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THE MEASURE ALGEBRA OF A LOCALLY COMPACT SEMIGROUP

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Let G be a locally compact idempotent, commutative, topological semigroup (semi-lattice). Let $\mathscr{M}(G)$ denote its measure algebra, i.e., $\mathscr{M}(G)$ consists of all countably additive regular Borel-measures defined on G and has the usual Banach algebra structure: pointwise linear operations, convolution, and total variation norm. To understand the structure of such a convolution algebra one studies its maximal ideals, the nature of the Gelfand transform, the structure of the closed ideal and the related question of spectral synthesis, etc.

In this paper G is the cartesian product of topological semigroups G_α of the following form: G_α is a linearly ordered set, locally compact in its order topology; multiplication in G_α is given by $xy = \max(x, y)$. The product semigroup is assumed locally compact in the product topology.

The main theorem of this paper gives a representation of the space of maximal ideals $\Delta\mathscr{M}(G)$, for a finite product, in terms of the dual semigroup $\hat{G}$. The multiplicative linear functionals of $\mathscr{M}(G)$ are integrals of fixed semi-characters

$$\tau(\mu) = \int_G \chi(x) d\mu(x) , \qquad \mu \in M(G) .$$

It is shown that this integral representation does not hold for infinite products because the semi-characters are usually not integrable.

This paper draws heavily upon the studies of Ross [14] and Hewitt and Zuckerman [5] in which linearly ordered semigroups of the present type were treated. Most of Ross' results generalize to the case of the finite product, in particular his description of the Gelfand transforms of measures. In Theorem 3.4 [14] Ross showed that for linearly ordered G spectral synthesis obtains in $\mathscr{M}(G)$ even though G is not compact (*cf.* 37C and 38A [8] as well as [9]). An example in this paper shows, on the other hand, that the compactness of the semigroup G does not imply spectral synthesis in $\mathscr{M}(G)$ in case G is the product of two linearly ordered semigroups.

For terminology not explained below in measure theory, topology, and harmonic analysis, see [2], [7], and [4] and [8], respectively.

1. Preliminaries.

1.1 Let X be a *partially ordered* set, i.e., a set ordered by a transitive, antisymmetric, and reflexive order relation $\leqq$. For $x \in X$ define $L_x = \{z \in X \mid z \leqq x\}$ and $M_x = \{z \in X \mid x \leqq z\}$. For $E \subset X$ define $L(E) = \cup\{L_x \mid x \in E\}$, and $M(E) = \cup\{M_x \mid x \in E\}$.

If X is *linearly ordered*, the following sets will be called *intervals*: for $x, y \in X$, $(x, y) = \{z \in X \mid x < z < y\}$, $[x, y] = \{z \in X \mid x \leqq z \leqq y\}$. The half-open intervals are defined analogously, and the notations $(-\infty, x]$ for L_x, $(-\infty, x)$ for $X \backslash M_x$, etc., will often be employed. The *order topology* for a linearly ordered set X has for a subbase the family $\{(-\infty, x)\}_{x \in X} \cup \{(x, \infty)\}_{x \in X}$.

For two sets A and B, $A \backslash B = \{x \mid x \in A \text{ and } x \notin B\}$, $A \Delta B = (A \backslash B) \cup (B \backslash A)$; the void set is denoted by ϕ. ξ_A denotes the *characteristic function* of the set A. $\bar{\bar{A}}$ is the *cardinal number* of A. For the *cartesian product* of a family $\{X_\alpha\}_{\alpha \in S}$ of sets we write $\boldsymbol{P}_{\alpha \in S} X_\alpha$ (or X^S if the X_α are all indentical). A point $x = (x_\alpha)_{\alpha \in S} \in \boldsymbol{P}_{\alpha \in S} X_\alpha$ has x_α for its α^{th} coordinate. π_α is the α^{th} projection function of the product $\boldsymbol{P}_{\alpha \in S} X_\alpha$.

1.2. In this section G will be any T_2 locally compact commutative idempotent topological semi-group. In particular, the multiplication $(x, y) \to xy$ in G is a continuous function of $G \times G$ onto G. $\mathscr{B}(G)$ will denote the set of all *Borel subsets* of G (11.1 [4]). A partial ordering on G is introduced by

DEFINITION 1.3 For $x, y \in G$ define $x \leqq y$ to mean $xy = y$.

LEMMA 1.4. *With the ordering of* 1.3, G *is a topological semi-lattice under* $x \vee y = xy$.

DEFINITION 1.5. (see [6]) A subset P of G is an *ideal* if $PG \subset P$. An ideal P is *prime* if $A = G \backslash P$ is a nonvoid subsemigroup of $G (A^2 \subset A)$.

The complement A of a prime ideal will be called a *prime subsemigroup* (pssg). Note that a nonvoid subsemi-group A of G is a pssg if and only if $L(A) = A$.

A *semi-character* of G is a bounded complex-valued function χ on G, not identically zero, which satisfies the functional equation $\chi(xy) = \chi(x)\chi(y)$ for all $x, y \in G$. The set of all semi-characters of G is denoted by $\hat{G}$.

THEOREM 1.6. $\chi \in \hat{G}$ *if and only if* χ *is the characteristic function of a pssg of* G.

Proof. Let $\chi \in \hat{G}$. Since $\chi^2(x) = \chi(x^2) = \chi(x)$ for all $x \in G$, χ assumes the values 0 and 1 only. The set $A = \{x \in G \mid \chi(x) = 1\}$ is the desired pssg. Given any pssg A, $\xi_A(xy) = \xi_A(x)\xi_A(y)$ for all $x, y \in G$ and $\xi_A \neq$

0; hence $\xi_A \in \hat{G}$.

This theorem points to ξ_{L_x}, $x \in G$, as an example of a Borel-measurable semi-character, because L_x is a closed pssg by Lemma 2 on page 361 ([16]).

1.7. Let $\mathscr{M}(G)$ denote the Banach space of all complex-valued countably additive regular Borel measures on G with the usual variation norm $\|\mu\| = |\mu|(G)$. Let $\mathscr{C}_0(G)$ be the Banach space of all continuous complex-valued functions on G which are arbitrarily small outside of compact sets, normed by the uniform norm. Then by the Riesz Representation theorem (19.12 [4]) $\mathscr{M}(G)$ is isometrically isomorphic to $\mathscr{C}_0^*(G)$ under the mapping $\mu \to M$, where $M(f) = \int_G f d\mu, f \in \mathscr{C}_0(G)$.

DEFINITION 1.8. For $\mu, \nu \in \mathscr{M}(G)$ define the *convolution* $\mu * \nu$ in $\mathscr{M}(G)$ by

$$\mu * \nu(E) = \int_G \int_G \xi_E(xy) d\mu(x) d\nu(y),\ E \in \mathscr{B}(G)\ . \tag{1.8.1}$$

With $*$ as multiplication $\mathscr{M}(G)$ is then a commutative Banach algebra which has an identity if G has an identity element (p. 351 [15]). The case in which G has no identity is discussed in 3.12, below.

1.9. It follows directly from the definition that for a Borel pssg A and for measures μ and ν in $\mathscr{M}(G)$

$$\mu * \nu(A) = \mu(A) \cdot \nu(A)\ . \tag{1.9.1}$$

1.10. For each $x \in G$ let δ_x denote the *point mass* at x (1.7 [14]). For $E \subset G$ and $x \in G$ let ${}_xE$ denote the set ${}_xE = \{y \in G \mid xy \in E\}$. For $A \in \mathscr{B}(G)$ let μ_A be the member of $\mathscr{M}(G)$ defined by $\mu_A(E) = \mu(E \cap A)$. The following formulae prove useful in the subsequent sections:

LEMMA 1.11. *For $\mu, \nu \in \mathscr{M}(G)$ and $x, y \in G$ we have*

$$\mu * \nu(E) = \int_G \mu({}_xE) d\nu(x),\ E \in \mathscr{B}(G) \tag{1.11.1}$$

$$\mu * \delta_x(E) = \mu({}_xE), \qquad E \in \mathscr{B}(G) \tag{1.11.2}$$

$$\delta_x * \delta_y = \delta_{xy} \tag{1.11.3}$$

(1.11.4) $\mu * \delta_x = \mu$ *if and only if* $S(\mu) \subset M_x$, *where* $S(\mu)$ *is the support of the measure* μ (11.25 [4]).

Proof. If $E \in \mathscr{B}(G)$ then ${}_xE \in \mathscr{B}(G)$ for all $x \in G$ and so

$$\int_G \xi_E(xy)\,d\mu(y) = \int_G \xi_{{}_xE}(y)\,d\mu(y) = \mu({}_xE)\ .$$

By the Fubini theorem (14.25 [4]) this function defined on G is Borel-measurable because $\mu*\nu \in \mathscr{M}(G)$, and (1.11.1) follows from (1.8.1). Setting $\nu = \delta_x$ in (1.11.1) we obtain (1.11.2) and similarly (1.11.3).

Next suppose $S(\mu) \subset M_x$. Then $\mu*\delta_x(E) = \mu({}_xE \cap M_x) = \mu(E \cap M_x) = \mu(E)$ for all $E \in \mathscr{B}(G)$, as ${}_xE \cap M_x = E \cap M_x$; thus $\mu*\delta_x = \mu$. If, conversely, $\mu*\delta_x = \mu$ and if $E \subset G\backslash M_x$ then ${}_xE = \phi$ and $\mu(E) = \mu*\delta_x(E) = \mu({}_xE) = 0$. It follows that $|\mu|(G\backslash M_x) = 0$ and $S(\mu) \subset M_x$, as required.

2. The representation of $\Delta\mathscr{M}(G)$.

2.1. Let $\Delta\mathscr{M}(G)$ denote the class of all algebra homomorphisms of $\mathscr{M}(G)$ onto the complex numbers, the structure space of the algebra $\mathscr{M}(G)$ (23A, [8]). It is the purpose of this section to show the impossibility of representing each member τ of $\Delta\mathscr{M}(G)$ as an integral of a fixed semi-character.

LEMMA 2.2. *Let $\tau \in \Delta\mathscr{M}(G)$, then the function $x \rightarrow \tau(\delta_x)$ is either a semi-character of G or is identically* 0.

Proof. (1.11.3) implies that the function $x \rightarrow \tau(\delta_x)$ is multiplicative and assumes the values 0 and 1 only.

DEFINITION 2.3. Let $\tau \in \Delta\mathscr{M}(G)$. If $A = \{x \in G \mid \tau(\delta_x) = 1\}$ then τ is said to *determine* the set A. By 2.2 A is either a pssg of G or $A = \phi$.

2.4. If G is compact then G is a *compact ordered space* in the sense of Nachbin [13], which can be embedded into a cube [i.e., a product of closed unit intervals] by means of a homeomorphism which also is a lattice isomorphism (see [12] and also [16]). Motivated by this theorem we shall, from now on, restrict our attention to the following type of semigroups:

For each α in an index set S let G_α be linearly ordered and topologized by the order topology. Let $G = \boldsymbol{P}_{\alpha \in S} G_\alpha$ be locally compact in the product topology. For $x, y \in G$ define $xy = (\max\{x_\alpha, y_\alpha\})_{\alpha \in S}$. G will then satisfy 1.2. In particular, the Tykhonov cube $G = [0, 1]^S$, ordered coordinatewise, is a semigroup of this type.

2.5. In an uncountable Tykhonov cube there exist examples of semi-characters which are nonmeasurable with respect to certain product measures. In the countable Tykhonov cube one can show by means of a cardinality argument that nonBorel measurable semi-characters abound. The method of constructing such semi-characters is employed in the following example. We consider here the 'minimal' infinite product space $G = \{0, 1\}^S$, where $\bar{\bar{S}} = \aleph_0$. G is a semigroup of the type under discussion as well as a compact Abelian group, the so-called *Cantor group*, under coordinatewise addition modulo 2. Utilizing a theorem of Hewitt (Theorem 47 [3]) which asserts that there exist 2^c distinct ultrafilters on the countably infinite set S, we will show that G possesses 2^c nonBorel pssg's which are in fact nonmeasurable with respect to some member of $\mathscr{M}(G)$.

2.6. Let $G = \{0, 1\}^S$. For $U \subset S$ define an element x_U of G by

(2.6.1) $$(x_U)_\alpha = 1 \text{ if } \alpha \in U \text{ and } (x_U)_\alpha = 0 \text{ if } \alpha \in S \backslash U .$$

Given any ultrafilter $\mathscr{A}$ on S, set

(2.6.2) $$A = \{x_U \mid U \subset S,\ U \notin \mathscr{A}\} .$$

THEOREM 2.7. *For distinct ultrafilters on* S, (2.6.2) *defines distinct pssg's of* G. *If* $\bar{\bar{S}} = \aleph_0$, *then the set of all pssg's of* G *has cardinality* 2^c, *and hence a pssg of* G *is usually not a Borel set.*

Proof. Let $\mathscr{A}$ and A be as in 2.6. If $x_U, x_V \in A$ then $x_U x_V = x_{U \cup V} \in A$, since $U \notin \mathscr{A}, V \notin \mathscr{A} \rightarrow U \cup V \notin \mathscr{A}$. Thus A is a sub-semigroup. Now A is a pssg because $L(A) = A$.

Let $\mathscr{A}, \mathscr{B}$ be distinct ultrafilters on S and A, B the corresponding pssg's. Let $U \in \mathscr{A} \backslash \mathscr{B}$. By (2.6.2) $x_U \in B$ and since $\mathscr{A}$ is closed under supersets, $x_U \notin A$. Therefore $A \neq B$. The last statement of the theorem follows from Hewitt's theorem (Th. 47 [3]) and the fact that G has only c Borel sets (p. 26 [2]).

EXAMPLE 2.8. Let μ be the Haar measure of the group G of 2.5. Let $\mathscr{A}$ be a free ultrafilter on the index set S and let A be the pssg given by (2.6.2). Consider the prime ideal $P = G \backslash A$. It will be shown that

(i) P is dense in G;

(ii) either $\mu(A) = 0$ or A is not μ-measurable;

(iii) $\mu(A) \neq 0$.

It follows then that A is not μ-measurable and hence $A \notin \mathscr{B}(G)$. It was shown in 2.7 that there exist 2^c distinct pssgs of this type in G.

To prove (i) let $N = \bigcap_{\alpha \in F} \pi_\alpha^{-1}(N_\alpha)$ be any nonvoid basic open set in G, i.e., F is a finite subset of S and $\phi \neq N_\alpha \subset \{0, 1\}$, for $\alpha \in F$.

Since $\mathscr{S}$ is a free ultrafilter, $F \notin \mathscr{S}$; hence $U = S \backslash F \in \mathscr{S}$ and $x_U \in P$. Let $x \in N$ and let $y = x x_U$ then y is in the ideal P and $y_\alpha = x_\alpha$ for all $\alpha \in F$, so that $y \in N \cap P$. Thus P meets every nonvoid open subset of G.

To establish (ii), assume that A is μ-measurable and $\mu(A) > 0$. Then by Steinhaus' theorem (20.17 [4]) there exists a nonvoid open set $0 \subset A - A = \{x - y;\ x, y \in A\}$.

Note, however, that $x_U, x_V \in A$ implies

$$x_U - x_V = x_U + x_V = x_{U \Delta V} \leqq x_{U \cup V} = x_U x_V \in A ,$$

therefore $0 \subset A - A \subset A$, a contradiction since by (i) $0 \cap P \neq \phi$.

For (iii) we note that $P - P \subset A$, and again apply Steinhaus' theorem to show that $\mu(K) = 0$ for each compact subset K of P. Hence $\mu(A) \neq 0$.

2.9. Suppose $\tau \in \Delta \mathscr{M}(G)$ and τ determines A as in 2.3; suppose also that for some measure $\mu \in \mathscr{M}(G)$, A is not μ-measurable, as in the above example. Then there exists no semi-character $\xi_B \in \hat{G}$ such that the formula

$$\tau(\nu) = \int_G \xi_B(x) d\nu(x) \tag{2.9.1}$$

holds for all $\nu \in \mathscr{M}(G)$. For if $B \neq A$ then

$$\tau(\delta_x) \neq \int \xi_B(t) d\delta_x(t) = \delta_x(B)$$

for all $x \in A \Delta B$; and if $B = A$, then B is not μ-measurable and (2.9.1) does not make sense for $\nu = \mu$.

On the other hand, we will show that each pssg A of G is determined by some $\tau \in \Delta \mathscr{M}(G)$; such a homomorphism $\tau \in \Delta \mathscr{M}(G)$ is then not representable by formula (2.9.1).

DEFINITION 2.10. Let μ_* denote the *inner measure* induced by $\mu \in \mathscr{M}(G)$; i.e., for any set $E \subset G$ and $\mu \geqq 0$

$$\mu_*(E) = \sup \{\mu(K) \mid K \subset E,\ K \text{ is compact}\} \tag{2.10.1}$$

and if $\mu \in \mathscr{M}(G)$ has the Jordan decomposition $\mu = \mu^1 - \mu^2 + i(\mu^3 - \mu^4)$ (p. 123 [2]), set

$$\mu_*(E) = \mu^1_*(E) - \mu^2_*(E) + i(\mu^3_*(E) - \mu^4_*(E)) . \tag{2.10.2}$$

THEOREM 2.11. *Let G be as in* 1.2 *and A a pssg of G. Then*

$$\tau_*(\mu) = \mu_*(A) \text{ for } \mu \in \mathscr{M}(G) \tag{2.11.1}$$

defines a member τ_* *of* $\Delta\mathscr{M}(G)$ *which determines* A *in the sense of* 2.3.

Proof. First let $\mu \geqq 0$. For a compact set $K \subset G$ set $S(K) = \bigcup\{L(K^n) \mid n = 1, 2, \cdots\}$, the smallest pssg of G which contains K. The order relation on G is continuous (p. 359 [16]) because the multiplication is continuous. Thus $L(K^n)$ is closed (p. 361 [16]), so that $S(K)$ is F_σ and hence a Borel pssg. Now form the family

$$\mathscr{S} = \{S(K) \mid K \subset A \text{ and } K \text{ is compact}\} . \tag{2.11.2}$$

Then $\mathscr{S}$ is a directed set (p. 65 [7]) under set inclusion and

$$\{\mu(S) \mid S \in \mathscr{S}\} \tag{2.11.3}$$

is a monotone net in the compact interval $[0, \mu(G)]$. Set

$$\tau_*(\mu) = \lim_{S \in \mathscr{S}} \mu(S) . \tag{2.11.4}$$

For arbitrary $\mu \in \mathscr{M}(G)$ we use the Jordan decomposition of μ and define

$$\tau_*(\mu) = \tau_*(\mu^1) - \tau_*(\mu^2) + i(\tau_*(\mu^3) - \tau_*(\mu^4)) . \tag{2.11.5}$$

Obviously (2.11.4) holds for arbitrary μ. By (1.9.1) τ_* is multiplicative. $\tau_* \neq 0$ since for $a \in A$, $L_a \in \mathscr{S}$ and therefore $\tau_*(\delta_a) \geqq \delta_a(\mathrm{L}_a) = 1$. Thus $\tau_* \in \Delta\mathscr{M}(G)$.

Each compact subset of A is contained in a member of $\mathscr{S}$. A straightforward calculation shows that $\tau_*(\mu) = \mu_*(A)$. For those $\mu \in \mathscr{M}(G)$ for which A happens to be μ-measurable $\tau_*(\mu) = \mu(A)$; in particular $\tau_*(\delta_x) = \delta_x(A)$ for all $x \in G$, so that the homomorphism τ_* determines the pssg A.

3. The structure space of $\mathscr{M}(G)$ for finite products.

3.1. From now on G will be a finite product, $G = \boldsymbol{P}_{k=1}^n G_k$, satisfying the hypothesis of 2.4; i.e., G is a product lattice. The symbol $\leqq$ will donote the order relation on G as well as that on the coordinate spaces G_k; likewise the meaning of L_x and M_x will vary according as $x \in G$ or $x \in G_k$. In context these usages will cause no confusion.

LEMMA 3.2. *Let* A *be a pssg of* G, *then* A *has the form* $A = \boldsymbol{P}_{k=1}^n A_k$ *where, for* $k = 1, \cdots, n$, A_k *is a pssg of the semigroup* G_k. *Thus* A *is a Borel set.*

Proof. Set $A_k = \pi_k(A)$, $k = 1, \cdots, n$; then A_k is a sub-semigroup of G_k and $L(A_k) = A_k$, so that A_k is a pssg of G_k.

Clearly, $A \subset \boldsymbol{P}_{k=1}^n A_k$. Given any $x = (x_k)_{k=1}^n \in \boldsymbol{P}_{k=1}^n A_k$ we choose $y^k \in A$, for $k = 1, \cdots, n$, such that $\pi_k y^k = x_k$. Then $y = y^1 y^2 \cdots y^n$ is a member of A and $x \leqq y$; hence $x \in A$. Therefore $A = \boldsymbol{P}_{k=1}^n A_k$; each A_k is either open or closed, so that A is a Borel set.

Note that in an infinite product lattice a pssg need not be of this form.

LEMMA 3.3. *Let A be a pssg of G. Let $\mu \in \mathscr{M}(G)$ and $\varepsilon > 0$ be given. Then there exists an element $a \in A$ such that $|\mu|(A \backslash L_a) < \varepsilon$.*

Proof. Use the regularity of μ to choose a compact set $K \subset A$ such that $|\mu|(A \backslash K) < \varepsilon$. If $a_k = \sup \pi_k K$, $k = 1, \cdots, n$, and if $a = (a_k)_{k=1}^n$ then $K \subset L_a$ and $a_k \in \pi_k K \subset \pi_k A$, hence by 3.2 $a \in A$, and the result follows.

LEMMA 3.4. *Let $\mu \in \mathscr{M}(G)$ be such that $\mu \geqq 0$ and suppose that $\tau \in \Delta\mathscr{M}(G)$ determines A. If there exists a number $\varepsilon > 0$ and an element $y \in G \backslash A$ such that $\mu(G \backslash M_y) < \varepsilon$ then $|\tau(\mu)| < \varepsilon$.*

Proof. Write $\mu = \mu_1 + \mu_{M_y}$ (see 1.10). Then $\|\mu_1\| = \mu_1(G) = \mu(G \backslash M_y) < \varepsilon$.

By (1.11.4) we have $\mu_{M_y} = \mu_{M_y} * \delta_y$, and so $\tau(\mu_{M_y}) = \tau(\mu_{M_y})\tau(\delta_y) = 0$, since $y \in G \backslash A$. Therefore $|\tau(\mu)| = |\tau(\mu_1)| \leqq \|\tau\| \cdot \|\mu_1\| < \varepsilon$.

We now state the main theorem of this section.

THEOREM 3.5. *The mapping $\tau \rightarrow \xi_A$ defined by*

$$\tau(\delta_x) = \xi_A(x) \text{ for all } x \in G \tag{3.5.1}$$

is a one-to-one mapping of $\Delta\mathscr{M}(G)$ onto $\hat{G}$. The formula

$$\tau(\mu) = \int \xi_A(x) d\mu(x) = \mu(A) \tag{3.5.2}$$

holds for all $\mu \in \mathscr{M}(G)$ and all $\tau \in \Delta\mathscr{M}(G)$.

Proof. By 2.2 $\xi_A \in \hat{G}$ provided $A \neq \phi$. Assuming, however, that $A = \phi$, i.e., $\tau(\delta_a) = 0$ for all $a \in G$, we shall conclude that $\tau(\mu) = 0$ for all $\mu \in \mathscr{M}(G)$, contradicting the hypothesis that $\tau \in \Delta\mathscr{M}(G)$. For, given $\mu \in \mathscr{M}(G)$ and $\varepsilon > 0$, there exists a compact set $K \subset G$ such that $|\mu|(G \backslash K) < \varepsilon$; hence $\|\mu_{G \backslash K}\| < \varepsilon$. Since K is compact there exists $y \in G$ such that $K \subset M_y$, and (1.11.4) implies that $\tau(\mu_K) = \tau(\mu_K * \delta_y) = 0$. Thus $|\tau(\mu)| = |\tau(\mu_{G \backslash K})| \leqq \|\mu_{G \backslash K}\| < \varepsilon$, and since ε was arbitrary, $\tau(\mu) = 0$.

To prove (3.5.2) let $\mu \geqq 0$ and consider the sets

(3.5.3) $P = G\backslash A$ and, for $k = 1, \cdots, n$, $P_k = G_k\backslash A_k$.

We show that $\tau(\mu_P) = 0$ by expressing P as in the union of the pairwise disjoint Borel sets $T_k = \pi_k^{-1}P_k\backslash\bigcup_{j=1}^{k-1}\pi_j^{-1}P_j$, and applying 3.4 to the measures μ_{T_k}. It follows that $\tau(\mu) = \tau(\mu_A)$.

Next, let $\varepsilon > 0$ be given and let $x \in A$ be such that $\mu(A\backslash L_x) < \varepsilon$ (by 3.3). By (1.11.2) there exists a measure $\nu \in \mathscr{M}(G)$ such that $\mu_A * \delta_x(E) = \mu_A(L_x)\delta_x(E) + \nu(E)$ and such that $\|\nu\| < \varepsilon$. It follows, since $x \in A$, that $\tau(\mu) = \tau(\mu_A) = \tau(\mu_A * \delta_x) = \mu_A(L_x) + \tau(\nu)$, and so

$$|\tau(\mu) - \mu(A)| \leqq |\mu_A(L_x) - \mu(A)| + |\tau(\nu)| \leqq |\mu(A\backslash L_x)| + \|\nu\| < 2\varepsilon ,$$

and (3.5.2) holds for $\mu \geqq 0$. For arbitrary $\mu \in \mathscr{M}(G)$ it holds because of the linearity of τ.

THEOREM 3.6. *The Banach-algebra $\mathscr{M}(G)$ is semi-simple.*

In view of (3.5.2) we need only show that if $\mu(A) = 0$ for all pssgs A then $\mu = 0$. This is most efficiently accomplished if one uses the machinery developed in § 4 and adapts the proof of 2.4 [14].

3.7. Having identified $\Delta\mathscr{M}(G)$ with $\hat{G}$ in 3.5, we will from now on use the notation

(3.7.1) $$\hat{G} = \{\tau_A \mid A \text{ is a pssg of } G\} ,$$

where $\tau_A(\mu) = \mu(A)$ for all $\mu \in \mathscr{M}(G)$. A partial order $\leqq$on $\hat{G}$ is given by

(3.7.2) $$\tau_A \leqq \tau_B \text{ if and only if } A \subset B .$$

Setting $\tau_0(\mu) = \mu(\phi) = 0$ for all $\mu \in \mathscr{M}(G)$ we write

(3.7.3) $$\hat{G}_0 = \hat{G} \cup \{\tau_0\}, \text{ and } \tau_0 < \tau_A \text{ for all } \tau_A \in \hat{G} .$$

τ_0 is the zero homomorphism on $\mathscr{M}(G)$ and is taken to correspond to the zero semi-character on G. $\hat{G}$ is further identified with $\boldsymbol{P}_{k=1}^n\hat{G}_k$ via the map

(3.7.4) $$\tau_A \longrightarrow (\tau_{A_k})_{k=1}^n$$

where $A_k = \pi_k A$ is a pssg of G_k and τ_{A_k} is regarded as a member of the structure space $\Delta\mathscr{M}(G_k)$. Following Ross' notation (2.5 [14]) we write

(3.7.5) $$\tau_{a_k)} \text{ for } \tau_{(-\infty,a_k)} \text{ and } \tau_{a_k]} \text{ for } \tau_{(-\infty,a_k]} .$$

Then $(\hat{G}, \leqq)$ and $(\hat{G}_0, \leqq)$ are lattices since the mapping in (3.7.4) is bi-isotone; and $\hat{G}$ is, in fact, a product lattice. We now topologize these lattices and obtain a characterization of $\Delta\mathscr{M}(G)$.

DEFINITION 3.8. For each measure $\mu \in \mathscr{M}(G)$ define the *Gelfand transform* $\hat{\mu}$ of μ by

$$\hat{\mu}(\tau_A) = \tau_A(\mu) = \mu(A) \text{ for all } \tau_A \in \hat{G} \text{ (respectively } \hat{G}_0) . \tag{3.8.1}$$

The *Gelfand topology* on $\hat{G}$ (resp. $\hat{G}_0$) is the weakest topology making each of the complex-valued functions $\hat{\mu}$ continuous on $\hat{G}$ ($\hat{G}_0$).

THEOREM 3.9. *The Gelfand topology on $\hat{G}_0$ equals the interval topology. With this topology $\hat{G}_0$ is a totally disconnected compact Hausdorff space; it is the one-point compactification of $\hat{G}$ with its Gelfand topology.*

Proof. Given a noneoid set $\mathscr{A} \subset \hat{G}_0$, let $B = \bigcap \{A \mid \tau_A \in \mathscr{A}\}$ and $C = \boldsymbol{P}_{k=1}^n \{\bigcup A_k \mid (\tau_{A_k})_{k=1}^n \in \mathscr{A}\}$. Then $\tau_B = \inf \mathscr{A}$ and $\tau_C = \sup \mathscr{A}$. It follows that $\hat{G}_0$ is a complete lattice and by Theorem 9 [1] is compact in its interval topology.

We now follow closely the proof of 2.7 [14] and thus show that the interval topology and the Gelfand topology are identical in both $\hat{G}$ and $\hat{G}_0$. The proof is completed by noting that a subbase for the closed sets in the interval topology for $\hat{G}_0$ is the collection

$$\{\tau \in \hat{G}_0 \mid \tau \leqq \tau_A\}_{\tau_A \in \hat{G}_0} \text{ and } \{\tau \in \hat{G}_0 \mid \tau \geqq \tau_A\}_{\tau_A \in \hat{G}_0} . \tag{3.9.1}$$

From this it follows that the interval topology of $\hat{G}$ is the relativized interval topology of $\hat{G}_0$. By Theorems 3 and 4 [1] the former is equal to the product topology of $\hat{G} = \boldsymbol{P}_{k=1}^n \hat{G}_k$, and is therefore a locally compact T_2-topology (2.7 [14]), having $\hat{G}_0$ as its one-point compactification. The proof that $\hat{G}_0$ is totally disconnected uses again the nature of $\hat{G}$ as a product space and Ross' result 2.8 [14].

DEFINITION 3.10. We define a multiplication in $\hat{G}_0$ by the rule

$$\tau_A \tau_B = \inf \{\tau_A, \tau_B\} \tag{3.10.1}$$

using the lattice operation given in 3.9. This multiplication is natural in that $\tau_A \tau_B$ corresponds to the semi-character $\xi_A \cdot \xi_B = \xi_{A \cap B}$.

THEOREM 3.11. *$\hat{G}$ with the Gelfand topology and the multiplication defined in* (3.10.1) *is a locally compact idempotent commutative Hausdorff semigroup; in fact, $\hat{G}$ is homeomorphic and semigroup= isomorphic with $\boldsymbol{P}_{k=1}^n G_k$ (viewed as a lower semilattice with the product topology).*

Proof. In view of 3.9 we need only show here that multiplication in $\hat{G}$ is continuous. This follows from Theorems 2 and 3 [1], which assert that the lattice $\boldsymbol{P}_{k=1}^{n}\hat{G}_k$, being distributive, is a topological lattice.

3.12. Under the present hypothesis it can now be shown that the algebra $\mathscr{M}(G)$ has an identity element if and only if G has a least element. For if $\mathscr{M}(G)$ has an identity then $\Delta\mathscr{M}(G)$ is compact (19B [8]), and hence $\hat{G} = \boldsymbol{P}_{k=1}^{n}\hat{G}_k$ is compact in its product topology. If τ_{A_k} is the least element of $\hat{G}_k$, $k = 1, \cdots, n$, then A_k must consist of a single element α_k of G_k and the point $\alpha = (\alpha_k)_{k=1}^{n}$ is the least element of G. The converse has already been discussed in 1.8.

That we cannot make the same claim under the more general hypothesis of 1.2 is seen from the following simple example.

3.13. Let $G = \{a, b, c\}$, where $ab = ac = bc = c$, and let G be otherwise as in 1.2. Then $\mathscr{M}(G) = \{\alpha\delta_a + \beta\delta_b + \gamma\delta_c \mid \alpha, \beta, \gamma$ are arbitrary complex numbers$\}$, and the measure $\nu = \delta_a + \delta_b - \delta_c$ is an identity for $\mathscr{M}(G)$. However, G itself has no identity element.

4. The Herglotz-Bochner theorem for $\mathscr{M}(G)$.

4.1. This section generalizes § 4 [14]. Under the hypothesis of 3.1 we first introduce the concept of a function of finite variation of the n variables $\tau_1, \cdots, \tau_n$, defined on $\hat{G}$. τ_0 will denote the zero functional on $\mathscr{M}(G_k)$ as well as on $\mathscr{M}(G)$.

DEFINITION 4.2. $\mathscr{A}$ will denote the set of all subsets R of $\hat{G}$, called *rectangles*, which are of the form

$$R = \boldsymbol{P}_{k=1}^{n} I_k, \text{ where for each } k,\ I_k = (\tau_{A_k}, \tau_{B_k}],\ \tau_{A_k} < \tau_{B_k}, \qquad \text{or } I_k = (\tau_0, \tau_{B_k}] \,. \tag{4.2.1}$$

The points $\tau_A = (\tau_{A_k})_{k=1}^{n}$, $\tau_B = (\tau_{B_k})_{k=1}^{n}$ of $\hat{G}$ appearing in (4.2.1) are called the *endpoints* of R.

We formally adopt the notation used by Munroe in his discussion of Stieltjes' measures (pp. 120–125 [10]) and adapt it for our purposes.

DEFINITION 4.3. Let g be any complex-valued function defined on $\hat{G} = \boldsymbol{P}_{k=1}^{n}\hat{G}_k$, let $R \in \mathscr{A}$ have its endpoints τ_A and τ_B in $\hat{G}$, and let $1 \leqq k \leqq n$.

Considering g as a function of the k^{th} coordinate τ_k of $\tau \in \hat{G}$, the operator δ_k, depending on R, is defined by

$$\delta_k(g) = g(\cdots \tau_{B_k} \cdots) - g(\cdots \tau_{A_k} \cdots) \,. \tag{4.3.1}$$

If $R = \boldsymbol{P}_{k=1}^{n} I_k$ and if for some k, $I_k = (\tau_0, \tau_{B_k}]$, we set

$$(4.3.2) \qquad \delta_k(g) = g(\cdots \tau_{B_k} \cdots)$$

$\delta_k(g)$ is thus a function defined on $\underset{\sim}{P}_{j\neq k}\hat{G}_j$.

DEFINITION 4.4. Let $h \in \mathscr{C}(\hat{G}_0)$, the space of all complex-valued continuous functions on $\hat{G}_0$. Define the function δh on $\mathscr{A}$ as follows

$$(4.4.1) \qquad \delta h(R) = \delta_1(\delta_2(\cdots \delta_n(h) \cdots)), \text{ for } R \in \mathscr{A},$$

the δ_k having been defined for each R by 4.3.

DEFINITION 4.5. For a subset C of G set

(4.5.1) $\tilde{C} = \{\tau \in \hat{G} \mid \tau = \tau_{L_a} \text{ for some } a \in C\}$. For subset S of $\hat{G}$ set

$$(4.5.2) \qquad \check{S} = \{a \in G \mid \tau_{L_a} \in S\} .$$

Let $\mathscr{B}$ denote the set of all $R \in \mathscr{A}$ whose endpoints belong to $\tilde{G}$. i.e., all $R = P_k I_k$ such that for all k, $I_k = (\tau_{a_k}], \tau_{b_k}]]$ or $I_k = (\tau_0, \tau_{b_k}]]$.

DEFINITION 4.6. A finite pairwise disjoint subcollection Δ of $\mathscr{A}$ is called a *partition* of $\hat{G}$. $\mathscr{D}$ denotes the set of all partitions. For $\Delta_1, \Delta_2 \in \mathscr{D}$ we say that

(4.6.1) $\Delta_1 \geqq \Delta_2$ (Δ_1 is *finer* than Δ_2) in case

(i) $\bigcup \Delta_1 \supset \bigcup \Delta_2$ and
(ii) $R \in \Delta_1$, $S \in \Delta_2$ implies $R \cap S = \phi$ or $R \subset S$.

We also set

$$(4.6.2) \qquad \mathscr{E} = \{\Delta \in \mathscr{D} \mid \Delta \subset \mathscr{B}\} ,$$

$\mathscr{B}$ having been defined in 4.5. With the ordering given in 4.6, $\mathscr{D}$ and $\mathscr{E}$ are easily seen to be directed sets.

DEFINITION 4.7. Let $h \in \mathscr{C}(\hat{G}_0)$. If $A \in \mathscr{A}$, or if $\hat{G}\backslash A \in \mathscr{A}$, or if $A = \hat{G}$ we define

$$(4.7.1) \qquad V(h; A) = \sup\{\textstyle\sum_{R\in\Delta} |\delta h(R)| \mid \Delta \in \mathscr{D} \text{ and } \bigcup \Delta \subset A\} ,$$

and set $V(h) = V(h; \hat{G})$. The function h is said to be of *finite variation* if $V(h) < \infty$.

DEFINITION 4.8. Let $h \in \mathscr{C}(\hat{G}_0)$ and let A be as in 4.7. Define (4.8.1) $V_c(h; A) = \sup\{\sum_{R\in\Delta} V(h; R) \mid \Delta \in \mathscr{E}, \bigcup \Delta \subset A,$ and R has compact closure in G for all $R \in \Delta\}$, and set $V_c(h) = V_c(h; \hat{G})$.

DEFINITION 4.9. Let $h \in \mathscr{C}(\hat{G}_0)$ be such that $\delta h(R) \geqq 0$ for all

$R \in \mathscr{A}$, and of finite variation. Let f be any complex-valued function defined on G. For each $\Delta \in \mathscr{E}$ define

$$S(f, \Delta) = \sum_{R \in \Delta} f(b)\delta h(R) , \tag{4.9.1}$$

where τ_{L_b} is the upper endpoint of R, i.e., $b = \sup \check{R}$.

THEOREM 4.10. *Let $f \in \mathscr{C}_0(G)$. Let $h \in \mathscr{C}(\hat{G}_0)$ be of finite variation and such that $\delta h(R) \geqq 0$ for all $R \in \mathscr{A}$, and $h(\tau_0) = 0$. Then $\{S(f, \Delta)\}_{\Delta \in \mathscr{E}}$ is a Cauchy net of complex numbers. We will write*

$$L(f) = \lim_{\Delta \in \mathscr{E}} S(f, \Delta) . \tag{4.10.1}$$

The function L defined in (4.10.1) *is a bounded, nonnegative linear functional on $\mathscr{C}_0(G)$.*

We apply the Riesz Representation theorem (19.12 [4]) to the above functional and obtain, as an extension of 4.6 [14], a theorem which characterizes those continuous functions on $\hat{G}_0$ which are Gelfand transforms of measures in $\mathscr{M}(G)$.

THEOREM 4.11. *Let $h \in \mathscr{C}(\hat{G}_0)$ have finite variation and suppose $h(\tau_0) = 0$. Then there exists a measure $\mu \in \mathscr{M}(G)$ such that $h = \hat{\mu}$ if and only if $V(h) = V_c(h)$.*

COROLLARY 4.12. *Suppose G is compact and $h \in \mathscr{C}(\hat{G}_0)$. Then h is the Gelfand transform of some measure $\mu \in \mathscr{M}(G)$ if and only if $V(h) < \infty$ and $h(\tau_0) = 0$.*

EXAMPLE 4.13. Let $G = E^n$ with coordinatewise ordering and the usual topology. Let $h \in \mathscr{C}(\hat{G}_0)$. Then h is the Gelfand transform of some measure $\mu \in \mathscr{M}(G)$ if and only if $V(h) < \infty$ and $h(\tau_0) = 0$.

Ross gives an example (4.7 [14]) of a function $h \in \mathscr{C}(\hat{G}_0)$ such that $V(h) < \infty$ but $V(h) > V_c(h)$, and which is not a transform.

It is possible to show that the variation functions used by Ross (2.9 and 4.4 [14]) for linearly ordered G are consistent, in all cases concerning this paper, with those defined in 4.7 and 4.8, and that Theorem 4.6 [14] is actually a corollary to our Theorem 4.11.

5. Consequences of the Herglotz-Bochner theorem.

5.1. Let I be a closed ideal in $\mathscr{M}(G)$. Let $h(I) = \{\tau \in \hat{G}_0 \mid \hat{\mu}(\tau) = 0$ for all $\mu \in I\}$ be the *hull* of I and $kh(I) = \{\mu \in \mathscr{M}(G) \mid \hat{\mu}(\tau) = 0$ for all $\tau \in h(I)\}$ the *kernel* of $h(I)$. *Spectral synthesis* obtains in $\mathscr{M}(G)$ if

$I = kh(I)$ for each closed ideal I in $\mathscr{M}(G)$. We show here by example that if G is not linearly ordered (see 3.4 [14]) spectral synthesis may not obtain, even though G is compact. In this section G satisfies the hypotheses of 3.1.

EXAMPLE 5.2. Let $G = [0, 1] \times [0, 1]$ be the unit square with the usual topology and coordinatewise order. Let $L = \{(t, s) \in G \mid s = 1 - t\}$; let $K = \{(t, s) \mid s \leqq 1 - t\}$, and let $H = G \backslash K$. Let $Q = \{r_i\}_{i=1}^{\infty}$ be the set of points in H which have two rational coordinates. Let λ be Lebesgue measure along the line segment L and define

$$\mu = \sum_{i=1}^{\infty} (1/2^i)\delta_{r_i} \tag{5.2.1}$$

$$I = \{\mu * \nu \mid \nu \in \mathscr{M}(G)\} . \tag{5.2.2}$$

It is easy to see that $\tau \in h(I)$ if and only if $\hat{\mu}(\tau) = 0$. This leads to a very simple description of $h(I)$.

$$h(I) = \{\tau_A \in \hat{G}_0 \mid \overline{\overline{A \cap L}} \leqq 1\} , \tag{5.2.3}$$

as $\overline{\overline{A \cap L}} \leqq 1$ if and only if $A \cap Q = \phi$ if and only if $\hat{\mu}(\tau_A) = \mu(A) = 0$.

Since $\hat{\lambda}(\tau_A) = \lambda(A) = 0$ for all $\tau_A \in h(I), \lambda \in kh(I)$. However, $\lambda \notin I$ and hence $I \neq kh(I)$, for by (1.11.2) we have

$$\mu * \nu(K) = \sum_{i=1}^{\infty} (1/2^i)(\delta_{r_i} * \nu)(K) = \sum_{i=1}^{\infty} (1/2^i)\nu({}_{r_i}K) = 0 ,$$

since ${}_rK = \phi$ for all $r \in Q$. But $\lambda(H) = 0$ and so λ and $\mu * \nu$ are mutually singular for all $\nu \in \mathscr{M}(G)$. It follows that the distance from λ to I is at least as large as $1/2 \,\|\lambda\| = \sqrt{2}/2$ and that therefore λ is not in I.

THEOREM 5.3. *Let $G = \boldsymbol{P}_{k=1}^{n} G_k$ and let $\mu \in \mathscr{M}(G)$. Then μ is idempotent, i.e., $\mu * \mu = \mu$, if and only if μ is a discrete measure of the form*

$$\mu = \sum_{k=1}^{m} \alpha_k \delta_{t_k} , \tag{5.3.1}$$

where the coefficients α_k are nonzero integers between -2^{n-1} and 2^{n-1} and have the property that for each $x \in G$

$$\sum_{t_k \leqq x} \alpha_k = 0 \text{ or } 1 . \tag{5.3.2}$$

It is interesting to note that the support $T = \{t_k\}_{k=1}^{m}$ of the idempotent measure μ need not be a sub-semigroup of G.

THEOREM 5.4. *$\mathscr{M}(G)$ is a symmetric algebra; i.e., if $\mu \in \mathscr{M}(G)$ then there exists a measure $\nu \in \mathscr{M}(G)$ such that $\hat{\nu}(\tau) = \overline{\hat{\mu}(\tau)}$ for all $\tau \in \hat{G}$. Here $\bar{z}$ denotes the complex conjugate of z.*

EXAMPLE 5.5. In contrast to the linearly ordered case (5.5 [14]) there exist positive measures in $\mathscr{M}(G)$ which have no square root in $\mathscr{M}(G)$. Let G be the unit square as in 5.2. For $n = 1, 2, \cdots$ let $x_n = (1/n, (n-1)/n) \in G$ and define

$$\mu = \sum_{k=1}^{\infty} (\delta_{x_k})/k^2 \tag{5.5.1}$$

Assume there exists a measure $\nu \in \mathscr{M}(G)$ such that $(\hat{\nu}(\tau))^2 = \hat{\mu}(\tau)$ for all $\tau \in \hat{G}_0$, then for each integer N we can find pairwise disjoint rectangles $R_1, \cdots, R_n \in \mathscr{A}$ such that

$$\delta\hat{\nu}(R_j) = \pm\sqrt{\hat{\mu}(\tau_{L_{x_j}})} = \pm(1/j) \, .$$

Hence $V(\hat{\nu}) \geqq \sum_{j=1}^{N} |\delta\hat{\nu}(R_j)| = \sum_{j=1}^{N} (1/j)$, for all N, contradiction 4.11. We conclude that $\mu \neq \nu * \nu$ for all $\nu \in \mathscr{M}(G)$.

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