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TRANSFORMATIONS OF FOURIER COEFFICIENTS

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Let A and B be function spaces on the unit circle and let F be a complex function defined in the plane. F is said to map A into B provided $\sum F(a_n)\, e^{in\theta}$ is the Fourier series of a function in B whenever $\sum a_n\, e^{in\theta}$ is the Fourier series of a function in A. For $1 \leqq q < \infty$, let L^q denote the usual space of functions on the unit circle normed by

$$(1) \qquad \|f\|_q = \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(e^{i\theta})|^q\, d\theta\right\}^{1/q} .$$

Let $2 \leqq q \leqq \infty$ and p be given by $p^{-1} + q^{-1} = 1$.

It follows from the Hausdorff-Young theorem that if $b(z)$ is bounded near the origin, then

$$(2) \qquad F(z) = c_1 z + c_2 \bar{z} + |z|^{2/p}\, b(z)$$

maps L^q into L^q.

In this paper it is shown that all functions mapping L^q into L^q have this form. In fact, all functions mapping the continuous functions into L^q have this form.

THEOREM 1. Let $2 \leqq q \leqq \infty$. The following are equivalent.

(i) F maps L^q into L^q.

(ii) F maps the continuous functions into L^q.

(iii) $F(z) = c_1 z + c_2 \bar{z} + |z|^{2/p}\, b(z)$ where $b(z)$ is bounded near the origin.

Rudin [2] proves that Theorem 1 is true provided F is an even function. Our proof consists primarily of applications of the method devised by Rudin.

$\mathscr{C}$ will denote the continuous functions on the unit circle. The Fourier coefficients of $f \in L^1$ are given by

$$(3) \qquad \hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta}) e^{-in\theta} d\theta \qquad (n = 0, \pm 1, \pm 2, \cdots) .$$

F maps A into B provided given $f \in A$ there is $g \in B$ such that $\hat{g} = F(\hat{f})$. This is written $g = F \circ f$.

2. Trigonometric polynomials with few coefficients. H. S. Shapiro in his Master's thesis [3], and, independently, Rudin [2], prove the existence of a sequence $\{\varepsilon_n\}$ with $\varepsilon_n = \pm 1$ such that

$$(4) \qquad \left|\sum_{n=1}^{N} \varepsilon_n\, e^{in\theta}\right| < 5N^{1/2} \qquad (0 \leqq \theta \leqq 2\pi;\ N = 1, 2, 3, \cdots) .$$

A similar construction yields

THEOREM 2. *Let r be a prime integer and $\alpha = \exp(2\pi i/r)$. There is a sequence $\{\varepsilon_r(n)\}$ with $\varepsilon_r(n)$ having for each n one of the values $1, \alpha, \cdots, \alpha^{r-1}$ such that*

$$(5)\qquad \left|\sum_{n=1}^{N} \varepsilon_r(n)\, e^{in\theta}\right| < r(1 + r^{(1/2)})N^{1/2} \qquad (0 \leqq \theta \leqq 2\pi;\ N = 1, 2, 3, \cdots)\,.$$

Proof. Let $A_0, A_1, \cdots, A_{r-1}$ be complex numbers. A simple calculation based on the identity $\sum_{j=0}^{r-1} \alpha^{sj} = \begin{cases} r, & s = 0 \\ 0, & s = 1, 2, \cdots, r-1 \end{cases}$ gives

$$(6)\qquad \sum_{s=0}^{r-1} \left|\sum_{j=0}^{r-1} \alpha^{sj} A_j\right|^2 = r \sum_{j=0}^{r-1} |A_j|^2\,.$$

For $r = 2$, this is just the parallelogram law used in [2] and [3] to prove the theorem for the special case $r = 2$.

Let $P_0^0(x) = P_0^1(x) = \cdots = P_0^{r-1}(x) = x$ and define polynomials P_k^s inductively by

$$(7)\qquad P_{k+1}^s(x) = \sum_{j=0}^{r-1} x^{jr^k} \alpha^{sj} P_k^j(x) \qquad (s = 0, 1, \cdots, r-1)\,.$$

P_k^s is a polynomial of degree r^k and it is easily seen by induction that each of its coefficients is a power of α and that P_k^0 is a partial sum of P_{k+1}^0. The sequence $\varepsilon_r(n)$ is defined by letting $\varepsilon_r(n)$ be the n^{th} coefficient of P_k^0 when $r^k > n$.

If $|x| = 1$, (6) and (7) yield

$$(8)\qquad \sum_{s=0}^{r-1} |P_{k+1}^s(x)|^2 = r \sum_{j=0}^{r-1} |P_k^j(x)|^2\,.$$

Since $\sum_{s=0}^{r-1} |P_0^s(x)|^2 = r$, we have

$$(9)\qquad \sum_{s=0}^{r-1} |P_k^s(e^{i\theta})|^2 = r^{k+1}\,.$$

Hence

$$(10)\qquad |P_k^0(e^{i\theta})| \leqq r^{1/2} r^{k/2}\,.$$

For $N = r^k$ this is stronger than (5). From it we can obtain (5) for all values of N by following the procedure of [2].

If we replace α by α^t $(t = 1, 2, \cdots, r-1)$ in (7) then we obtain a sequence $\{\varepsilon_{r,t}(n)\}$ such that

$$(11)\qquad \varepsilon_{r,t}(n) = (\varepsilon_r(n))^t$$

and

$$(12)\qquad \left|\sum_{n=1}^{N} \varepsilon_{r,t}(n) e^{in\theta}\right| < r(1 + r^{1/2})N^{1/2}\,.$$

Now let $\delta_r(n) = \sum_{t=1}^{r-1} \varepsilon_{r,t}(n)$. Since $\varepsilon_r(n)$ is an rth root of unity, it follows from (11) that $\delta_r(n) = r - 1$ or -1. Thus (12) yields

THEOREM 3. *If r is a prime there is a sequence $\{\delta_r(n)\}$ with $\delta_r(n) = r - 1$ or -1 such that*

$$\left|\sum_{n=1}^{N} \delta_r(n)e^{in\theta}\right| < (r-1)r(1+r^{1/2})N^{1/2} \quad (0 \leqq \theta \leqq 2\pi; N = 1, 2, 3, \cdots) . \tag{13}$$

3. Proof of Theorem 1. To prove Theorem 1, we need only show that (ii) implies (iii). Furthermore, by [2, Theorem 4], we can assume that F is odd. For $q = 2$, Theorem 1 follows from [2, Theorem 4]. For if F maps $\mathscr{C}$ into L^2 then $H(z) = |F(z)| + |F(-z)|$ is an even function mapping $\mathscr{C}$ into L^2 so that $|F(z)/z|$ is bounded near the origin. In this section F will map $\mathscr{C}$ into L^q $(q > 2; 1/p + 1/q = 1)$.

The proof of the theorem relies primarily on the following lemma similar to [1; Lemma 3.2].

LEMMA 1. *Let F map $\mathscr{C}$ into L^q. There are constants $\delta > 0$ and $M < \infty$ such that*

$$\| F \circ f \|_q \leqq M \tag{14}$$

whenever $f \in \mathscr{C}$ and $\| f \|_\infty < \delta$.

Proof. It is sufficient to show that (14) holds for trigonometric polynomials.

For let $f \in \mathscr{C}$, $\| f \|_\infty < (1/3)\delta$, and define

$$K_m(e^{i\theta}) = \sum_{n=-2m}^{2m} \min\left(1, 2 - \frac{|n|}{m}\right)e^{in\theta} \quad (m = 1, 2, 3, \cdots) . \tag{15}$$

If $*$ denotes ordinary convolution then $f * K_m$ is a polynomial such that $\| f * K_m \|_\infty < \delta$. Hence $\| F \circ (f * K_m) \|_q \leqq M$. But a subsequence of $\{F \circ (f * K_m)\}$ approaches $F \circ f$ weakly as elements of L^q. Hence $\| F \circ f \|_q \leqq M$.

Thus if the lemma is false there is a sequence of polynomials $\{f_m\}$ with $\| f_m \|_\infty < 1/m^2$ and $\| F \circ f_m \|_q \to \infty$ as $m \to \infty$. Clearly we may assume that $\hat{f}_m(k) = 0$ if $k < 0$. Let N_m be the degree of f_m and choose integers n_m so that

$$n_m + 3N_m < n_{m+1} - N_{m+1} . \tag{16}$$

The series

$$f(e^{i\theta}) = \sum_{m=1}^{\infty} e^{in_m\theta} f_m(e^{i\theta}) \tag{17}$$

converges uniformly to a continuous function. Let

$$H_m(e^{i\theta}) = e^{i(n_m+N_m)\theta}K_{N_m}(\theta) .$$

The choice of $\{n_m\}$ implies that

$$(F\circ f)*H_m = e^{in_m\theta}(F\circ f_m) . \tag{18}$$

Since $\| H_m \|_1 < 3$, it follows that

$$\| F\circ f_m \|_q < 3 \| F\circ f \|_q . \tag{19}$$

But this is impossible since $\| F\circ f \|_q \rightarrow \infty$.

LEMMA 2. $| F(z/2) - (1/2)F(z) | | z |^{-2/p}$ *is bounded near the origin.*

Proof. If the lemma is false there are numbers $z_m \neq 0$ ($m = 1, 2, 3, \cdots$) such that $mz_m \rightarrow 0$ and

$$\left| F\left(\frac{z_m}{2}\right) - \frac{1}{2}F(z_m)\right| > m^3 | z_m |^{2/p} . \tag{20}$$

Let $N_m = [m^{-2}z_m^{-2}]$ and define

$$T_m(e^{i\theta}) = \frac{z_m}{2}\sum_{n=1}^{N_m} \delta_3(n)e^{in\theta} \tag{21}$$

where $\delta_3(n)$ is the sequence of Theorem 3 for $r = 3$. From Theorem 3 and the definition of N_m it follows that $\| T_m \|_\infty \rightarrow 0$ as $m \rightarrow \infty$. Hence, by Lemma 1, $\| F\circ T_m \|_q$ is bounded as $m \rightarrow \infty$.

Since F is an odd function

$$(F\circ T_m)(e^{i\theta}) = F(z_m)\sum_{1\leqq n\leqq N_m, \delta_3(n)=2} e^{in\theta} - F\left(\frac{z_m}{2}\right)\sum_{1\leqq n\leqq N_m, \delta_3(n)=-1} e^{in\theta} . \tag{22}$$

Thus

$$\begin{aligned} | F\circ T_m(e^{i\theta}) | \geqq \frac{2}{3}\left|\frac{1}{2}F(z_m) - F\left(\frac{z_m}{2}\right)\right| \left|\sum_{n=1}^{N_m} e^{in\theta}\right| \\ - \frac{1}{3}\left|F(z_m) + F\left(\frac{z_m}{2}\right)\right| \left|\sum_{n=1}^{N_m} \delta_3(n)e^{in\theta}\right| . \end{aligned} \tag{23}$$

Now if F maps $\mathscr{C}$ to L^q, $q > 2$, then, *a fortiori*, F maps $\mathscr{C}$ to L^2. Thus the truth of Theorem 1 for $q = 2$ implies that $| F(z)/z |$ is bounded near the origin. Thus, since $\| \sum_{n=1}^{N_m} e^{in\theta} \|_q \geqq C_q N_m^{1/p}$, it follows that $| (1/2)F(z_m) - F(z_m/2) | \cdot N_m^{1/p}$ is bounded as $m \rightarrow \infty$. However this is a contradiction to (20).

LEMMA 3. $F(z) = F_1(z) + F_2(z)$ *where*

(a) *F_1 and F_2 map $\mathscr{C}$ into L^q.*
(b) *$|F_2(z)|\,|z|^{-2/p}$ is bounded near the origin.*
(c) *$F_1(z/2) = (1/2)F_1(z)$ for all z.*

REMARK. F_2 is the "small" part of F. Lemmas 5 and 6 show that because of (a) and (c)

$$F_1(z) = c_1 z + c_2 \bar{z} .$$

Proof. By Lemma 2 there are finite positive constants B and C such that for $|z| \leqq B$

$$\begin{aligned} \left|F\left(\frac{z}{2^k}\right) - \frac{1}{2^k}F(z)\right| &\leqq \sum_{j=0}^{k-1} \frac{1}{2^j}\left|F\left(\frac{z}{2^{k-j}}\right) - \frac{1}{2}F\left(\frac{z}{2^{k-j-1}}\right)\right| \\ &\leqq C\sum_{j=0}^{k-1} \frac{1}{2^j}\left|\frac{z}{2^{k-j-1}}\right|^{2/p} \\ &\leqq C'\frac{|z|^{2/p}}{2^k} \qquad (k = 1, 2, 3, \cdots) . \end{aligned} \tag{24}$$

Define

$$F_1(z) = \lim_{n\to\infty} 2^n F\left(\frac{z}{2^n}\right) . \tag{25}$$

This limit exists. For if $n > j$ and we apply (24) to $z/2^j$ with $k = n - j$ and multiply by 2^n then

$$\left|2^n F\left(\frac{z}{2^n}\right) - 2^j F\left(\frac{z}{2^j}\right)\right| \leqq 2^j C' \left|\frac{z}{2^j}\right|^{2/p} . \tag{26}$$

Since $p < 2$, the right side of (26) $\to 0$ as j and $n \to \infty$.

It is clear from the definition of F_1 that (c) holds. $F_2(z) = F(z) - F_1(z)$ and (b) is a result of (24). F_2 maps $\mathscr{C}$ into L^q because of (b). Thus F_1 does also. Note that F_1 is odd (since F is).

LEMMA 4. *F_1 is continuous.*

Proof. It is sufficient to show it is continuous at 1. If not, there is a sequence $z_m \to 1$ such that $F_1(z_m) \to F_1(1)$. The z_m can be chosen so that

$$|1 - z_m| < 2^{-m} . \tag{27}$$

Let $N_m = [2^{2m} \cdot m^{-2}]$ and define

$$T_m(e^{i\theta}) = 2^{-m} \sum_{n=1}^{N_m} \left\{\varepsilon_2(n) + \frac{(1 - z_m)}{2}\,(1 - \varepsilon_2(n))\right\} e^{in\theta} \tag{28}$$

where $\{\varepsilon_2(n)\}$ is the sequence of Theorem 2.

Theorem 2, (27) and the choice of N_m imply that $\| T_m \|_\infty = 0(1/m)$ so that, by Lemma 1, $\| F \circ T_m \|_q$ is bounded as $m \to \infty$. But then since $F_1(z/2) = (1/2)F_1(z)$,

$$
\begin{aligned}
| F_1 \circ T_m(e^{i\theta}) | &= 2^{-m} \left| F_1(1) \sum_{1 \leq n \leq N_m, \varepsilon_2(n)=1} e^{in\theta} - F_1(z_m) \sum_{1 \leq n \leq N_m, \varepsilon_2(n)=-1} e^{in\theta} \right| \\
&\geqq 2^{-m-1} | F_1(1) - F_1(z_m) | \left| \sum_{n=1}^{N_m} e^{in\theta} \right| \\
&\quad - 2^{-m-1} | F_1(1) + F_1(z_m) | \left| \sum_{n=1}^{N_m} \varepsilon_2(n) e^{in\theta} \right| .
\end{aligned} \tag{29}
$$

As in Lemma 2 this implies that $| F_1(1) - F_1(z_m) | N_m^{1/p} \cdot 2^{-m}$ is bounded as $m \to \infty$, which is impossible unless $F_1(z_m) \to F_1(1)$. Hence F_1 is continuous.

LEMMA 5. *There are continuous functions C_1 and C_2 on $(0, \infty)$ such that*

$$F_1(xe^{i\theta}) = C_1(x)e^{i\theta} + C_2(x)e^{-i\theta} \qquad (0 < x < \infty) .$$

Proof. We will show that if r is an integer $(r \neq 0, 1)$ and z a complex number then

$$\sum_{j=1}^{r} F_1\left(z \exp \frac{2\pi i j}{r}\right) = 0 . \tag{30}$$

Now consider $F_1(xe^{i\theta}) = G_x(e^{i\theta})$ for a fixed x. G_x is a continuous function of θ by Lemma 4. It follows from (30) that for each integer $r \neq 0, 1$.

$$\sum_{j=1}^{r} G_x\left(\exp i\left(\theta + \frac{2\pi j}{r}\right)\right) = 0 \qquad (0 \leqq \theta \leqq 2\pi) . \tag{31}$$

By considering the Fourier coefficients of G_x it is easily seen that $G_x(e^{i\theta}) = C_1(x)e^{i\theta} + C_2(x)e^{-i\theta}$. C_1 and C_2 are continuous because of Lemma 4.

To prove (30) it is sufficient to assume that $z = 1$. It is also sufficient to assume r is prime. For if $r = pq$ where p is a prime then (30) can be written

$$\sum_{s=1}^{q} \sum_{j=1}^{p} F_1\left(z \exp \frac{2\pi i(jq + s)}{pq}\right) . \tag{32}$$

If (30) holds for primes then each summand of the outer sum of (32) is zero.

Let $N_m = [2^{2m} m^{-2}]$ and define

$$T_m^t(e^{i\theta}) = 2^{-m} \sum_{n=1}^{N_m} \{\varepsilon_r(n)\}^t e^{in\theta} \qquad (t = 1, 2, \cdots, r-1)\ , \tag{33}$$

where $\{\varepsilon_r(n)\}$ is the sequence of Theorem 2. $\| T_m^t \|_\infty = 0(1/m)$ so that if $\beta = \sum_{j=1}^r F_1(\exp 2\pi ij/r)$ and

$$H_m(e^{i\theta}) = \sum_{t=1}^{r-1} \left\{ F_1 \circ T_m^t + \left\{ \frac{\beta}{r} - F_1(1) \right\} T_m^t \right\} \tag{34}$$

then, by Lemma 1, $\| H_m \|_q$ is bounded as $m \to \infty$. Now since $F_1(z/2) = (1/2)F(z)$

$$\begin{aligned} | H_m(e^{i\theta}) | = 2^{-m} \Bigg| \sum_{t=1}^{r-1} \Bigg\{ & \sum_{n=1}^{N_m} F_1\{(\varepsilon_r(n))^t\} e^{in\theta} \\ & + \left\{ \frac{\beta}{r} - F_1(1) \right\} \sum_{n=1}^{N_m} \{\varepsilon_r(n)\}^t e^{in\theta} \Bigg\} \Bigg|\ . \end{aligned} \tag{35}$$

Suppose $\varepsilon_r(n) = 1$. The coefficient of $e^{in\theta}$ in (35) is then

$$(r-1)F_1(1) + (r-1)\left\{ \frac{\beta}{r} - F_1(1) \right\} = \left(1 - \frac{1}{r}\right)\beta\ . \tag{36}$$

Suppose $\varepsilon_r(n) \neq 1$, so that $\varepsilon_r(n)$ is a primitive r^{th} root of unity. Then $\sum_{t=1}^{r-1} F_1\{(\varepsilon_r(n))^t\} = \beta - F_1(1)$ and $\sum_{t=1}^{r-1} (\varepsilon_r(n))^t = -1$ so that the coefficient of $e^{in\theta}$ is

$$\beta - F_1(1) - \left\{ \frac{\beta}{r} - F_1(1) \right\} = \left(1 - \frac{1}{r}\right)\beta\ . \tag{37}$$

Hence

$$| H_m(e^{i\theta}) | = 2^{-m}\left(1 - \frac{1}{r}\right) | \beta | \left| \sum_{n=1}^{N_m} e^{in\theta} \right| \tag{38}$$

so that

$$\| H_m \|_q \geqq \left(1 - \frac{1}{r}\right) \frac{|\beta|}{2^m} C_q N_m^{1/p} \qquad (m = 1, 2, \cdots)\ . \tag{39}$$

But this is impossible unless $\beta = 0$. That is

$$\sum_{j=1}^{r} F_1\left(\exp \frac{2\pi ij}{r}\right) = 0 \tag{40}$$

which was to be proved

LEMMA 6. $C_j(x) = xC_j(1) \qquad (0 < x < \infty;\ j = 1, 2)$.

Proof. Fix x and φ. Let r be a prime, $N_m = [2^{2m}m^{-2}]$, and define

$$T_m(e^{i\theta}) = \frac{xe^{i\varphi}}{(r-1)2^m} \sum_{n=1}^{N_m} \delta_r(n) e^{in\theta} \tag{41}$$

where $\{\delta_r(n)\}$ is the sequence of Theorem 3.

Since F_1 is odd and $F_1(z) = 2F_1(z/2)$ we can write

$$\begin{aligned} F_1 \circ T_m(e^{i\theta}) &= \frac{1}{2^m r}\left\{F_1(xe^{i\varphi}) - (r-1)F_1\left(\frac{xe^{i\varphi}}{r-1}\right)\right\} \sum_{n=1}^{N_m} e^{in\theta} \\ &+ \frac{1}{2^m r}\left\{F_1(xe^{i\varphi}) + F_1\left(\frac{xe^{i\varphi}}{r-1}\right)\right\} \sum_{n=1}^{N_m} \delta_r(n) e^{in\theta} . \end{aligned} \tag{42}$$

As in the proofs of Lemma 2 and 4, $\| F_1 \circ T_m \|_q$ and $2^{-m} \| \sum \delta_r(n) e^{in\theta} \|_q$ are bounded. Hence $2^{-m} N_m^{1/p} \,|\, F_1(xe^{i\varphi}) - (r-1)F_1(xe^{i\varphi}/r-1) \,|$ is bounded. But $2^{-m} N_m^{1/p}$ is unbounded so that

$$F_1(xe^{i\varphi}) - (r-1)F_1\left(\frac{xe^{i\varphi}}{r-1}\right) = 0 \qquad (0 < x < \infty;\ 0 \leqq \varphi \leqq 2\pi) . \tag{43}$$

By Lemma 5, (43) can be written

$$\begin{aligned} &\left\{C_1(x) - (r-1)C_1\left(\frac{x}{r-1}\right)\right\} e^{i\varphi} \\ &\qquad + \left\{C_2(x) - (r-1)C_2\left(\frac{x}{r-1}\right)\right\} e^{-i\varphi} = 0 . \end{aligned} \tag{44}$$

Clearly this possible only if

$$C_j(x) = (r-1)C_j\left(\frac{x}{r-1}\right) \qquad (0 < x < \infty;\ j = 1, 2) . \tag{45}$$

Thus, if r and q are primes and n is an integer,

$$C_j\left(\left(\frac{r-1}{q-1}\right)^n\right) = \left(\frac{r-1}{q-1}\right)^n C_j(1) \qquad (j = 1, 2) . \tag{46}$$

Now $\{(r-1/q-1)^n \colon r, q, \text{ primes}; n \text{ an integer}\}$ is dense in the positive real numbers. This is true since given $\varepsilon > 0$ there are infinitely many pairs of consecutive primes q_n, q_{n+1} such that $q_{n+1} < (1+\varepsilon)q_n$.

Since C_j is continuous (46) then implies $C_j(x) = xC_j(1)$ for all x.

The proof of Theorem 1 follows from Lemmas 3, 5, and 6.

4. The general case. We remark here that Theorem 1 holds if we consider any compact Abelian group G. If Γ, the dual group of G, has elements of arbitrarily large order then it is possible to construct polynomials as in § 2 and the proof proceeds as in § 3. When, Γ, and hence G, has an exponent the construction of the polynomials is slightly different (it depends on the structure of Γ) but the remainder of the proof is similar.

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