discrete groups due to Rudin and have applications to interpolation problems for bounded analytic functions.

Various conditions which guarantee that $\mathfrak{A} = C(\Omega)$ are known. One, due to Glicksberg [5], is the following.

(1) Assume $\mathfrak{A}$ is sup-norm closed, contains the constants, and in addition assume that the restriction of $\mathfrak{A}$ to each closed subset F of Ω is a closed subalgebra of $C(F)$.

Another, due to the present authors [1], is the following:

(2) Assume Ω is a totally disconnected F-space and that Ω is the maximal ideal space for $\mathfrak{A}$.

A compact space Ω is an F-space if disjoint open F_σ sets in Ω have disjoint closures. This class of spaces was introduced by Gillman and Henriksen in [4] and includes stonian and σ-stonian spaces as well as their closed subsets. There are also connected examples.

The results in this paper center around extensions of these conditions as well as others due to Katznelson [11, 12]. Many of the techniques apply equally well in a Banach space setting, and are discussed in this way where possible.

To begin the discussion we need the following definition: given $\varepsilon > 0$, call a subset $\mathfrak{F} \subset C(\Omega)$ an *ε-normal family* if for each pair F_1, F_2 of disjoint compact subsets of Ω there exists an $x \in \mathfrak{F}$ satisfying

(i) $|x(\omega)-1| < \varepsilon, \qquad \omega \in F_1,$

(ii) $|x(\omega)| < \varepsilon, \qquad \omega \in F_2.$

By a *Banach subalgebra* of $C(\Omega)$ we will mean a subalgebra of $C(\Omega)$, not necessarily containing the constants, which is a Banach algebra under some norm.

The result of the paper which illustrates the unusual properties of F-spaces is the following:

THEOREM A. *Let Ω be a compact F-space and $\mathfrak{A}$ be a Banach subalgebra of $C(\Omega)$. If $\mathfrak{A}$ is an ε-normal subalgebra for an $\varepsilon < 1/2$ (in particular if $\mathfrak{A}$ is dense) then $\mathfrak{A} = C(\Omega)$.*

In addition to extending the result of [1] mentioned in (2) above, this theorem contains Rudin's characterization of Sidon sets in discrete groups and has applications to interpolation problems for bounded analytic functions.

For general compact spaces Ω, we will call an ε-normal family contained in a Banach subalgebra of $C(\Omega)$, or more generally contained in the continuous image in $C(\Omega)$ of a Banach space X, *locally bounded* if for each point ω there exists a compact neighborhood N_ω such that whenever the sets F_1, F_2 belong to N_ω, the x may be chosen to have $\mathfrak{A}$-norm (or X norm) less than a constant depending on ω.

Our extension of (1) and the results of Katznelson is basically contained in the following:

THEOREM B. *Let Ω be a compact Hausdorff space, $\mathfrak{A}$ a Banach subalgebra of $C(\Omega)$. Then $\mathfrak{A} = C(\Omega)$ if the following conditions are satisfied.*

(1) *$\mathfrak{A}$ is an ε-normal family for some $\varepsilon < 1/2$.*

(2) *$\mathfrak{A}$ is locally bounded.*

If Ω is an F-space (1) implies (2), thus proving Theorem A. Indeed for F-spaces much can be said when $\mathfrak{A}$ is only assumed to be the continuous image of a Banach space X.

THEOREM C. *Let Ω be a compact F-space and T a continuous linear map of a Banach space X into $C(\Omega)$ such that TX forms an ε-normal family for some $\varepsilon < 1/4$. Then there exists a finite open covering $U_1, \cdots, U_n$ of Ω such that*

$$TX \mid \bar{U}_i = C(\bar{U}_i),\ i = 1, \cdots, n .$$

In general, linear subspaces of $C(\Omega)$ may be ε-normal, $\varepsilon < 1/4$, without being dense, but for totally disconnected spaces density and ε-normality are equivalent. A short argument shows that ε-normality and density are also equivalent for arbitrary F-spaces.

Theorem C raises the question whether a continuous linear map T of a Banach space X into $C(\Omega)$, Ω an F-space, which has dense range, must be onto $C(\Omega)$. Section 4 contains an example due to J. Lindenstrauss which shows that $TX \neq C(\Omega)$ in general. G. Seever [16] has proved that $TX = C(\Omega)$ if one has the stronger assumption that TX is normal on Ω. We give a new proof of Seever's theorem. An unpublished result of Beurling which covers the case when X is an adjoint space, $C(\Omega) = l_\infty$, and T is weak star continuous is proved by an argument of Helson.

We are indebted to J. Lindenstrauss and Y. Katznelson for the elegant examples in § 4, and to Katznelson for illuminating discussions of the problems of this paper.

1. Preliminaries. In this section we collect some facts about approximation and onto maps. Given a bounded linear map $T: X \to C_0(\Omega)$, where Ω is locally compact and X is a Banach space, one seeks conditions to insure that $TX = C_0(\Omega)$. These results are not new for the most part, and the techniques appear in a variety of settings. We first consider the general case $T: X \to Y$, where Y is a Banach space. Recall that a subset $E \subseteq Y$ is equilibrated if y in E and $|\alpha| \leqq 1$ implies $\alpha y \in E$. The smallest convex equilibrated set containing E is denoted by $coe(E)$, the smallest convex set by $co(E)$, their respective closures by $\overline{coe}(E)$ and $\overline{co}(E)$.

LEMMA 1.1. *Let E be a subset of Y. Then $\overline{coe}(E)$ contains the closed unit ball S of Y if and only if*

$$\| y^* \| \leqq \sup_{y \in E} | y^*(y) |$$

for each $y^* \in Y^*$.

Proof. Note $coe(E)$ consists of all sums $\sum_{i=1}^n \alpha_i y_i$, where $y_i \in E$ and $\sum_{i=1}^n |\alpha_i| \leqq 1$. If $\overline{coe}(E) \supseteq S$, then for each $y^* \in Y^*$ and $\varepsilon > 0$ we can find $y_0 = \sum_{i=1}^n \alpha_i y_i \in coe(E)$ such that

$$| y^*(y_0) | > \| y^* \| - \varepsilon .$$

For some i we must have $| y^*(y_i) | > \| y^* \| - \varepsilon$. Conversely, if $y_0 \in S$, but $y_0 \notin \overline{coe}(E)$, the separation theorem [3, p. 417] yields a functional y_0^* such that

$$\| y_0^* \| \geqq \operatorname{Re} y_0^*(y_0) > \sup \{\operatorname{Re} y_0^*(y) : y \in \overline{coe}(E)\} .$$

Since $\overline{coe}(E)$ is equilibrated, we have

$$\| y_0^* \| > \sup | y_0^*(y) | .$$

THEOREM 1.2. *Let $T: X \to Y$ be linear and continuous. Suppose* *there exists a set* $E \subseteq Y$ *and constants* $k < 1$ *and* K *such that*

(1) $\overline{coe}(E)$ *contains the closed unit ball* S *of* Y.

(2) *For each* $y \in E$ *there exists an* $x \in X$ *such that*

$$\| Tx - y \| \leqq k, \| x \| \leqq K .$$

Then $TX = Y$. *If* T *is one to one,* $\| T^{-1} \| \leqq K(1-k)^{-1}$.

Proof. For any $y^* \in Y^*$ and $y \in E$, select x by (2) and note

$$\begin{aligned} | y^*(y) | &\leqq | y^*(y - Tx) | + | y^*(Tx) | \\ &\leqq k \| y^* \| + | (T^* y^*)(x) | \\ &\leqq k \| y^* \| + K \| T^* y^* \| . \end{aligned}$$

Taking sups on y in E yields

$$\| y^* \| \leqq k \| y^* \| + K \| T^* y^* \|$$

by Lemma 1. Thus

$$\| T^* y^* \| \geqq K^{-1}(1 - k) \| y^* \| , y^* \in Y^* ,$$

showing T^*, and hence T has a closed range [3, p. 488]. However the argument above shows TX is dense in Y, since if $y^*(TX) = 0$, $y^* \neq 0$, then

$$| y^*(y) | \leqq k \| y^* \| < \| y^* \| , y \in E,$$

in contradiction of Lemma 1.1. The result now follows.

Theorem 1.2 is due to Katznelson [11] in slightly different form. He constructs a solution of $Tx = y$ by successive approximations. The proof above follows an argument of Rudin to prove a theorem on Helson sets [15, p. 116], (Corollary 1.3 below) which is a special case of Theorem 1.1. Recall that if G is a locally compact abelian group, a compact set $P \subseteq G$ is a *Helson set* if each continuous function on P is the restriction $\hat{f} | P$ of the Fourier transform of an element of L_1 on the dual group Γ.

COROLLARY 1.3. *Suppose* P *is compact in* G *and there exist constants* $k < 1$ *and* K *such that to each* $F \in C(P)$ *with* $| F(t) | = 1$, $t \in P$, *there corresponds an element* $f \in L_1(\Gamma)$ *such that* $\| f \|_1 \leqq K$ *and*

$$\sup_{t \in P} | \hat{f}(t) - F(t) | \leqq k .$$

Then P *is a Helson set.*

Proof. Define $T: L_1(\Gamma) \rightarrow C(P)$ by $Tf = \hat{f} | P$. The extreme points of the unit ball S in $C(P)$ are precisely the functions of absolute value one. Lemma 1.1 and Lemma 5.5.1 of [15] show S is the norm closed convex hull of its set of extreme points. The result now follows directly from Theorem 1.2.

We next consider the case that Y is $C_0(\Omega)$, the continuous functions vanishing at infinity on a locally compact Hausdorff space. A subset E of $C_0(\Omega)$ is a *normal family* if for each pair F_1, and F_2 of disjoint compact sets in Ω there exists $f \in E$ with $f(F_1) = 0, f(F_2) = 1$. We denote the nonnegative functions in the closed unit ball S of $C_0(\Omega)$ by S_+.

LEMMA 1.4. *If E is a bounded normal family, then $\overline{co}(E)$ contains S_+.*

Proof. It suffices to approximate any element f of S_+ having compact support by elements of $co(E)$. Let n be any positive integer and $L = \sup\{\|f\|: f \in E\}$. Let C be a compact set whose interior contains the support of f. Define

$$U_k = \left\{\omega \in \Omega: f(\omega) \geqq \frac{k}{n}\right\},$$
$$V_k = \left\{\omega \in C: f(\omega) \leqq \frac{k-1}{n}\right\}, \qquad k = 1, 2, \cdots, n-1 .$$

Choose $f_1 \in E$ with

$$f_1(U_1) = 1 , \qquad f_1(V_1) = 0 .$$

Let $W_1 = \{\omega \notin C: |f_1(\omega)| \geqq 1/n^2\}$. Then W_1 is compact. Choose $f_2 \in E$ with

$$f_2(U_2) = 1 , \qquad f_2(V_2 \cup W_1) = 0 .$$

Let $W_2 = \{\omega \notin C: |f_2(\omega)| \geqq 1/n^2\}$, etc. Continuing, we finally obtain $f_{n-1} \in E$ with

$$\begin{aligned} f_{n-1}(\omega) &= 1 , \quad \omega \in U_{n-1} \\ &= 0 , \quad \omega \in V_{n-1} \cup W_1 \cup \cdots \cup W_{n-2} . \end{aligned}$$

Define the function $g = 1/n \sum_{k=1}^{n-1} f_k$. We estimate $\|f - g\|_\infty$. Note that

$$|f(\omega) - g(\omega)| \leqq \frac{L+1}{n} , \qquad \omega \in C .$$

Now $f(\omega) = 0, \omega \notin C$, the sets W_i are disjoint and compact, and

$$|g(\omega)| \leqq \frac{1}{n^2} , \qquad \omega \notin \cup_{i=1}^{n-2} W_i .$$

If $\omega \in W_i$,

$$f_j(\omega) = 0\ , \quad j > i$$
$$|f_j(\omega)| \leqq \frac{1}{n^2}\ , \quad j < i$$
$$|f_i(\omega)| \leqq L\ .$$

Thus

$$|g(\omega)| \leqq \frac{1}{n}\sum_{j=1}^{i}|f_j(\omega)| \leqq \frac{1}{n^2} + \frac{L}{n}\ , \quad \omega \in W_i\ ,$$

and we have proved

$$\|f - g\|_\infty \leqq \frac{L+1}{n}\ .$$

Now $h = n(n-1)^{-1} g \in co(E)$. Since n is arbitrary, the result follows.

THEOREM 1.5. *Let Ω be locally compact and T: $X \rightarrow C_0(\Omega)$ be linear and continuous. Suppose there exist constants $\varepsilon < 1/4$ and M such that if F_1 and F_2 are any disjoint compact sets in Ω, these exists an $x \in X$ such that*
(i) $|(Tx)(\omega)| \leqq \varepsilon\ , \quad \omega \in F_1$
(ii) $|(Tx)(\omega) - 1| \leqq \varepsilon\ , \quad \omega \in F_2$
(iii) $\|x\| \leqq M$.
Then $TX = C_0(\Omega)$. If T is one-to-one then $\|T^{-1}\| \leqq 4M(1-4\varepsilon)^{-1}$.

Proof. Let $\varepsilon < \varepsilon' < 1/4$ and E be any bounded normal family in $C_0(\Omega)$. If F_1 and F_2 are any disjoint compact sets, we can find $x \in X$ with
$\|x\| \leqq M$, $|(Tx)(F_1) - 1| \leqq \varepsilon$, $|(Tx)(F_2)| \leqq \varepsilon$, and $f \in E$ with

$$f(F_1) = 1\ , \qquad f(F_2) = 0\ .$$

Since $C(F_1 \cup F_2)$ is isometrically isomorphic with the quotient of $C_0(\Omega)$ by the ideal of functions zero on $F_1 \cup F_2$, we can select $g_f \in C_0(\Omega)$ with $g_f(F_1 \cup F_2) = 0$ and

$$\|Tx - (f + g_f)\| \leqq \varepsilon'\ .$$

Then $E' = \{f + g_f : f \in E\}$ is a bounded normal family, $\overline{co}(E') \supseteq S_+$ by Lemma 1.4, and hence $\overline{coe}(4E') \supseteq S$. For each $g \in 4E'$ we can find an x with

$$\|Tx - g\| \leqq 4\varepsilon'\ , \qquad \|x\| \leqq 4M\ .$$

Theorem 1.5 now follows from Theorem 1.2.

REMARK. Often in applications it is just as convenient to verify that for each compact set $F \subseteq \Omega$ and $f \in C(F)$ there exists $x \in X$ satisfying $\|x\| \leqq M$ and $\|Tx - f\|_\infty \leqq \varepsilon \|f\|_\infty$. This of course implies the condition of Theorem 1.5.

If TX satisfies (i, ii) we shall call TX an *ε-normal family*. If (iii) is satisfied we call TX *boundedly ε-normal*. If TX is ε-normal and for each $\omega \in \Omega$ (iii) holds when F_1, F_2 belong to a suitably small compact neighborhood of ω, then we call TX *locally* bounded. Note that Theorem 1.5 yields:

COROLLARY 1.6. *If Ω is compact and TX is ε-normal and locally bounded, there exist finitely many compact sets $S_1, \cdots, S_n$ whose interiors cover Ω and such that $TX \mid S_i = C(S_i)$.*

COROLLARY 1.7. *If Ω is compact and totally disconnected and $\mathfrak{M}$ is a linear subspace of $C(\Omega)$ which is ε-normal for $\varepsilon < 1/4$, then $\mathfrak{M}$ is dense in $C(\Omega)$.*

Proof. Let X be the closure of $\mathfrak{M}$ in the sup-norm and T be the natural injection map of X into $C(\Omega)$. Since F_1 and F_2 may be enclosed in disjoint open and closed sets having Ω as their union condition (iii) of Theorem 1.5 is satisfied with $M = 1 + \varepsilon$.

It is important to note in the absence of boundedness 1.4 and 1.5 are false. The recent example of McKissick [13] of a sup-norm closed normal function algebra which is not $C(\Omega)$ provides a counter example.

We note in passing that if Ω is compact and $\omega \in \Omega$, then $TX = C(\Omega)$ if and only if $(TX)_\omega = C_0(\Omega \sim \{\omega\})$, where for a linear subspace $\mathfrak{M} \subseteq C(\Omega)$, $\mathfrak{M}_\omega = \{f \in C(\Omega): f(\omega) = 0\}$. Also, as shown by Seever [16], $TX = C(\Omega)$ if and only if for each measure μ on Ω, the restriction of TX to the carrier of μ is all continuous function on the carrier.

2. We now specialize to the situation when X is a Banach algebra of continuous functions vanishing at ∞ on Ω. In this case we shall write $\mathfrak{A}$ for X. Then of course $|x(\omega)| \leqq \|x\|$ for $x \in \mathfrak{A}$. Hence $\|x\|_\infty \leqq \|x\|$ and the embedding of $\mathfrak{A}$ into $C_0(\Omega)$ is continuous. Also if an algebra is ε-normal for some $\varepsilon < 1/2$, it is ε-normal for every such ε. This is clear since the function f defined by

$$\begin{aligned} f(z) &= 0\,, \quad |z| \leqq \varepsilon \\ &= 1\,, \quad |1 - z| \leqq \varepsilon \end{aligned}$$

can be uniformly approximated on these sets by polynomials without constant term. Indeed, a condition equivalent to ε-normality for algebras is the following: For each pair of disjoint compact sets S_0, S_1

of Ω, there exists an $x \in \mathfrak{A}$ and disjoint open sets V_0, V_1 of complex numbers such that $x(S_1) \subset V_i$, and $C \sim \overline{(V_0 \cup V_1)}$ is connected and contains the origin.

That this is equivalent to ε-normality is easily seen. If $\varepsilon < 1/6$ and $|x(S_0)| \leqq \varepsilon$, $|1 - x(S_1)| \leqq \varepsilon$, $|1 - y(S_0)| \leqq \varepsilon$, and $|y(S_1)| \leqq \varepsilon$, then $2x + y$ has the desired property. For the converse note that without loss of generality we may assume that $\bar{V}_0 \cap \bar{V}_1 = \varnothing$. Let W be an open neighborhood of 0 satisfying $W \subset \bar{W} \subset C \sim \overline{(V_0 \cup V_1)}$ and such that $C \sim \overline{(V_0 \cup V_1 \cup W)}$ is connected. Then by the theorem of Mergelyan, the function f defined by $f(V_0 \cup W) = 0, f(V_1) = 1$ may be uniformly approximated on $\overline{V_0 \cup V_1 \cup W}$ by polynomials $p(z)$. Since these polynomials may be taken to have no constant term, if $x(S_i) \subset V_i$, $i = 1, 2$; then $p_n(x) \in \mathfrak{A}$, $|p_n(x)(S_0)| < \varepsilon$, and $|1 - p_n(x)(S_1)| < \varepsilon$.

Suppose now Ω is compact and the ε-normal Banach subalgebra of $C(\Omega)$ is locally bounded. Then by Corollary 1.6 there exist finitely many compact sets $S_1, \cdots, S_n$ whose interiors cover Ω such that $\mathfrak{A} \mid S_i = C(S_i)$. We shall prove $\mathfrak{A} = C(\Omega)$. Incidentally the ability to match each continuous function on the sets of a covering is not sufficient to prove $\mathfrak{A} = C(\Omega)$ without the assumption of ε-normality. To see this let

$$\mathfrak{A} = \{x \in l_\infty : x(n) = x(-n)\} .$$

For $\mathfrak{A}$ restricted to the negative or to the nonnegative integers yields all bounded sequences on these sets, but $\mathfrak{A} \neq l_\infty$. We begin with the following

THEOREM 2.1. *Let Ω be a compact Hausdorff space and $\mathfrak{A}$ be an ε-normal Banach subalgebra of $C(\Omega)$. Suppose there exists a finite covering of Ω by open sets U_i such that $\mathfrak{A} \mid \bar{U}_i = C(\bar{U}_i)$. Then $\mathfrak{A} = C(\Omega)$.*

Proof. We first make a definition.

Let $U_1, \cdots, U_n$ be an open covering of a topological space. An ε-partition of unity subordinate to this covering is a set of continuous functions $f_1, \cdots, f_n$ such that

$$\begin{aligned} &|f_i(\omega)| \leqq \varepsilon \qquad \omega \notin U_i \\ &f_1 + \cdots + f_n = 1 . \end{aligned}$$

LEMMA 2.2. *Let $\mathfrak{A}$ satisfy the hypothesis of Theorem 2.1. Then for each $\varepsilon > 0$ $\mathfrak{A}$ contains an ε-partition of unity subordinate to the given covering.*

Proof. We observe first the following identity for complex numbers $\lambda_1, \cdots, \lambda_n$

$$(\sharp)\quad \prod_{i=1}^{n}(1-\lambda_i) = 1 - \lambda_1 \prod_{i=2}^{n}(1-\lambda_i) - \lambda_2 \prod_{i=3}^{n}(1-\lambda_i) - \cdots - \lambda_n$$

which is easily proved by induction. Next we assert that if U is one of the open sets U_i, then there is a constant M such that for each compact set $S \subseteq U$ and $\varepsilon > 0$ there exists $x \in \mathfrak{A}$ satisfying $|x(\omega)| \leqq \varepsilon$ off U, $x(S) = 1$ and $\|x\|_\infty \leqq M$. To prove this, note that by the closed graph theorem there is a constant C such that if $f \in C(\bar{U})$, then $\hat{f}$ may be chosen in $\mathfrak{A}$ satisfying

$$\hat{f} = f \text{ on } \bar{U}, \quad \text{and}$$
$$\|\hat{f}\| \leqq C\|f\|_\infty .$$

Let $0 < \varepsilon < 1/2$. Pick $h \in \mathfrak{A}$ satisfying

$$|1 - h(\omega)| \leqq \varepsilon \qquad \omega \in S$$
$$|h(\omega)| \leqq \varepsilon \qquad \omega \notin U .$$

Let $T = \{\omega : |h(\omega)| \geqq 1/2\}$. By the Tietze extension theorem there exists $g \in C(\bar{U})$ satisfying $g(\omega) = h^{-1}(\omega)$, $\omega \in T$,

$$\|g\|_\infty \leqq 2 .$$

Therefore if $\hat{g} \in \mathfrak{A}$, $\hat{g} = g$ on U, and $\|\hat{g}\| \leqq 2C$, we may take $x = \hat{g}h$. Then $|x(\Omega \sim U)| \leqq 2C\varepsilon$, $x(T) = 1$, and $\|x\|_\infty \leqq C$.

Now select open sets V_i, $i = 1, \cdots, n$, covering Ω and satisfying $V_i \subset \bar{V}_i \subset U_i$. Choose $f_n, \cdots, f_1 \in \mathfrak{A}$ in turn such that

$$|f_k(\omega)| \leqq \varepsilon_k \text{ off } U_k$$
$$f_k(\omega) = 1 \text{ on } V_k$$

where $\varepsilon_n = \varepsilon$, $\varepsilon_k = \varepsilon \|\prod_{k+1}^{n}(1 - f_i)\|^{-1}$, $k = 1, \cdots, n-1$. Let $x_k = f_k \prod_{k+1}^{n}(1 - f_i)$; $x_k \in \mathfrak{A}$ and $|x_k(\omega)| \leqq \varepsilon$ off U_k. Since $\prod_{k=1}^{n}(1 - f_k)(\omega) = 0$, $\omega \in \Omega$, it follows from ($\sharp$) that $1 = x_1 + \cdots + x_n$.

To finish the proof of the theorem we apply 1.2. That is, we assert for $0 < \varepsilon < 1$ there is a constant M such that for $f \in C(\Omega)$, $\|f\|_\infty \leqq 1$, we may choose $\hat{f} \in \mathfrak{A}$ satisfying

$$\|f - \hat{f}\| \leqq \varepsilon$$
$$\|\hat{f}\| \leqq M .$$

To do this observe that there exists a constant K and functions $f_1, \cdots, f_n \in \mathfrak{A}$ satisfying $f_i = f$ on U_i and $\|f_i\| \leqq K$. Let $x_1, \cdots, x_n$ be the ε-partition constructed via Lemma 2.2. If $\hat{f} = f_1x_1 + \cdots + f_nx_n$, then $\hat{f} \in \mathfrak{A}$. Also $\|\hat{f}\| \leqq K \sum_{i=1}^{n}\|x_i\|$ which we may take for M. To estimate $\|f - \hat{f}\|_\infty$ note that if $\omega \in U_i$, then $(fx_i - f_ix_i)(\omega) = 0$. If $\omega \notin U_i$, then

$$|(fx_i - f_ix_i)(\omega)| \leqq (\|f\| + \|f_i\|_\infty)|x_i(\omega)| \leqq (K+1)\varepsilon .$$

Therefore

$$\|f - \hat{f}\|_\infty \leqq \sum_{i=1}^{n} \|fx_i - f_i x_i\|_\infty \leqq n(K+1)\varepsilon < 1$$

for a suitable choice of ε. This finishes the proof.

COROLLARY 2.3. *Let $\mathfrak{A}$ be an ε-normal Banach subalgebra of $C(\Omega)$, Ω compact Hausdorff. If $\mathfrak{A}$ is locally bounded then $\mathfrak{A} = C(\Omega)$.*

We note that this corollary is a local version of the main theorem of [12]. For, Katznelson's condition that for each closed set F of a compact space Ω there exists an $\varepsilon = \varepsilon(F)$ such that whenever N is both closed and open in F, $\mathfrak{A}$ contains an element h of norm one satisfying $\mathrm{Re}(x(\omega)) < 0$, $\omega \in N$, $\mathrm{Re}(x(\omega)) > \varepsilon$, $\omega \in F \sim N$ is easily seen to be equivalent to 'bounded ε-normality'. That his condition implies the latter is implicit in Lemma 1 of [12], and the same sort of argument yields the converse.

Another related result is the theorem of Glicksberg [5] that if $\mathfrak{A}$ is a closed separating subalgebra of $C(\Omega)$, Ω compact, containing the constants such that the restriction to each closed subset $F \subset \Omega$ is a closed subalgebra of $C(F)$, then $\mathfrak{A} = C(\Omega)$. Theorem 2.2 yields a local version of this result which is as follows.

THEOREM 2.4. *Let $\mathfrak{A}$ be a Banach subalgebra of $C(\Omega)$ which strongly separates the points of Ω. Then $\mathfrak{A} = C(\Omega)$ if the following two conditions are satisfied:*

(i) *To each pair of points ω_1, ω_2 there exist disjoint compact neighborhoods N_1, N_2 of ω_1, ω_2 respectively such that $\mathfrak{A} \mid N_1 \cup N_2$ is closed in $C(N_1 \cup N_2)$*

(ii) *Each point ω has a compact neighborhood N_ω such that for each compact set $F \subset N_\omega$, $\mathfrak{A} \mid F$ is closed in $C(F)$.*

Proof. We shall apply 2.1. First note that the hypothesis of ε-normality in 2.1 can be weakened slightly. We need only to insist that for disjoint nonempty compact subsets F_1, F_2 of Ω there exists an $x \in \mathfrak{A}$ satisfying $|x(F_1)| < \varepsilon$, $|1 - x(F_2)| < \varepsilon$. Indeed the argument on pp. 158–9 of [5] shows that (i) implies that for disjoint nonempty compact sets F_1, F_2 of Ω there exists an $x \in \mathfrak{A}$ for which $x(F_1) = 0$, and $x(F_2) = 1$.

To complete the proof we observe that for each ω there exists a compact neighborhood N_ω such that $\mathfrak{A}_\omega$ is boundedly normal on $N_\omega \sim \{\omega\}$. This follows from condition (ii) by the same argument as in [5, Lemma 3]. To wit: If there exist neighborhoods $U_i \subset \bar{U}_i \subset U_{i-1}$ of ω_i; disjoint compact sets $S_{0,i}, S_{1,i} \subset U_{i-1} \sim \bar{U}_i$ such that if $x_i \in \mathfrak{A}_\omega$, $x(S_{0,i}) = 0$,

$x(S_{1,i}) = 1$, then $||x_i|| \geqq i$, the closed graph theorem implies that it cannot be the case that $\mathfrak{A} \mid E$ is closed in $C(F)$ where $F = \overline{\cup_i S_{0,i} \cup S_{1,i} \cup \{\omega\}}$. Therefore $\mathfrak{A} \mid N_\omega \sim \{\omega\} = C_0(N \sim \{\omega\})$ and consequently $\mathfrak{A} \mid N_\omega = C(N_\omega)$. The result now follows from 2.2.

As an application of these techniques we consider the following question:

Let Ω be compact as before and assume that $\Omega = \Omega_1 \cup \Omega_2$, Ω_i compact. The example at the beginning of Section 2 shows that if $\mathfrak{A}$ is a Banach subalgebra of $C(\Omega)$ satisfying $\mathfrak{A} \mid \Omega_i = C(\Omega)$, $i = 1, 2$, then it is not necessarily true that $\mathfrak{A} \mid \Omega = C(\Omega)$ even if Ω_1 and Ω_2 are disjoint. If, however, $\mathfrak{A}$ is normal or even ε-normal, then the disjointness of Ω_1 and Ω_2 implies $\mathfrak{A} \mid \Omega = C(\Omega)$. This is trivial in the first case and the second is a special case of Theorem 2.2.

If $\Omega_1 \cap \Omega_2 \neq \emptyset$, then it is not known whether normality of $\mathfrak{A}$ is sufficient of guarantee that $\mathfrak{A} = C(\Omega)$. In particular let $\mathfrak{A}$ be the algebra of Fourier transforms of $L_1(\Gamma)$; and Ω_1, Ω_2 be Helson sets in G (cf. Section 1). It is not known whether $\Omega_1 \cup \Omega_2$ is a Helson set.

The following theorem shows that $\mathfrak{A} = C(\Omega)$ if one assumes a certain extension property for ideals.

THEOREM 2.5. *Let $\Omega = \Omega_1 \cup \Omega_2$; Ω_i compact, and assume $\mathfrak{A}$ is a normal Banach subalgebra of $C(\Omega)$ such that $\mathfrak{A} \mid \Omega_i = C(\Omega_i)$ $i = 1, 2$. For a closed set $F \subset \Omega$, let $F_i = \Omega_i \cap F$. Let*

$$J_F = \{x \in \mathfrak{A} : x(F) = 0\},\ \tilde{J}_{F_i} = \{x \in C(\Omega_i) : x(F_i) = 0\} .$$

If $J_F \mid \Omega_i = \tilde{J}_{F_i}$, $i = 1, 2$, for each closed set F, then $\mathfrak{A} = C(\Omega)$.

Proof. Let us establish first the following:

LEMMA 2.6. *For each $\omega \in \Omega$ there exists a neighborhood N_ω of ω and a constant M such that if H_1, H_2 are disjoint compact subsets of $N_\omega \sim \{\omega\}$ and $H_i \cap \Omega_i = \emptyset$, then there exists $x \in \mathfrak{A}_\omega$ satisfying*

$$x(H_1) = 0$$
$$x(H_2) = 1$$
$$||x|| \leqq M .$$

Proof. Note if there exists a neighborhood $N_\omega \ni \{\omega\}$ such that $N_\omega \cap \Omega_i = \emptyset$ for $i = 1$ or $i = 2$ there is nothing to prove. If the lemma is false, there exists a decreasing sequence of open sets $U_i \ni \omega$ such that $U_{i+1} \subset \bar{U}_{i+1} \subset U_i$, and disjoint closed sets $H_{i,1}$, $H_{i,2}$ with $H_{i,j} \subseteqq U_i \sim \bar{U}_{i+1}$, such that for each i $H_{i,j} \cap \Omega_i = \emptyset$, $j = 1, 2$, and if $x_i \in \mathfrak{A}_\omega$; $x_i(H_{i,1}) = 0$, $x_i(H_{i,2}) = 1$ then $||x_i|| \geqq i$. Let $H = \overline{\cup H_{i,1} \cup \{\omega\}}$, and $H_1 = H \cap \Omega_1$. Applying the hypothesis together with the closed graph theorem, we see that if $\tilde{x} \in C(\Omega_1)$, $\tilde{x}(H_1) = 0$, and $\|\tilde{x}\|_\infty \leqq 1$, then there is an $x \in J_H$ satisfying $x \mid \Omega_1 = \tilde{x}$, and $\|x\| \leqq K$. Consequently, since for each i there exists $\tilde{x}_i \in C(\Omega_1)$ satisfying $\tilde{x}_i(H_1) = 0$, $\tilde{x}_i(H_{i,2}) = 1$, $\|\tilde{x}_i\|_\infty = 1$, it follows that there exists $x_i \in J_H$, $\|x_i\| \leqq K$, satisfying $x_i \mid \Omega_1 = \tilde{x}_i$. This is a contradiction.

To prove the theorem in view of the normality of $\mathfrak{A}$, it suffices to show that for each ω there exists a compact neighborhood $N_\omega \ni \omega$ such that $\mathfrak{A}_\omega \mid N_\omega = C_0(N_\omega \sim \{\omega\})$. To prove the latter statement it suffices, by the remark following 1.5, to show that there exist constants $M_1, k, k < 1$, such that for each compact set $F \subset N_\omega \sim \{\omega\}$ and $f \in C(F)$, $\|f\|_\infty \leqq 1$, there exists $\hat{f} \in \mathfrak{A}_\omega$, $\|\hat{f}\| \leqq M_1$, and $|(f - \hat{f})(\omega)| \leqq k$ for $\omega \in F$.

Choose N_ω so that 2.6 is satisfied. There exists a constant N depending only on Ω_1 and Ω_2, so that if $F \subseteq N_\omega \sim \{\omega\}$ and $f \in C(F)$, $\|f\|_\infty \leqq 1$, then there exist $f_1, f_2 \in \mathfrak{A}_\omega$ such that

$$f_1(\omega) = f(\omega), \omega \in F \cap \Omega_i, \|f_i\| \leqq N , \quad i = 1, 2.$$

Let M be the constant of Lemma 2.6 and $\varepsilon < (2M)^{-1}$. Let

$$H_i = \{\omega \in F: |f_i(\omega) - f(\omega)| \geqq \varepsilon\} .$$

Then $H_i \cap \Omega_i = \varnothing$, $i = 1, 2$, and $H_1 \cap H_2 = \varnothing$. We know by Lemma 2.6 there exist functions $x_i \in \mathfrak{A}_\omega$ satisfying $\|x_i\| \leqq M$ and

$$\begin{aligned} x_1(H_1) &= 0 , \qquad x_1(H_2) = 1 \\ x_2(H_1) &= 1 , \qquad x_2(H_2) = 0 . \end{aligned}$$

Let $\hat{f} = f_1x_1 + f_2x_2$. Then $\hat{f} \in \mathfrak{A}_\omega$, $\|\hat{f}\| \leqq 2MN$, and

$$\begin{aligned} |(\hat{f} - f)(\omega)| &\leqq |(f - f_1)(\omega)x_1(\omega)| + |(f - f_2)(\omega)x_2(\omega)| \\ &\leqq 2M\varepsilon , \quad \omega \in F. \end{aligned}$$

3. Let us return to the situation when a Banach space X is continuously imbedded by the linear mapping T as an ε-normal family in $C(\Omega)$, Ω compact Hausdorff. As was remarked in Section 1 some sort of boundedness condition was essential to guarantee that the mapping was onto. We show next that an appropriate condition can be imposed on Ω which will guarantee that T is locally bounded thus proving that locally T must be onto. The condition we need is due to Gillman and Henriksen [4].

DEFINITION 3.1. A compact Hausdorff space Ω is called an *F-space* if disjoint open F_σ sets in Ω have disjoint closures.

The Stone spaces of complete or σ-complete Boolean algebras as well as their closed sub-spaces have this property. If $C(\Omega)$ is an adjoint space, Ω is extremely disconnected, [7, Theorem 2], and is therefore an F-space. There are connected examples such as $\beta(R_+)\sim R_+$, where R_+ is the nonnegative reals [4].

REMARK. If Ω is a compact F-space, a subspace M of $C(\Omega)$ which is ε-normal for $\varepsilon < 1/4$ is necessarily dense in $C(\Omega)$. To prove this it clearly suffices to show that the restriction of M to the carrier of any measure is dense in all continuous functions on the carrier. However, K. Hoffman and (independently) G. Seever [16] have proved that the carrier of any measure on an F-space is extremely disconnected. The result now follows from Corollary 1.7. Thus in the next theorems the hypothesis of ε-normality is no gain in generality over density. However it is easier to verify in applications. The result of Hoffman and Seever shows it would suffice to prove these theorems in the case Ω is extremely disconnected and supports a measure. This observation does not simplify the present proofs.

THEOREM 3.2. *Let Ω be a compact F-space, T a continuous imbedding of a Banach space X into $C(\Omega)$ such that TX is ε-normal for some $\varepsilon < 1/4$. Then there exist finitely many compact sets $S_1, \cdots, S_n$ whose interiors cover Ω such that*

$$TX \mid S_i = C(S_i) \qquad i = 1, \cdots, n\ .$$

Proof. By Corollary 1.6 it suffices to prove that TX is boundedly ε-normal in a neighborhood of each point ω_0 of Ω. Since TX is dense in $C(\Omega)$, $(TX)_{\omega_0}$ is dense in $C_0(\Omega \sim \{\omega_0\})$ (see [17]), and by 1.5 it suffices to verify that $(TX)_{\omega_0}$ is boundedly ε-normal in a deleted neighborhood of ω_0. Suppose, on the contrary, that there exists a sequence of open neighborhoods V_i of ω_0, $V_{i+1} \subset \overline{V_{i+1}} \subset V_i$, and disjoint compact sets $F_{i,1}, F_{i,2} \subset V_i \sim \overline{V_{i+1}}$ such that if

$$(1) \qquad |(Tx-1)(\omega)| < \varepsilon \qquad \omega \in F_{i,1}\ ,$$

$$(2) \qquad |(Tx)(\omega)| < \varepsilon \qquad \omega \in F_{i,2}\ ,$$

$$(3) \qquad (Tx)(\omega_0) = 0\ ,$$

then $\|x\| \geqq i$. But there exist disjoint open F_σ sets $G_{i,1}, G_{i,2}$ containing $F_{i,1}, F_{i,2}$ respectively and contained in $V_i \sim \overline{V_{i+1}}$. Now let $G_1 = \cup_i G_{i,1}$, $G_2 = \cup_i G_{i,2}$. Then G_1, G_2 are open F_σ sets with disjoint closures H_1, H_2 respectively. We may assume $\omega_0 \notin H_1 \cup H_2$. For if $\omega_0 \in H_1$ say, replace G_1 by $G_1' = \cup_i G_{2i,1}$ or $G_1'' = \cup G_{2i+1,1}$ depending on whether $\omega_0 \notin \overline{G_1'}$ or $\omega_0 \notin \overline{G_1''}$. Choose $x \in X$ such that $(Tx)(\omega_0) = 0$, $|(Tx-1)(\omega)| < \varepsilon$, $\omega \in H_1$; $|(Tx)(\omega)| < \varepsilon$, $\omega \in H_2$. Then (1) and (2) and (3) are satisfied by x, so

$\|x\| = \infty$. This contradiction completes the proof.

Combining this with Theorem 2.1 we obtain:

THEOREM 3.3. *If $\mathfrak{A}$ is an ε-normal Banach subalgebra of $C(\Omega)$, Ω a compact F-space, then $\mathfrak{A} = C(\Omega)$.*

A known special case of 3.3 is Rudin's interpolation theorem for Sidon sets [15, Theorem 5.74], (Corollary 3.4 below). Recall that a set E in a discrete group Γ is a Sidon set if for each bounded function φ on E there exists a measure μ on the dual group G satisfying $\hat{\mu}(\gamma) = \varphi(\gamma)$. The restriction of $\hat{\mu}$ to E defines a mapping T of $M(G)$ into $l_\infty(E) = C(\beta E)$. Since βE is an F space, (c.f. [4] p. 369), an application of 3.3 yields the following.

COROLLARY 3.4. *If G is a compact group with dual group Γ, then $E \subset \Gamma$ is a Sidon set if for each function φ satisfying $\varphi(\gamma) = \pm 1$, $\gamma \in E$; there exists a measure μ on G satisfying*

$$\sup_{\gamma \in E} |\hat{\mu}(\gamma) - \varphi(\gamma)| < 1 .$$

A similar statement can be made for interpolating sequences for bounded analytic functions. Following Hoffman [9], call $E \equiv \{z_n\} \subsetneqq \{|z| < 1\}$ an *interpolating sequence* if for each bounded function φ on E there exists a bounded analytic function on the open unit disc such that

$$f(z_n) = \varphi(z_n) , \quad z_n \in E .$$

Again the restriction of φ to E defines a mapping T of the bounded analytic functions into $l_\infty = C(\beta N)$. Since βN is an F space, Theorem 3.3 yields the following extension of a result of Hayman [8] (see also Hoffman [9, p. 205]).

COROLLARY 3.5. *A sequence E in the open unit disc is an interpolating sequence if for each function φ on E such that $\varphi^2 = \varphi$ there exists a bounded analytic function f satisfying*

$$\sup_{z \in E} |f(z) - \varphi(z)| < \frac{1}{2} .$$

The intrinsic condition that E be an interpolating sequence, proved by Carleson [2], is that

$$\prod_{j \neq k} \left| \frac{z_k - z_j}{1 - \bar{z}_j z_k} \right| \geqq \delta > 0 \qquad k = 1, 2, \cdots$$

To prove Carleson's theorem by the methods discussed here it would suffice to produce for each subset $E_1 \subset E$ a bounded analytic function vanishing on E_1 such that $f(E \sim E_1)$ is contained in a compact set not containing the origin and having connected complement. If the sequence E is real this can be accomplished by the Blaschke product having E_1 as its set of zeroes. Whether the Blaschke products provide this separation in the general case is not known. In view of the known behavior of these functions on the boundary of the unit disc, this is perhaps too much to expect.

COROLLARY 3.6. *If $\mathfrak{A}$ is a Banach subalgebra of $C(\Omega)$, Ω a compact F space, then $\mathfrak{A} = C(\Omega)$ if for each pair of points ω_1, ω_2 there exists an $x \in \mathfrak{A}$ which vanishes in a neighborhood of ω_1 and equals one in a neighborhood of ω_2.*

Proof. As in the proof of 2.4. the condition implies that for each pair of disjoint nonempty compact subsets F_1, F_2 there exists an $x \in \mathfrak{A}$ satisfying $x(F_1) = 0; x(F_2) = 1$. By 3.3 for each $\omega \in \Omega$, there is a compact neighborhood N_ω of ω such that $\mathfrak{A} \mid N_\omega = C(N_\omega)$. An application of 2.1 completes the proof.

One should not expect that the condition of 3.6 can be substantially weakened. For recently Hoffman and Ramsey [10] have shown that if one assumes the continuum hypothesis then separating closed subalgebras of l_∞ exist in great abundance.

4. This final section contains some results and examples concerned with the problem of extending Theorem 3.2. In an earlier version of the manuscript we had conjectured that if Ω is a compact F-space and $T: X \to C(\Omega)$ is a continuous linear map of a Banach space with dense range, then $TX = C(\Omega)$. We are grateful to J. Lindenstrauss for the following elegant counterexample to this conjecture:

There exists a continuous linear map Φ from $L_\infty(0, 1)$ onto l_2, since $L_1(0, 1)$ contains a subspace isomorphic to l_2 (e.g. the subspace generated by the Rademacher functions). Let $\{e_n\}$, $n = 1, 2, \cdots$, be an orthonormal basis in l_2 and $E_n = sp\{e_i, \cdots, e_n\}$. Let $X_n = \Phi^{-1}(E_n)$ and let X be the Banach space of all sequences $x = \{x_n\}$, with $x_n \in X_n$ and $\| x \| = \sup_n \| x_n \| < \infty$. Let $T: X \to L_\infty(0, 1)$ be defined by

$$Tx = T(\{x_n\}) = \sum_{n=1}^{\infty} \frac{x_n}{n!} .$$

Then $L_\infty(0, 1)$ is a space $C(\Omega)$, where Ω is stonian, and TX is dense in $L_\infty(0, 1)$ as it contains $\cup_{n=1}^{\infty} X_n$. Also TX is not the whole of $L_\infty(0, 1)$, since if there were an $x = \{x_n\}$ in X such that

$$\Phi(Tx) = \sum_{n=1}^{\infty} \frac{\Phi(x_n)}{n!} = \sum_{n=1}^{\infty} \frac{e_n}{n}$$

then one must have

$$\left|\left| \sum_{n=N+1}^{\infty} \frac{\Phi(x_n)}{n!} \right|\right| \geqq \left|\left| \sum_{n=N+1}^{\infty} \frac{e_n}{n} \right|\right|$$

in l_2, for $N = 1, 2, \cdots$, as $\Phi(x_n) \in E_n$. Since $\sup_n || \Phi(x_n) || < \infty$, we obtain the desired contradiction.

We know of two cases one can conclude that $TX = C(\Omega)$ under additional hypotheses. G. Seever [16] has proved that $TX = C(\Omega)$ if TX is normal on Ω. We shall give a new proof.

THEOREM 4.1. (Seever) *Let Ω be a compact F-space and $T: X \to C(\Omega)$ be a continuous linear map of a Banach space such that TX is normal on Ω. Then $TX = C(\Omega)$.*

By earlier remarks one can suppose that Ω is totally disconnected. Seever proves that if TX is normal on Ω, $T^*: C(\Omega)^* \to X^*$ has a closed range. This fact rests on a uniform boundedness theorem for measures on totally disconnected F-spaces which is derived from a theorem of R.S. Phillips [15, p. 525] on convergence of finitely additive measures on the subsets of the integers. Since TX is dense in $C(\Omega)$ and has a closed range it follows that $TX = C(\Omega)$. The proof we shall give also relies on Seever's reduction to the case that Ω is totally disconnected. The necessary element of uniformity is supplied by the following lemma.

LEMMA 4.2. *Let Ω be a totally disconnected F-space and TX be normal on Ω. For each point $\omega_0 \in \Omega$ there exists an open and closed neighborhood V and constant K such that if E_1 and E_2 are any disjoint compact and open subsets of $V \sim \{\omega_0\}$, one can find $x \in X$ such that*

$$(Tx)(E_1) = 1\ , \quad (Tx)(E_2) = 0\ , \quad (Tx)(V') = 0,$$
$$(Tx)(\omega_0) = 0\ , \quad || x || \leqq K\ .$$

Proof. Suppose the lemma is false. If W_1 is an open and closed neighborhood of ω_0, there exist disjoint compact and open subsets E_{11}, E_{12} of $W_1 \sim \{\omega_0\}$ such that if $x \in X$ and $(Tx)(E_{11}) = 1$, $(Tx)(E_{12} \cup W_1') = 0$, $(Tx)(\omega_0) = 0$, then $|| x || \geqq 2$. Let k_E denote the characteristic function of a set E. Using the assumption of normality, select $x_1 \in X$ such that $Tx_1 = k_{E_{11}}$, and let W_2 be an open and closed neighborhood of ω_0 with $W_2 \subseteq W_1 \sim (E_{11} \cup E_{12})$. Proceeding inductively we construct decreasing open and closed neighborhoods W_n of ω_0 and disjoint open and closed subsets E_{n1}, E_{n2} of $W_n \sim W_{n+1}$ and elements $x_n \in X$ such that if

$$(\sharp) \qquad (Tx)(E_{n1}) = 1, (Tx)(E_{n2} \cup W_1') = 0, (Tx)(\omega_0) = 0 ,$$

then

$$\| x \| \geqq 2^n + \sum_{i=1}^{n-1} \| x_i \| .$$

Note that

$$E_{11} \subseteq E_{11} \cup E_{21} \subseteq E_{11} \cup E_{21} \cup E_{31} \subseteq \cdots \subseteq W_3 \cup E_{11} \cup E_{21} \subseteq W_2 \cup E_{11} \subseteq W_1 .$$

Since Ω is an F-space, there exists an open and closed set F_0 such that

$$\bigcup_{i=1}^{n} E_{i1} \subseteq F_0 \subseteq W_n \cup \bigcup_{i=1}^{n-1} E_{i1} , \quad n = 1, 2, \cdots .$$

Then

$$k_{F_0}(\omega) = \sum_{n=1}^{\infty} (Tx_n)(\omega) = \sum_{n=1}^{\infty} k_{E_{n1}}(\omega)$$

for $\omega \in \bigcup_{i=1}^{\infty} W_1'$. By dropping, if necessary, to the subsequences for n even or n odd we can suppose $\omega_0 \notin F_0$. Choose x_0 with $Tx_0 = k_{F_0}$ and define $z_n = x_0 - \sum_{i=1}^{n-1} x_i$. Then Tz_n satisfies ($\sharp$), so $\| z_n \| \geqq 2^n + \sum_{i=1}^{n-1} \| x_i \|$. This implies $\| x_0 \| \geqq 2^n$, $n = 1, 2, \cdots$, which is the required contradiction. It follows now by the arguments of Theorem 1.5 that TX contains all continuous functions which vanish outside of V. A covering argument completes the proof of Theorem 4.1.

One also has $TX = C(\Omega)$ is the special case that X is a conjugate space, $C(\Omega)$ is l_∞ and T is weak star continuous. This theorem is essentially an unpublished result of Beurling.

THEOREM 4.3. (Beurling) *Let S be an arbitrary set and X be a Banach space. If $T: l_1(S) \to X$ is linear and continuous and $T^*: X^* \to l_\infty(S)$ has dense range then $T^*X^* = l_\infty(S)$.*

Proof. The density of T^*X^* shows T is one to one. We prove that T^{-1} is bounded, so T and hence T^* has a closed range. If T^{-1} is not bounded we can find a sequence $\{\xi_n\}$ of elements of $l_1(S)$ of norm one such that $T\xi_n$ converges to zero. Thus

$$(T^*x^*)(\xi_n) = \sum_S \xi_n(s)(T^*x^*)(s) \to 0 , \quad x^* \in X^* ,$$

as $n \to \infty$. Since T^*X^* is dense in $l_\infty(S)$, $\{\xi_n\}$ converges weakly to zero. But then $\{\xi_n\}$ converges strongly to zero ([3], p. 295), giving the desired contradiction.

In a seminar in 1960 Beurling gave a proof (unpublished) of the following theorem equivalent to the result of Rudin (Corollary 3.4) for the case of the circle group.

THEOREM 4.4. *Let $\{n_j\}$ be a sequence of integers, and suppose that for each $\varepsilon > 0$ and $\{\alpha_i\}$ in l_∞ there is a measure μ_ε on the circle Γ such that*

$$|\hat{\mu}_\varepsilon(n_j) - \alpha_j| < \varepsilon\,, \quad j = 1, 2, \cdots .$$

Then for each $\{\alpha_j\} \in l_\infty$ there exists a measure μ such that $\hat{\mu}(n_j) = \alpha_j$, $j = 1, 2, \cdots$.

Helson observed that the essential argument of Beurling's proof gave a proof of 4.3. On the other hand, to prove Theorem 4.4 from 4.3 one defines $T(\{\xi_n\}) = \sum_{j=1}^{\infty} \xi_j e^{in_jt}$. Then $T: l_1 \to C(\Gamma)$, Γ the unit circle, and $T^*(C(\Gamma)^*)$ is dense in l_∞, so the restrictions of the Fourier transforms of measures must yield all bounded sequences. One may also base a proof of Corollary 3.5 on Theorem 4.2.

Finally we give an example of a compact Hausdorff space Γ, which is not an F-space, and a continuous map $T: X \to C(\Gamma)$ such that TX is dense and normal on Γ and enjoys the local matching property of Theorem 3.2, yet for which $TX \neq C(\Gamma)$. This example is due to Y. Katznelson. We denote the n-th Fourier coefficient of a continuous function x on the unit circle Γ by $\hat{x}_n$. Let X be the subspace of $C(\Gamma)$ for which $\sum_{n=-\infty}^{\infty} |\hat{x}_{2n+1}| < \infty$. We may write $X = Y \oplus Z$, where

$$Y = \{y \in C(\Gamma): \hat{y}_{2n+1} = 0\,, \quad n = 0, \pm 1, \pm 2, \cdots\}$$
$$Z = \left\{z \in C(\Gamma): \hat{z}_{2n} = 0\,, \quad n = 0, \pm 1, \pm 2, \cdots, \sum_{n=-\infty}^{\infty} |\hat{z}_{2n+1}| < \infty\right\}.$$

For $x = y + z$ define

$$\|x\| = \sup_{t \in \Gamma} |y(t)| + \sum_{n=-\infty}^{\infty} |\hat{z}_{2n+1}|\,.$$

Then TX is a dense and normal subspace of $C(\Gamma)$ complete in this norm. If f is any continuous function defined on an arc of length less than π, we may construct a continuous function y of period π and hence in Y, which extends f. Thus if T is the injection map of X into $C(\Gamma)$, Γ is covered by closed arcs S_1, S_2, S_3 such that $TX | S_i = C(S_i)$ but $TX \neq C(\Gamma)$.

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