PACIFIC JOURNAL OF MATHEMATICS
Vol. 19, No. 3, 1966

AN APPLICATION OF THE BOTT SUSPENSION MAP TO THE TOPOLOGY OF EIV

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Consider the compact simply connected symmetric pair (E_6, F_4). By a slight abuse of the notation of E. Cartan, the corresponding symmetric space is denoted by EIV. Let W be the Cayley projective plane. The Bott suspension map $E: \Sigma(W) \to EIV$ (where Σ denotes the nonreduced suspension) is defined by means of the set of minimal geodesic segments joining the two nontrivial points of the "center" of EIV. In this paper a map $q: S^{25} \to \Sigma(W)$ is constructed and E is extended to a homeomorphism of $\Sigma(W) \cup {}_q e_{26}$ onto EIV. Among other things, this gives canonical isomorphisms $\pi_j(EIV) \approx \pi_j(\Sigma(W))$, $0 \leqq j \leqq 24$. These groups are explicitly determined.

Statement of results. The maps E and q will be constructed in § 2 and the following theorems will be proven.

THEOREM 1.1. *The map E extends to a homeomorphism E': $\Sigma(W) \bigcup_q e_{26} \to EIV$.*

COROLLARY 1.2. *$E_*: \pi_j(\Sigma(W)) \to \pi_j(EIV)$ is a bijection for $j \leqq 24$, and a surjection for $j = 25$.*

THEOREM 1.3. *$\operatorname{Im}(q_*) = \operatorname{Ker}(E_*)$ in homotopy in dimensions $\leqq$ 32, and*

$$0 \longrightarrow \pi_{25}(S^{25}) \xrightarrow[q_*]{} \pi_{25}(\Sigma(W)) \xrightarrow[E_*]{} \pi_{25}(EIV) \longrightarrow 0$$

is exact and canonically split, with $\pi_{25}(EIV)$ a finite 2-primary group.

Having by (1.2) reduced the problem of computing $\pi_j(EIV)$, $j \leqq$ 24, to a somewhat easier problem, we devote the remaining sections of the paper to deducing the consequences listed below. We do not list $\pi_j(EIV)$ for $j \leqq 15$, since isomorphisms $\pi_j(EIV) \approx \pi_j(S^9)$, together with the explicit values of these latter groups, are already well known for that range.

(1.4) $$\pi_{16}(EIV) = 0$$

(1.5) $$\pi_{17}(EIV) = \boldsymbol{Z} + (\boldsymbol{Z}_2)^2$$

(1.6) $$\pi_{18}(EIV) = (\boldsymbol{Z}_2)^3$$

(1.7) $$\pi_{19}(EIV) = \boldsymbol{Z}_6$$

(1.8) $$\pi_{20}(EIV) = \boldsymbol{Z}_{1512} + \boldsymbol{Z}_2$$

(1.9) $$\pi_{21}(EIV) = 0$$

(1.10) $$\pi_{22}(EIV) = \boldsymbol{Z}_3$$

(1.11) $$\pi_{23}(EIV) = \boldsymbol{Z}_4$$

(1.12) $$\pi_{24}(EIV) = \boldsymbol{Z}_{225} + (\text{2-primary group}) .$$

REMARKS. (1.4) was communicated to the author some time ago by Shôrô Araki who proved it by a somewhat different method (unpublished). The present paper actually resulted from attempts to verify this formula. (1.9) was proven in a different way in [8] and (1.5) and (1.10) remove the ambiguities from the partial determinations of these groups in that same paper. In (1.1) one gets a fully explicit cellular structure by recalling that

$$\Sigma(W) = e_0 \cup_p e_9 \cup_g e_{17}$$

where $p: S^8 \rightarrow e_0$ is the only map possible and $g: S^{16} \rightarrow e_0 \cup_p e_9 \approx S^9$ is the suspension of the standard Hopf map $f: S^{15} \rightarrow S^8$.

In the course of this paper we will repeatedly (and without further reference) make use of the values of $\pi_i(S^n)$ as found in [14].

2. The maps E and q. Let $\mathfrak{e}_6$ be the Lie algebra of E_6 and $\beta: \mathfrak{e}_6 \rightarrow \mathfrak{e}_6$ the involution corresponding to EIV. Let $\mathfrak{m} \subset \mathfrak{e}_6$ be the -1 eigenspace of β. Let $\mathfrak{t} \subset \mathfrak{m}$ be a maximal abelian subalgebra (a two dimensional real vector space) and consider the root system of EIV relative to $\mathfrak{t}$. This is a proper root system (in the sense of [2]) isomorphic to the root system of A_2, each root having multiplicity 8. Let Δ be a fundamental simplex in $\mathfrak{t}$.

The symmetric space EIV is canonically imbedded in E_6 as $\exp(\mathfrak{m})$. The adjoint action of F_4 on $\mathfrak{m}$ passes over, under exp, to the adjoint action of F_4 on $EIV \subset E_6$.

$\operatorname{Exp} | \Delta$ is one-to-one (since EIV is simply connected) and $\exp(\Delta)$ intersects each F_4-orbit on EIV in one and only one point.

Let B denote the union in $\mathfrak{m}$ of the F_4-orbits of points of Δ. By the above remarks $\exp: B \rightarrow EIV$ is onto. Let $s(t)$, $0 \leqq t \leqq 1$, describe the edge of Δ opposite the vertex 0. Then $x_0 = \exp(s(0))$ and $x_1 = \exp(s(1))$ coincide with the nontrivial elements of the center Z_3 of E_6, while $\exp \circ s$ is a minimal geodesic joining x_0 and x_1. The following lemma and its corollary are completely straightforward.

LEMMA 2.1. *B is homeomorphic to the standard closed cell e_{26} and the boundary $\partial B \approx S^{25}$ is the union of the F_4-orbits of $s(t)$, $0 \leqq t \leqq 1$.*

COROLLARY 2.2. *Under the homeomorphism $B \approx e_{26}$, $\exp|B$ defines a surjection $e_{26} \to EIV$ which is a homeomorphism on the interior of e_{26}.*

LEMMA 2.3. $\exp(\partial B) \approx \Sigma(W)$.

Proof. From [1] one knows that the centralizer in F_4 of $\exp(s(t))$, $0 < t < 1$, is the symmetric subgroup Spin $(9) \subset F_4$, while for $t = 0, 1$ the centralizer is clearly all of F_4. Since $W = F_4/\text{Spin}(9)$, the lemma follows.

COROLLARY 2.4. *The inclusion $\exp(\partial B) \subset EIV$ is a Bott suspension $E\colon \Sigma(W) \to EIV$.*

Proof. Let $\Omega = \Omega(EIV; x_0, x_1)$, the space of paths on EIV joining x_0 and x_1. From the proof of (2.3) it is clear that the subspace of shortest geodesics in Ω is homeomorphic to W. The adjoint of the inclusion map $W \subset \Omega$ is precisely the Bott suspension [4], is one-to-one, and its image is $\exp(\partial B)$.

Of course, we define q as $\exp|\partial B$ and immediately obtain (1.1) and (1.2).

REMARK. The loop space Ω of EIV is homology commutative, hence the theory of [5] can be applied to the Pontrjagin ring $H_*(\Omega)$. $W \subset \Omega$ proves to be a generating variety contributing generators $x_8, x_{16} \in H_*(\Omega) \approx \boldsymbol{Z}[x_8, x_{16}]$, $\dim(x_i) = i$. The diagram

$$\begin{array}{ccc} H_i(\Omega) & \xrightarrow[\sigma]{} & H_{i+1}(EIV) \\ \beta_*\downarrow & & \beta_*\downarrow \\ H_i(\Omega) & \xrightarrow[\sigma]{} & H_{i+1}(EIV) \end{array}$$

is commutative, where σ is homology suspension and the homomorphisms β_* are induced by the involution β of E_6. β_* is -1 on $H_9(EIV) \approx \boldsymbol{Z}$ [9] and $\sigma(x_8)$ generates this group. Thus $\beta_*(x_8) = -x_8$ and $\beta_*(x_8^2) = x_8^2$. β_* is -1 on $H_{17}(EIV) \approx \boldsymbol{Z}$ [9], so $\sigma(x_8^2) = 0$. $\sigma H_{16}(\Omega) = H_{17}(EIV)$, hence $\sigma(x_{16})$ generates that group. From the known homology of EIV [9], it follows that $E_*\colon H_i(\Sigma(W)) \to H_i(EIV)$ is bijective, $i \leqq 25$. (1.2) then follows by the Whitehead theorem. One can also deduce a map q (defined up to homotopy) and a weakened version of (1.1) in which E' is only a homotopy equivalence. In point of fact, it was this somewhat roundabout line of thought that suggested (1.1).

We now take up the proof of (1.3). Consider the homomorphisms

$$q_*\colon \pi_j(S^{25}) \longrightarrow \pi_j(\Sigma(W))$$
$$\partial\colon \pi_{j+1}(EIV, \Sigma(W)) \longrightarrow \pi_j(\Sigma(W)) .$$

LEMMA 2.5. *For* $j \leqq 32$ *there is a natural bijection* $h\colon \pi_j(S^{25}) \to \pi_{j+1}(EIV, \Sigma(W))$ *such that* $\partial \circ h = q_*$.

Proof. q defines a map $\bar{q}\colon (e_{26}, S^{25}) \to (EIV, \Sigma(W))$ and by [11, Chapter XI, Ex. B-3] (cf. the references given there to [10] and [16]), $\bar{q}_*$ is bijective in dimensions $\leqq 33$. Let

$$\gamma\colon \pi_j(S^{25}) \longrightarrow \pi_{j+1}(e_{26}, S^{25}),\ j \leqq 32 ,$$

be the inverse of the boundary map. Then $h = \bar{q}_* \circ \gamma$ is as desired.

The first assertion of (1.3) follows immediately from (2.5). For the exactness of

$$0 \longrightarrow \pi_{25}(S^{25}) \xrightarrow[q_*]{} \pi_{25}(\Sigma(W)) \xrightarrow[E_*]{} \pi_{25}(EIV) \longrightarrow 0$$

we need only the following.

LEMMA 2.6. $\partial\colon \pi_{26}(EIV, \Sigma(W)) \to \pi_{25}(\Sigma(W))$ *is one-to-one.*

Proof. From [8], $\pi_j(EIV, S^9) \approx \pi_{j-1}(S^{16})$, $j \leqq 31$. Thus, since $\pi_{26}(S^9)$ and $\pi_{25}(S^{16})$ are finite groups, so is $\pi_{26}(EIV)$. Since $\pi_{26}(EIV, \Sigma(W)) \approx \boldsymbol{Z}$ by (2.5), the map $\pi_{26}(EIV) \to \pi_{26}(EIV, \Sigma(W))$ is zero. The lemma follows by exactness.

The fact that $\pi_{25}(EIV)$ is a finite 2-primary group also follows from the results in [8], so we are left with the task of proving that the above sequence splits. (If it splits at all, the splitting is canonical, since $\pi_{25}(EIV)$ will have to be identified with the torsion subgroup of $\pi_{25}(\Sigma(W))$.)

The imbedding $S^9 \to EIV$ studied in [8] defines a generator of $\pi_9(EIV) \approx \boldsymbol{Z}$, hence E can be assumed to define a map

$$i\colon (\Sigma(W), S^9) \to (EIV, S^9),\ i \mid S^9 = 1 ,$$

where $S^9 \subset \Sigma(W)$ is given by our standard cellular decomposition of

$\Sigma(W)$. Using $\pi_{25}(EIV, S^9) \approx \pi_{24}(S^{16}) \approx (\boldsymbol{Z}_2)^2$ [8], we obtain a commutative diagram

$$\begin{array}{ccccc}
 & & 0 & & \\
 & & \downarrow & & \\
 & & \boldsymbol{Z} & & \\
 & & \downarrow & & \\
\pi_{25}(S^9) & \xrightarrow{r} & \pi_{25}(\Sigma(W)) & \xrightarrow{j} & \pi_{25}(\Sigma(W), S^9) \\
\downarrow & & \downarrow{\scriptstyle E_*} & & \downarrow{\scriptstyle i_*} \\
\pi_{25}(S^9) & \xrightarrow{r'} & \pi_{25}(EIV) & \xrightarrow{j'} & (\boldsymbol{Z}_2)^2 \\
 & & \downarrow & & \\
 & & 0 & &
\end{array}$$

where the second column and both rows are exact. Extending this diagram two more terms to the right, one easily establishes the surjective half of the five lemma.

LEMMA 2.7. $i_*\colon \pi_{25}(\Sigma(W), S^9) \rightarrow (\boldsymbol{Z}_2)^2$ *is surjective and* $\operatorname{Ker}(i_*) \subset \operatorname{Im}(j)$.

LEMMA 2.8. $j^{-1}(\operatorname{Ker}(i_*)) = \operatorname{Ker}(E_*) \oplus \operatorname{Im}(r)$.

Proof. $j^{-1}(\operatorname{Ker}(i_*)) = \operatorname{Ker}(i_* \circ j) = \operatorname{Ker}(j' \circ E_*)$. Now $\operatorname{Ker}(E_*)$ is infinite cyclic while $\operatorname{Im}(r)$ is a torsion group. Thus $\operatorname{Ker}(E_*) \cap \operatorname{Im}(r) = 0$. Furthermore, if $j'(E_*(a)) = 0$, then $E_*(a) \in \operatorname{Im}(r')$ and $a = b + c$, $b \in \operatorname{Ker}(E_*)$, $c \in \operatorname{Im}(r)$.

COROLLARY 2.9. $\operatorname{Ker}(i_*)$ *is the infinite cyclic group* $j(\operatorname{Ker}(E_*))$.

LEMMA 2.10. $(\boldsymbol{Z}_2)^2 \subset \pi_{25}(\Sigma(W), S^9)$.

Proof. In $\Sigma(W) = S^9 \cup_g e_{17}$, the attaching map g defines the characteristic map

$$\bar{g}\colon (e_{17}, S^{16}) \longrightarrow (\Sigma(W), S^9)\ .$$

Since suspension $\Sigma\colon \pi_{24}(S^{16}) \rightarrow \pi_{25}(S^{17})$ is one-to-one, it follows [11, p. 333] that

$$\bar{g}_*\colon \pi_{25}(e_{17}, S^{16}) \longrightarrow \pi_{25}(\Sigma(W), S^9)$$

is one-to-one. But $\pi_{25}(e_{17}, S^{16}) \approx \pi_{24}(S^{16}) \approx (\boldsymbol{Z}_2)^2$.

PROPOSITION 2.11. Ker (E_*) is a direct summand of $\pi_{25}(\Sigma(W))$.

Proof. Write Ker $(E_*) \subset \boldsymbol{Z}^1$, where $\boldsymbol{Z}^1$ stands for a maximal infinite cylic subgroup of $\pi_{25}(\Sigma(W))$. Im $(r) \cap \boldsymbol{Z}^1 = 0$, so $j \mid \boldsymbol{Z}^1$ is one-to-one. Thus $j(\boldsymbol{Z}^1) \cap (\boldsymbol{Z}_2)^2 = 0$, and, by (2.9), Im $(i_*) \supset (\boldsymbol{Z}_2)^2 \oplus j(\boldsymbol{Z}^1)/j(\text{Ker}\,(E_*))$. Thus $j(\text{Ker}\,(E_*)) = j(\boldsymbol{Z}^1)$, so Ker $(E_*) = \boldsymbol{Z}^1$.

This completes the proof of (1.3). It also proves

$$\pi_{25}(\Sigma(W), S^9) \approx \boldsymbol{Z} + (\boldsymbol{Z}_2)^2 . \tag{2.12}$$

3. **The homotopy sequence of $(\Sigma(W), S^9)$.** For the computation of $\pi_j(EIV)$, $j \leqq 24$, we are reduced to computing $\pi_j(\Sigma(W))$. We begin the attack on this latter problem by investigating the boundary operator ∂ in the homotopy sequence of $(\Sigma(W), S^9)$.

Recall that $\Sigma(W) = S^9 \cup_g e_{17}$ where g is the suspension of the standard Hopf map $f: S^{15} \to S^8$. By [11, p. 334] one shows that

$$\bar{g}_*: \pi_j(e_{17}, S^{16}) \longrightarrow \pi_j(\Sigma(W), S^9)$$

is bijective for $j \leqq 24$, $\bar{g}$ the characteristic map determined by g.

Let

$$F: \pi_j(\Sigma(W), S^9) \longrightarrow \pi_{j-1}(S^{16}),\ j \leqq 24\ , \tag{3.0}$$

be the natural bijection obtained by composing $(\bar{g}_*)^{-1}$ with the natural isomorphism $\pi_j(e_{17}, S^{16}) \approx \pi_{j-1}(S^{16})$.

LEMMA 3.1. *$\partial: \pi_j(\Sigma(W), S^9) \to \pi_{j-1}(S^9)$ is given by $g_* \circ F$ if $j \leqq 24$.*

Next consider the commutative diagram ($n \leqq 29$)

$$\begin{array}{ccc} \pi_n(S^{16}) & \xrightarrow{g_*} & \pi_n(S^9) \\ \approx \uparrow & & \Sigma \uparrow \\ \pi_{n-1}(S^{15}) & \xrightarrow{f_*} & \pi_{n-1}(S^8) \end{array}$$

where the vertical maps are suspensions.

LEMMA 3.2. Ker $\{\partial: \pi_j(\Sigma(W), S^9) \to \pi_{j-1}(S^9)\} \approx$ Im $(f_*) \cap$ Ker (Σ) *in* $\pi_{j-2}(S^8)$, $j \leqq 24$.

Proof. By (3.1) we are reduced to finding Ker (g_*). In the above diagram f_* is injective (because it has Hopf invariant one [7, exposé 6, Proposition 5]). This immediately yields the assertion.

We study $\mathrm{Im}(f_*) \cap \mathrm{Ker}(\Sigma)$ by means of the exact suspension sequence [7, expose 6]:

$$\cdots \xrightarrow[\Sigma]{} \pi_{n+1}(S^9) \xrightarrow[H]{} \pi_n(\Omega(S^9), S^8) \xrightarrow[\Delta]{} \pi_{n-1}(S^8) \xrightarrow[\Sigma]{} \pi_n(S^9) \xrightarrow[H]{} \cdots .$$

This gives $\mathrm{Ker}(\Sigma) = \mathrm{Im}(\Delta)$. In order to study Δ we will consider the topology of $\Omega(S^9)$ in lower dimensions.

Let i_8 generate $\pi_8(S^8)$ and consider the Whitehead product $[i_8, i_8] \in \pi_{15}(S^8)$. Let $h: S^{15} \to S^8$ be in this homotopy class and set $X = S^8 \bigcup_h e_{16}$. It is known [7, exposé 5] that $\Omega(S^9)$ has the homotopy type of a CW complex obtained by attaching to X cells of dimensions $\geqq 24$. Thus the inclusion $(X, S^8) \subset (\Omega(S^9), S^8)$ is a homotopy equivalence in dimensions $\leqq 22$, and in this range we can consider Δ as defined on $\pi_n(X, S^8)$. h determines a characteristic map

$$\bar{h}: (e_{16}, S^{15}) \longrightarrow (X, S^8) .$$

By [**11**, p. 334] we obtain

LEMMA 3.3. *$\bar{h}_*: \pi_n(e_{16}, S^{15}) \to \pi_n(X, S^8)$ is bijective, $n \leqq 22$.*

COROLLARY 3.4. *$\Delta = h_* \circ \partial \circ \bar{h}_*^{-1}$ in $\dim \leqq 22$, where*

$$\partial: \pi_n(e_{16}, S^{15}) \approx \pi_{n-1}(S^{15}) .$$

COROLLARY 3.5. $\mathrm{Ker}\{\partial: \pi_j(\Sigma(W), S^9) \to \pi_{j-1}(S^9)\} \approx \mathrm{Im}(f_*) \cap \mathrm{Im}(h_*)$ *in* $\pi_{j-2}(S^8)$, $j \leqq 23$.

4. $\pi_j(\Sigma(W))$, $j \leqq 18$. For the simple proof of the following lemma I am indebted to S. Araki.

LEMMA 4.1. *Let g be the suspension of the standard Hopf map $f: S^{15} \to S^8$. The class $[g]$ generates $\pi_{16}(S^9) \approx \boldsymbol{Z}_{240}$.*

Proof. Let $\sigma \in \pi_7(SO(8))$ be the element defined by the natural action on $\boldsymbol{R}^8$ of the unit sphere of Cayley numbers. Let $\sigma' \in \pi_7(SO(9))$ be the image of σ under the standard inclusion $SO(8) \subset SO(9)$. Then σ' generates $\pi_7(SO(9)) \approx \boldsymbol{Z}$ [15]. The J-homomorphism

$$J: \pi_7(SO(9)) \longrightarrow \pi_{16}(S^9) \approx \boldsymbol{Z}_{240}$$

is surjective [12] and $J(\sigma') = [g]$.

COROLLARY 4.2. $\pi_{16}(\Sigma(W)) = 0$

This establishes (1.4). For (1.5) and (1.6) we will need to make use of (3.5).

For h and f as in § 3, the class $[\zeta] = [h] - 2[f]$ is a torsion element in $\pi_{15}(S^8)$, hence $\zeta\colon S^{15} \to S^8$ is the suspension of some map [7, exposé 6].

LEMMA 4.3. *Let $\beta \in \pi_{16}(S^{15}) \approx \boldsymbol{Z}_2$ be the generator. Then $h_*(\beta)$ is a suspension class.*

Proof. Since β is a suspension class, $h_*(\beta) = 2f_*(\beta) + \zeta_*(\beta) = \zeta_*(\beta)$ and this is a suspension class.

COROLLARY 4.4. $\operatorname{Ker}\{\partial\colon \pi_{18}(\Sigma(W), S^9) \to \pi_{17}(S^9)\} = 0$.

Proof. By (4.3), $\operatorname{Im}(h_*)$ in $\pi_{16}(S^8)$ is contained in the image of the suspension. Therefore $\operatorname{Im}(f_*) \cap \operatorname{Im}(h_*) = 0$ in $\pi_{16}(S^8)$. The conclusion follows by (3.5).

COROLLARY 4.5. $\pi_{17}(\Sigma(W)) \approx \boldsymbol{Z} + (\boldsymbol{Z}_2)^2$.

Proof. $\pi_{18}(\Sigma(W), S^9) \approx \pi_{17}(S^{16}) \approx \boldsymbol{Z}_2$ by (3.0), and $\pi_{17}(S^9) \approx (\boldsymbol{Z}_2)^3$. From the exact sequence of $(\Sigma(W), S^9)$ and (4.4) one obtains

$$0 \longrightarrow (\boldsymbol{Z}_2)^2 \longrightarrow \pi_{17}(\Sigma(W)) \longrightarrow \pi_{17}(\Sigma(W), S^9) \longrightarrow \pi_{16}(S^9)\ .$$

Since $\pi_{17}(\Sigma(W), S^9) \approx \boldsymbol{Z}$ and $\pi_{16}(S^9)$ is finite, this gives an exact sequence

$$0 \longrightarrow (\boldsymbol{Z}_2)^2 \longrightarrow \pi_{17}(\Sigma(W)) \longrightarrow \boldsymbol{Z} \longrightarrow 0\ .$$

This completes the proof of (1.5).

Proceeding analogously as above, let $\beta \in \pi_{17}(S^{15}) \approx \boldsymbol{Z}_2$ be the generator and show that $h_*(\beta) \in \operatorname{Im}(\Sigma)$. Then

$$\partial\colon \pi_{19}(\Sigma(W), S^9) \longrightarrow \pi_{18}(S^9)$$

is one-to-one. Since, by (3.0), $\pi_{19}(\Sigma(W), S^9) \approx \boldsymbol{Z}_2$, and $\pi_{18}(S^9) \approx (\boldsymbol{Z}_2)^4$, one obtains

$$0 \longrightarrow (\boldsymbol{Z}_2)^3 \longrightarrow \pi_{18}(\Sigma(W)) \longrightarrow \pi_{18}(\Sigma(W), S^9) \stackrel{\partial}{\longrightarrow} \cdots$$

where ∂ is one-to-one by (4.4). This yields the following proposition and so proves (1.6).

PROPOSITION 4.6. $\pi_{18}(\Sigma(W)) \approx (\boldsymbol{Z}_2)^3$.

5. **Partial determinations of $\pi_j(\Sigma(W))$, $j = 19, 20$.** The 3-primary components of these two groups present a special problem. The

ambiguities left by the partial determinations in this section will be removed in § 7 by cohomological methods.

LEMMA 5.1. *$\Delta\colon \pi_{17}(X, S^8) \rightarrow \pi_{16}(S^8)$ is one-to-one.*

Proof. Consider the exact sequence

$$\pi_{17}(X, S^8) \xrightarrow{\Delta} \pi_{16}(S^8) \xrightarrow{\Sigma} \pi_{17}(S^9) \xrightarrow{H} \cdots .$$

H is zero since $\pi_{17}(S^9)$ is finite. Thus Σ is onto. Also $\pi_{16}(S^8) \approx (\boldsymbol{Z}_2)^4$, $\pi_{17}(S^9) \approx (\boldsymbol{Z}_2)^3$, so, by (3.3), $\operatorname{Im}(\Delta) \approx \boldsymbol{Z}_2 \approx \pi_{17}(X, S^8)$. It follows that Δ is one-to-one.

COROLLARY 5.2. *$\Delta\colon \pi_{18}(X, S^8) \rightarrow \pi_{17}(S^8)$ is one-to-one.*

Proof. By (5.1) the sequence

$$\pi_{18}(X, S^8) \xrightarrow{\Delta} \pi_{17}(S^8) \xrightarrow{\Sigma} \pi_{18}(S^9) \longrightarrow 0$$

is exact. Since $\pi_{17}(S^8) \approx (\boldsymbol{Z}_2)^5$, $\pi_{18}(S^9) \approx (\boldsymbol{Z}_2)^4$, we obtain $\operatorname{Im}(\Delta) = \operatorname{Ker}(\Sigma) \approx \boldsymbol{Z}_2 \approx \pi_{18}(X, S^8)$.

COROLLARY 5.3. *$\Delta\colon \pi_{19}(X, S^8) \rightarrow \pi_{18}(S^8)$ is one-to-one.*

Proof. By (5.2)

$$\pi_{19}(X, S^8) \xrightarrow{\Delta} \pi_{18}(S^8) \xrightarrow{\Sigma} \pi_{19}(S^9) \longrightarrow 0$$

is exact.

$$\pi_{18}(S^8) \approx (\boldsymbol{Z}_{24})^2 + \boldsymbol{Z}_2,\ \pi_{19}(S^9) \approx \boldsymbol{Z}_{24} + \boldsymbol{Z}_2, \quad \text{and} \quad \pi_{19}(X, S^8) \approx \pi_{18}(S^{15}) \approx \boldsymbol{Z}_{24} .$$

The assertion follows.

By (5.3) and (3.4), $h_*\colon \pi_{18}(S^{15}) \rightarrow \pi_{18}(S^8)$ is one-to-one. Let β generate $\pi_{18}(S^{15}) \approx \boldsymbol{Z}_{24}$. Then β is a suspension class and

$$h_*(\beta) = 2f_*(\beta) + \zeta_*(\beta)$$

is of order 24. Since f_* is known to be one-to-one in all dimensions, $f_*(\beta)$ is also of order 24. It follows that $\zeta_*(\beta)$ is of order 24 or 8. This ambiguity affects the rest of this section.

LEMMA 5.4. *$\partial\colon \pi_{20}(\Sigma(W), S^9) \rightarrow \pi_{19}(S^9)$ has kernel 0 or $\boldsymbol{Z}_3$.*

Proof. If $\zeta_*(\beta)$ is order 24, then $\operatorname{Im}(f_*) \cap \operatorname{Im}(h_*)$ is 0 in $\pi_{18}(S^8)$. If $\zeta_*(\beta)$ is of order 8, then $\operatorname{Im}(f_*) \cap \operatorname{Im}(h_*) \approx \boldsymbol{Z}_3$ in $\pi_{18}(S^8)$. The lemma follows by (3.5).

PROPOSITION 5.5. $\pi_{19}(\Sigma(W)) \approx \boldsymbol{Z}_2$ or $\boldsymbol{Z}_6$.

Proof. Consider the exact sequence

$$0 \longrightarrow \mathrm{Ker}\,(\partial) \longrightarrow \pi_{20}(\Sigma(W), S^9) \xrightarrow{\partial} \pi_{19}(S^9) \longrightarrow \pi_{19}(\Sigma(W)) \longrightarrow 0$$

where exactness holds on the right by the proof of (4.6).

$$\pi_{20}(\Sigma(W), S^9) \approx \pi_{19}(S^{16}) \approx \boldsymbol{Z}_{24} \quad \text{and} \quad \pi_{19}(S^9) \approx \boldsymbol{Z}_{24} + \boldsymbol{Z}_2 \,.$$

The proposition follows by (5.4).

PROPOSITION 5.6. There is an exact sequence

$$0 \longrightarrow \boldsymbol{Z}_{504} + \boldsymbol{Z}_2 \longrightarrow \pi_{20}(\Sigma(W)) \longrightarrow \pi_{19}(\Sigma(W)) \otimes \boldsymbol{Z}_3 \longrightarrow 0 \,.$$

Proof. By (5.4) and (5.5) the kernel of $\partial\colon \pi_{20}(\Sigma(W), S^9) \to \pi_{19}(S^9)$ is $\pi_{19}(\Sigma(W)) \otimes \boldsymbol{Z}_3$. This, together with $\pi_{21}(\Sigma(W), S^9) \approx \pi_{20}(S^{16}) \approx 0$ and $\pi_{20}(S^9) \approx \boldsymbol{Z}_{504} + \boldsymbol{Z}_2$, yields the proposition.

6. $\pi_j(\Sigma(W))$, $21 \leqq j \leqq 23$. One has $\pi_{21}(S^9) \approx 0$ and $\pi_{21}(\Sigma(W), S^9) \approx \pi_{20}(S^{16}) \approx 0$, so the exact homotopy sequence of the pair yields the following proposition, completing the proof of (1.9).

PROPOSITION 6.1. $\pi_{21}(\Sigma(W)) \approx 0$.

Now let β generate $\pi_{21}(S^{15}) \approx \boldsymbol{Z}_2$. As usual, $h_*(\beta) = \zeta_*(\beta)$ so that $\mathrm{Im}\,(f_*) \cap \mathrm{Im}\,(h_*)$ is 0 in $\pi_{21}(S^8)$. Thus $\partial\colon \pi_{23}(\Sigma(W), S^9) \to \pi_{22}(S^9)$ is one-to-one.

PROPOSITION 6.2. $\pi_{22}(\Sigma(W)) \approx \boldsymbol{Z}_3$.

Proof. $\pi_{23}(\Sigma(W), S^9) \approx \pi_{22}(S^{16}) \approx \boldsymbol{Z}_2$, $\pi_{22}(S^9) \approx \boldsymbol{Z}_6$, and

$$\pi_{22}(\Sigma(W), S^9) \approx \pi_{21}(S^{16}) \approx 0 \,.$$

By the above remarks we obtain an exact sequence

$$0 \longrightarrow \boldsymbol{Z}_2 \longrightarrow \boldsymbol{Z}_6 \longrightarrow \pi_{22}(\Sigma(W)) \longrightarrow 0 \,.$$

This also establishes (1.10). In order to prove (1.11) a slight change in approach is needed. The difficulty is that we are now out of the range of validity of (3.5).

There is an exact sequence

$$\pi_{24}(\Sigma(W), S^9) \xrightarrow{\partial} \pi_{23}(S^9) \longrightarrow \pi_{23}(\Sigma(W)) \longrightarrow 0 \tag{6.3}$$

where exactness on the right follows from the fact that ∂ is one-to-one on $\pi_{23}(\Sigma(W), S^9)$. Substituting the known values of the first two groups (note that we are still in the range of validity for (3.0)) we obtain

$$\text{(6.3a)} \qquad \boldsymbol{Z}_5 + \boldsymbol{Z}_3 + \boldsymbol{Z}_{16} \xrightarrow{\partial} \boldsymbol{Z}_{16} + \boldsymbol{Z}_4 \longrightarrow \pi_{23}(\Sigma(W)) \longrightarrow 0\ .$$

Our problem will be to compute Ker (∂) in (6.3a).

LEMMA 6.4. *$\Delta: \pi_{22}(X, S^8) \to \pi_{21}(S^8)$ is one-to-one.*

Proof. By (3.3), $\pi_{22}(X, S^8) \approx \pi_{21}(S^{15}) \approx \boldsymbol{Z}_2$, and $\pi_{21}(S^8) \approx \boldsymbol{Z}_6 + \boldsymbol{Z}_2$, $\pi_{22}(S^9) \approx \boldsymbol{Z}_6$. The suspension sequence of § 3 then yields

$$\boldsymbol{Z}_2 \xrightarrow{\Delta} \boldsymbol{Z}_6 + \boldsymbol{Z}_2 \xrightarrow{\Sigma} \boldsymbol{Z}_6$$

which necessitates $\Delta \neq 0$.

COROLLARY 6.5. *$\Sigma: \pi_{22}(S^8) \to \pi_{23}(S^9)$ is onto.*

Recall that $f_*: \pi_{22}(S^{15}) \to \pi_{22}(S^8)$ and $\Sigma: \pi_{21}(S^7) \to \pi_{22}(S^8)$ are one-to-one and

$$\pi_{22}(S^8) = \mathrm{Im}\,(f_*) \oplus \mathrm{Im}\,(\Sigma)$$

Furthermore,

$$\mathrm{Im}\,(f_*) \approx \boldsymbol{Z}_5 + \boldsymbol{Z}_3 + \boldsymbol{Z}_{16}$$
$$\mathrm{Im}\,(\Sigma) \approx \boldsymbol{Z}_3 + \boldsymbol{Z}_8 + \boldsymbol{Z}_4$$
$$\pi_{23}(S^9) \approx \boldsymbol{Z}_{16} + \boldsymbol{Z}_4$$

It now follows from (6.5) that $\Sigma: \pi_{22}(S^8) \to \pi_{23}(S^9)$ must vanish on $\boldsymbol{Z}_5 + \boldsymbol{Z}_3 \subset \mathrm{Im}(f_*)$ but must be one-to-one on $\boldsymbol{Z}_{16} \subset \mathrm{Im}(f_*)$. The following lemma now holds by (3.2).

LEMMA 6.6. *Ker (∂) in (6.3a) is $\boldsymbol{Z}_5 + \boldsymbol{Z}_3$.*

PROPOSITION 6.7. *$\pi_{23}(\Sigma(W)) \approx \boldsymbol{Z}_4$.*

Proof. By (6.6), Im $(\partial) \approx \boldsymbol{Z}_{16}$ in (6.3a). Regardless of how the imbedding Im $(\partial) \subset \boldsymbol{Z}_{16} + \boldsymbol{Z}_4$ is realized, the quotient must be $\boldsymbol{Z}_4$.

This completes the proof of (1.11).

7. The 3-primary components in $\pi_j(EIV)$, $j = 19, 20$. Our present aim is to complete the proofs of (1.7) and (1.8) which were begun in

§5. Let Ω denote the space of loops on EIV. From the spectral sequence one easily obtains:

LEMMA 7.1. *In dimensions* <32, $H^*(\Omega; \boldsymbol{Z}_3)$ *has a basis* $\{1, x_8, x_{16}, x_8^2, x_8x_{16}, x_{24}\}$, $\dim(x_i) = i$. *Furthermore*, $x_8^3 = 0$.

In order to compute the 3-primary components of $\pi_{18}(\Omega)$ and $\pi_{19}(\Omega)$, we proceed by the method of killing cohomology classes in $H^*(\Omega; \boldsymbol{Z}_3)$ via successive fibrations with appropriate Eilenberg-MacLane complexes as fibers. This yields the values of $\pi_j(\Omega) \otimes \boldsymbol{Z}_3$, $j = 18, 19$, and this information, together with §5, will prove (1.7) and (1.8). In the computations of this section we will also set the stage for computation of $\pi_{23}(\Omega) \otimes \boldsymbol{Z}_3$ which will be completed in §8.

A description the of $\boldsymbol{Z}_3$-algebra $H^*(\pi, n; \boldsymbol{Z}_3)$, π a finitely generated abelian group, will be essential. Since, in §8, we will also need a description of $H^*(\pi, n; \boldsymbol{Z}_5)$, we here discuss the general case of $H^*(\pi, n; \boldsymbol{Z}_p)$, p an odd prime. For the proofs of our assertions cf. [6], especially exposés 9, 15, and 16.

Let $I = (a_1, a_2, \cdots)$, a sequence of integers almost everywhere zero. I will be called admissible if

$$a_i \equiv 0 \text{ or } 1 \bmod (2p-2)$$
$$a_i \geqq pa_{i+1} .$$

The degree of I is defined as $q(I) = \Sigma a_i$. I is said to be of the first kind if $a_i \neq 1$, $\forall i$. Otherwise I is said to be of the second kind. If $I = (a_1, \cdots, a_r, 0, 0, \cdots)$ is of the first kind, then one obtains an I' of the second kind by setting

$$I' = (a, \cdots, a_r, 1, 0, \cdots) .$$

Define the numbers

$$g(I) = [pa_1/(p-1)] - q(I)$$
$$n(I) = \{pa_1/(p-1)\} - q(I)$$

where $[b]$ denotes the greatest integer $\leqq b$ and $\{b\}$ denotes the least integer $\geqq b$. Finally, let P^i, $i = 0, 1, 2, \cdots$, denote the Steenrod reduced p-powers, β the mod p Bockstein, and define cohomology operations

$$St^a = P^k,\ b = 2k(p-1)$$
$$St^b = \beta P^k,\ b = 2k(p-1) + 1$$
$$St^I = St^{(a_1)} \circ St^{(a_2)} \circ \cdots,\ I \text{ admissible.}$$

THEOREM 7.2. (*H. Cartan*) *If* I *is admissible of the first kind and if* $n(I') \leqq n$, *then*

$$St^I\colon H^{n+1}(\pi, n; \boldsymbol{Z}_p) \longrightarrow H^{n+q(I')}(\pi, n; \boldsymbol{Z}_p)$$

is a monomorphism. If also $n(I) \leqq n$, *then*

$$St^I\colon H^n(\pi, n; \boldsymbol{Z}_p) \longrightarrow H^{n+q(I)}(\pi, n; \boldsymbol{Z}_p)$$

is a monomorphism. Let $A^*(\pi, n; \boldsymbol{Z}_p)$ *be the direct sum of the images of all of the above monomorphisms, graded by* $n + q(I')$ *and* $n + q(I)$ *respectively. Then the operations* St^I *define a graded homomorphism*

$$A^*(\pi, n; \boldsymbol{Z}_p) \longrightarrow H^*(\pi, n; \boldsymbol{Z}_p)$$

which is an isomorphism onto the image of suspension

$$\sigma\colon H^*(\pi, n + 1; \boldsymbol{Z}_p) \longrightarrow H^*(\pi, n; \boldsymbol{Z}_p) .$$

Let $M_n \subset A^*(\pi, n; \boldsymbol{Z}_p)$ *be the graded subspace consisting of the direct sum of the images of those of the above monomorphisms where* I' *(respectively* I*) is required to satisfy the additional condition* $g(I') < n$ *(respectively* $g(I) < n$*). Then the algebra* $H^*(\pi, n; \boldsymbol{Z}_p)$ *is the free graded commutative* $\boldsymbol{Z}_p$*-algebra generated by* M_n.

A further remark that is of use is that

$$H^n(\pi, n; \boldsymbol{Z}_p) \approx \mathrm{Hom}\,(\pi, \boldsymbol{Z}_p)$$
$$H^{n+1}(\pi, n; \boldsymbol{Z}_p) \approx \mathrm{Hom}\,({}_p\pi, \boldsymbol{Z}_p)$$

where ${}_p\pi \subset \pi$ is the subgroup of elements of order p. One also notes that if ${}_p\pi = \pi$, then

$$\beta\colon H^n(\pi, n; \boldsymbol{Z}_p) \longrightarrow H^{n+1}(\pi, n; \boldsymbol{Z}_p)$$

is a bijection.

In the remainder of this section we understand p to be 3. By the Adem relations [13] one has $P^2 = P^1P^1$. P^1, P^3, and β are trivial on $H^*(\Omega; \boldsymbol{Z}_3)$ since the nontrivial dimensions in this graded vector space are all of the form $8k$. Consequently P^2 is also trivial on $H^*(\Omega; \boldsymbol{Z}_3)$.

We kill the class $x_8 \in H^8(\Omega; \boldsymbol{Z}_3)$ by a fibration

$$K(\boldsymbol{Z}, 7) \longrightarrow X_1 \longrightarrow \Omega .$$

An application of (7.2) gives the following classes as a basis of $H^*(\boldsymbol{Z}, 7; \boldsymbol{Z}_3)$ in dimensions $\leqq 25$ (where $\dim(y) = 7$): $1, y\, P^1(y), \beta P^1(y), P^2(y), \beta P^2(y), P^3(y), \beta P^3(y), P^3P^1(y), \beta P^3P^1(y), y \cdot P^1(y), y \cdot \beta P^1(y), y \cdot P^2(y), y \cdot \beta P^2(y), P^1(y) \cdot \beta P^1(y), (\beta P^1(y))^2$. By straightforward computations using the spectral sequence of this fibration, one obtains

LEMMA 7.3. *In* dim $\leqq 25$, $H^*(X_1; \boldsymbol{Z}_3)$ *has basis* $\{1, u_{11}, \beta(u_{11}), P^1(u_{11}), \beta P^1(u_{11}), x_{16}, u_{19}, \beta(u_{19}), P^3(u_{11}), u_{11}\cdot\beta(u_{11}), u_{23}, \beta P^3(u_{11}), (\beta(u_{11}))^2, x_{24}\}$, *where the dimension of an element is indicated by its subscript.*

In (7.3) the classes x_{16}, x_{24} are the pull-backs of the classes in the base Ω that were denoted by the same symbols. u_{11} and u_{19} restrict respectively to $P^1(y)$ and $P^3(y)$ in the fiber. u_{23} corresponds to $y\cdot x_8^2$ in the E^2 term of the spectral sequence. Using these facts and the Adem relations [13] one verifies the following relations:

$$\begin{aligned} \beta P^1\beta(u_{11}) &= 0 \\ P^2(u_{11}) &= 0 \\ P^2\beta(u_{11}) &= -\beta(u_{19}) \\ \beta P^2\beta(u_{11}) &= 0 \\ P^3\beta(u_{11}) &= \beta P^3(u_{11}) \\ \beta P^3\beta(u_{11}) &= 0\ . \end{aligned}$$

Next kill u_{11} by a fibration

$$K(\boldsymbol{Z}_3, 10) \longrightarrow X_2 \longrightarrow X_1\ .$$

By (7.2), a basis for $H^*(\boldsymbol{Z}_3, 10; \boldsymbol{Z}_3)$ in dimensions $\leqq 24$ is given by the following classes ($\dim(y) = 10$) : $1, y, \beta(y), P^1(y), \beta P^1(y), P^1\beta(y), \beta P^1\beta(y), P^2(y), \beta P^2(y), P^2\beta(y), \beta P^2\beta(y), y^2, y\cdot\beta(y), P^3(y), \beta P^3(y), P^3\beta(y), \beta P^3\beta(y), y\cdot P^1(y)$.

LEMMA 7.4. *Transgression*

$$t : H^{15}(\boldsymbol{Z}_3, 10; \boldsymbol{Z}_3) \longrightarrow H^{16}(X_1; \boldsymbol{Z}_3)$$

is bijective.

Proof. Otherwise the first nonvanishing $H^i(X_2; \boldsymbol{Z}_3)$ for $i > 0$ occurs for $i = 15$, and this would give $\pi_{15}(\Omega)\otimes\boldsymbol{Z}_3 \approx \pi_{15}(X_2)\otimes\boldsymbol{Z}_3 \neq 0$, contradicting (1.4).

Applying all of this information to the spectral sequence of the fiber space X_2 we obtain.

LEMMA 7.5. *In* dim $\leqq 24$, $H^*(X_2; \boldsymbol{Z}_3)$ *has a basis* $\{1, u_{16}, u_{18}, \beta(u_{18}), u_{19}, P^1(u_{16}), P^1\beta(u_{18}), u_{23}, P^2(u_{16}), x_{24}\}$.

These classes satisfy the following relations:

$$\begin{aligned} P^2(u_{16}) &\equiv -\beta P^1\beta(u_{18}) \bmod x_{24} \\ \beta(x_{24}) &= 0 \\ \beta P^2(u_{16}) &= 0 \text{ (a consequence of the above two)} \\ \beta(u_{19}) &= 0 \\ P^1(u_{19}) &\equiv 0 \bmod u_{23} \\ \beta P^1(u_{19}) &\equiv 0 \bmod x_{24} . \end{aligned}$$

Note that, by (1.5), $\pi_{16}(\Omega) \approx \boldsymbol{Z} + (\boldsymbol{Z}_2)^2$, hence to kill u_{16} we need a fibration

$$K(\boldsymbol{Z}, 15) \longrightarrow X_3 \longrightarrow X_2 .$$

Using (7.2), (7.5), and the above relations, we obtain.

LEMMA 7.6. *In* dim $\leqq 24$, $H^*(X_3; \boldsymbol{Z}_3)$ *has a basis* $\{1, u_{18}, \beta(u_{18}), u_{19}, u_{20}, P^1\beta(u_{18}), u_{23}, P^1(u_{20}), x_{24}\}$ *satisfying the relations*: $\beta P^1\beta(u_{18}) \equiv 0 \bmod x_{24}$; $\beta(u_{19}) = 0$; $P^1(u_{19}) \equiv 0 \bmod u_{23}$; $\beta P^1(u_{19}) \equiv 0 \bmod x_{24}$.

COROLLARY 7.7. $\pi_{18}(\Omega) \approx \boldsymbol{Z}_6$.

Proof. By (7.6), $\pi_{18}(\Omega) \otimes \boldsymbol{Z}_3 \approx \boldsymbol{Z}_3$. By (5.5), $\pi_{18}(\Omega) \approx \boldsymbol{Z}_2$ or $\boldsymbol{Z}_6$.

This completes the proof of (1.7).

Next we kill u_{18} by

$$K(\boldsymbol{Z}_3, 17) \longrightarrow X_4 \longrightarrow X_3 .$$

Using the spectral sequence and (7.6) one readily obtains:

LEMMA 7.8. $H^j(X_4; \boldsymbol{Z}_3) \approx 0$, $0 < j < 19$, *and* $H^{19}(X_4; \boldsymbol{Z}_3) \approx \boldsymbol{Z}_3$.

COROLLARY 7.9. $\pi_{19}(\Omega) \approx \boldsymbol{Z}_{1512} + \boldsymbol{Z}_2$.

Proof. By (5.6) and (7.7) there is an exact sequence

$$0 \longrightarrow \boldsymbol{Z}_9 + \boldsymbol{Z}_8 + \boldsymbol{Z}_7 + \boldsymbol{Z}_2 \longrightarrow \pi_{19}(\Omega) \longrightarrow \boldsymbol{Z}_3 \longrightarrow 0 .$$

By (7.8), $\pi_{19}(\Omega) \otimes \boldsymbol{Z}_3 \approx \boldsymbol{Z}_3$. Hence $\pi_{19}(\Omega) \approx \boldsymbol{Z}_{27} + \boldsymbol{Z}_8 + \boldsymbol{Z}_7 + \boldsymbol{Z}_2$.

This completes the proof of (1.8). Evidently in the above lemmas we have obtained information on the cohomology of the spaces X_i in dimensions higher than necessary for the purposes of this section. This information will be used in the next section to help prove (1.12).

8. Partial determination of $\pi_{24}(EIV)$. Notice that by the theory of [8] there is an exact sequence

$$\pi_{24}(S^{16}) \longrightarrow \pi_{24}(S^9) \longrightarrow \pi_{24}(EIV) \longrightarrow \pi_{23}(S^{16}) \longrightarrow \pi_{23}(S^9)$$

which gives explicitly

$$(8.1) \qquad (\boldsymbol{Z}_2)^2 \longrightarrow \boldsymbol{Z}_{240} + (\boldsymbol{Z}_2)^3 \longrightarrow \pi_{24}(EIV) \longrightarrow \boldsymbol{Z}_{240} \longrightarrow \boldsymbol{Z}_{16} + \boldsymbol{Z}_4 .$$

Thus, to prove (1.12) we must compute $\pi_{24}(EIV) \otimes \boldsymbol{Z}_5$ and $\pi_{24}(EIV) \otimes \boldsymbol{Z}_3$.

Recall the fibration $K(\boldsymbol{Z}_3, 17) \rightarrow X_4 \rightarrow X_3$. Recall also from (7.6) the relation $\beta P^1\beta(u_{18}) \equiv 0 \bmod x_{24}$. Replacing x_{24} with its negative if necessary, we obtain just two possibilities:

$$\beta P^1\beta(u_{18}) = 0$$

or

$$\beta P^1\beta(u_{18}) = x_{24} .$$

In order to determine a basis for $H^*(X_4; \boldsymbol{Z}_3)$ it will be necessary to consider these two possibilities.

LEMMA 8.2. *If* $\beta P^1\beta(u_{18}) = 0$, *then, in* $\dim \leqq 24$, $H^*(X_4; \boldsymbol{Z}_3)$ *has as a basis* $\{1, u_{19}, u_{20}, u_{21}, \beta(u_{21}), u_{23}, P^1(u_{20}), w_{23}, x_{24}\}$. *The following relations are also satisfied:* $\beta(u_{19}) = 0$; $P^1(u_{19}) \equiv 0 \bmod u_{23}$; $\beta P^1(u_{19}) \equiv 0 \bmod x_{24}$.

LEMMA 8.3. *If* $\beta P^1\beta(u_{18}) = x_{24}$, *then, in* $\dim \leqq 24$, $H^*(X_4; \boldsymbol{Z}_3)$ *has as a basis* $\{1, u_{19}, u_{20}, u_{21}, \beta(u_{21}), P^1(u_{20}), u_{23}\}$ *with* $\beta(u_{19}) = 0$, $\beta P^1(u_{19}) = 0$, $P^1(u_{19}) \equiv 0 \bmod u_{23}$.

We kill u_{19} by

$$K(\boldsymbol{Z}_{27}, 18) \longrightarrow X_5 \longrightarrow X_4 .$$

The use of $K(\boldsymbol{Z}_{27}, 18)$ is dictated by (7.9). The 3-primary component of $\pi_{19}(X_5)$ is 0.

Note that by (7.2) a basis of $H^*(\boldsymbol{Z}_{27}, 18; \boldsymbol{Z}_3)$ is given by $\{1, y_{18}, y_{19}, P^1(y_{18}), \beta P^1(y_{18}), P^1(y_{19}), \beta P(y_{19})\}$ in $\dim \leqq 24$. Here $\beta(y_{18}) = 0$.

LEMMA 8.4. *Transgression*

$$t\colon H^{19}(\boldsymbol{Z}_{27}, 18; \boldsymbol{Z}_3) \longrightarrow H^{20}(X_4; \boldsymbol{Z}_3)$$

is bijective.

Proof. Otherwise, $\pi_{19}(X_5) \otimes \boldsymbol{Z}_3 \approx \boldsymbol{Z}_3$, contradicting the construction of X_5.

COROLLARY 8.5. $H^i(X_5, \boldsymbol{Z}_3) \approx 0$, $0 < i < 21$, *while* $H^{21}(X_5; \boldsymbol{Z}_3) \approx \boldsymbol{Z}_3$ *and is generated by (the pull-back of)* u_{21}. $\beta(u_{21}) \neq 0$.

LEMMA 8.6. $t(P^1(y_{18})) = \pm u_{23}$.

Proof. In either the hypothesis of (8.2) or of (8.3), $t(P^1(y_{18})) = P^1(u_{19}) \equiv 0 \bmod u_{23}$. We must show $P^1(u_{19}) \neq 0$. Suppose the contrary. Then, killing u_{21} by $K(\boldsymbol{Z}_3, 20) \to X_6 \to X_5$, one shows that $H^i(X_6; \boldsymbol{Z}_3) \approx 0$, $0 < i < 22$, and $H^{22}(X_6; \boldsymbol{Z}_3) \approx \boldsymbol{Z}_3$. Thus $\pi_{22}(\Omega) \otimes \boldsymbol{Z}_3 \approx \pi_{22}(X_6) \otimes \boldsymbol{Z}_3 \approx \boldsymbol{Z}_3$, contradicting (1.11).

LEMMA 8.7. *In the hypothesis of* (8.2), $t(\beta P^1(y_{18})) = \pm x_{24}$.

Proof. By (8.2), $t(\beta P^1(y_{18})) = \beta P^1(u_{19}) \equiv 0 \bmod x_{24}$. We must show $\beta P^1(u_{19}) \neq 0$. Suppose the contrary. Kill $u_{21} \in H^{21}(X_5; \boldsymbol{Z}_3)$ by $K(\boldsymbol{Z}_3, 20) \to X_6 \to X_5$. Using (8.2), (8.4), (8.5), and (8.6), one shows $\pi_{23}(\Omega) \otimes \boldsymbol{Z}_3 \approx \pi_{23}(X_6) \otimes \boldsymbol{Z}_3 \approx \boldsymbol{Z}_3 + \boldsymbol{Z}_3$. Here the two generators of $H^{23}(X_6; \boldsymbol{Z}_3)$ come from the w_{23} of (8.2) and from $\beta P^1(y_{18})$. This information, together with (8.1), implies that the 3-component of $\pi_{23}(\Omega)$ is $\boldsymbol{Z}_3 + \boldsymbol{Z}_3$. Thus if $w_{23}, v_{23} \in H^{23}(X_6; \boldsymbol{Z}_3)$ are the two generators, $\beta(w_{23})$ and $\beta(v_{23})$ will be linearly independent. But $\beta(w_{23})$ and $\beta(v_{23})$ are $\equiv 0 \bmod x_{24}$, so that we have reached a contradiction.

LEMMA 8.8. *In the hypothesis of* (8.3), $t(\beta P^1(y_{18})) = 0$.

Proof. $t(\beta P^1(y_{18}) = \beta P^1(u_{19}) = 0$ by (8.3).

Putting all of this information together, one obtains.

LEMMA 8.9. *In either the hypothesis of* (8.2) *or of* (8.3), $H^*(X_5, \boldsymbol{Z}_3)$ *has as a basis in* $\dim \leqq 23$ *classes* $1, u_{21}, \beta(u_{21}), w_{23}$.

PROPOSITION 8.10. The 3-primary component of $\pi_{23}(\Omega)$ is $\boldsymbol{Z}_9$.

Proof. By (8.9) and the process of killing u_{21}, one finds $\pi_{23}(\Omega) \otimes \boldsymbol{Z}_3 \approx \boldsymbol{Z}_3$. The assertion now follows by (8.1).

There remains the task of finding the 5-primary component of $\pi_{24}(EIV)$. Here we make use of (1.2) and of the mod 5 Steenrod algebra. Recall from [3, 19.6] that if x_i generates $H^i(\Sigma(W); \boldsymbol{Z}_5)$, $i = 9, 17$, then $P^1(x_9) = \pm 2x_{17}$.

Kill x_9 by

$$K(\boldsymbol{Z}, 8) \longrightarrow X_1 \longrightarrow \Sigma(W) .$$

This gives the following lemma.

LEMMA 8.11. *In* dim $\leqq 25$, $H^*(X_1; \boldsymbol{Z}_5)$ *has a basis* $\{1, u_{17}, u_{24}, \beta(u_{24}), u_{25}\}$ *with relations* $\beta(u_{17}) = 0$, $P^1(u_{17}) \equiv \beta(u_{24}) \bmod u_{25}$.

Since $\pi_{17}(\Sigma(W)) \approx \boldsymbol{Z} + (\boldsymbol{Z}_2)^2$, one needs

$$K(\boldsymbol{Z}, 16) \longrightarrow X_2 \longrightarrow X_1$$

to kill u_{17}.

LEMMA 8.12. $H^i(X_2; \boldsymbol{Z}_5) \approx 0$, $0 < i < 24$, *and* $H^{24}(X_2; \boldsymbol{Z}_5) \approx \boldsymbol{Z}_5$.

COROLLARY 8.13. *The* 5-*primary component of* $\pi_{24}(\Sigma(W))$ *is* $\boldsymbol{Z}_{25}$.

Proof. By (8.12), $\pi_{24}(\Sigma(W)) \otimes \boldsymbol{Z}_5 \approx \boldsymbol{Z}_5$. The corollary now follows by (8.1).

Now by (8.1), (8.10), and (8.13) we can conclude (1.12).

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Received August 31, 1965.

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