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ITERATES OF BERNSTEIN POLYNOMIALS

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$B_n(f)$ transforms each function defined on $[0, 1]$ into its Bernstein polynomial of degree n. In this paper we study the convergence of the iterates $B_n^{(k)}(f)$ as $k \to \infty$ both in the case that k is independent of n and (for polynomial f) when k is a function of n.

To each $f(x)$ defined on I: $0 \leqq x \leqq 1$ there is associated its Bernstein polynomial of degree n defined by

$$B_n(f; x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k(1 - x)^{n-k} . \tag{1.1}$$

It is well known that if f is continuous on I, then

$$\lim_{n \to \infty} B_n(f; x) = f(x) \tag{1.2}$$

uniformly on I. (Cf., Lorentz [2] for this and other properties of the Bernstein polynomials used here.) Let $B_n(f)$ denote the (polynomial) function defined by (1.1), then for $k > 1$, $B_n^{(k)}(f; x) = B_n(B_n^{(k-1)}(f); x)$ defines, by mathematical induction, a sequence of iterates of the Bernstein polynomials. Our purpose is to study the convergence behavior of this sequence as $k \to \infty$, both in the case that k is independent of n and when it is a nonconstant function of n.

We show in § 2 that $B_n^{(k)}(f; x)$ converges (uniformly) for fixed n, to the line segment joining $(0, f(0))$ to $(1, f(1))$, and in § 3 that the sequence $B_n^{(g(n))}(x^s; x)$ with appropriate assumptions on $g(n)$, also converges, for each $s = 0, 1, 2, \cdots$ to a polynomial of degree s whose coefficients we determine explicitly. Finally, in § 4 arbitrary iterates are defined as a natural generalization of the positive integral iterates.

When (1.1) is rewritten in conventional polynomial form, it becomes

$$\begin{aligned} B_n(f; x) &= \sum_{q=0}^{n} \left\{\binom{n}{q} \sum_{k=0}^{q} f\left(\frac{k}{n}\right)\binom{q}{k}(-1)^{q-k}\right\}x^q \\ &= \sum_{q=0}^{n} \Delta_{1/n}^{q} f(0)\binom{n}{q}x^q \end{aligned} \tag{1.3}$$

which reveals that if f is a polynomial of degree m, then $B_n(f)$ is a polynomial whose degree is at most $\min(m, n)$. Let s be a fixed positive integer satisfying $s \leqq n$. (There is no loss of generality in this restriction on s for $k > 1$, since for $s > n$, $B_n^{(k)}(x^s) = B_n^{(k-1)}(B_n(x^s))$ and $B_n(x^s)$ is of degree at most n.) We consider $f(x) = x^j, j = 1, \cdots, s$. (1.3) implies that

$$(1.4)\qquad B_n(x^j) = a_{1j}x + a_{2j}x^2 + \cdots + a_{jj}x^j = \sum_{q=1}^{j} \pi_q \sigma_j^q \frac{1}{n^{j-q}} x^q ,\qquad j = 1, \cdots, s ,$$

where σ_j^q are the Stirling numbers of the second kind (Cf., Jordan [1, pp. 168-173]) defined by

$$(1.5)\qquad \sigma_j^q = \frac{(-1)^q}{q!} \sum_{k=1}^{q} k^j \binom{q}{k} (-1)^k ,$$

and

$$(1.6)\qquad \begin{cases} \pi_q = \left(1 - \dfrac{1}{n}\right)\left(1 - \dfrac{2}{n}\right) \cdots \left(1 - \dfrac{q-1}{n}\right), & q = 2, \cdots, s \\ \pi_1 = 1 . \end{cases}$$

2. **Limit of the iterates.** The study of the iterates of $B_n(f; x)$ for $f(x) = x^s$ is considerably simplified if we use the language of linear algebra. There is no loss of generality in this choice of $f(x)$ since B_n replaces f by a polynomial.

Let A denote the $s \times s$ upper triangular matrix whose entries a_{ij} are defined in (1.4), i.e.,

$$(2.1)\qquad a_{ij} = \begin{cases} \pi_i \sigma_j^i n^{i-j} , & i \leqq j \\ 0 & i > j . \end{cases}$$

Let e_s be the column vector of s components, the first $s - 1$ components being zero and the last one. Then

LEMMA 1. *If* $A^k e_s = (\alpha_1^{(k)}, \cdots, \alpha_s^{(k)})^T$, *then*

$$(2.2)\qquad B_n^{(k)}(x^s) = \alpha_1^{(k)}x + \alpha_2^{(k)}x^2 + \cdots + \alpha_s^{(k)}x^s , \qquad k = 1, 2, \cdots .$$

Proof. If $p(x) = c_1x + c_2x^2 + \cdots + c_sx^s$ (for example, $p(x) = B_n^{(j)}(x^s)$) and

$$B_n(p) = d_1x + d_2x^2 + \cdots + d_sx^s = \sum_{j=1}^{s} c_j(a_{1j}x + \cdots + a_{sj}x^s)$$
$$= \sum_{l=1}^{s} \sum_{j=1}^{s} c_j a_{lj} x^l ,$$

then $(d_1, \cdots, d_s)^T = A(c_1, \cdots, c_s)^T$. The lemma now follows by mathematical induction on k.

LEMMA 2. *The eigenvalues of A are* $\pi_1, \pi_2, \cdots, \pi_s$.

Proof. $a_{ii} = \pi_i$, $i = 1, \cdots, s$, and $a_{ij} = 0$ if $i > j$.

Let Λ denote the $s \times s$ matrix with the eigenvalues of $A, \pi_1, \cdots, \pi_s$ on the main diagonal and zeros everywhere else. Let V denote the matrix of eigenvectors of A, normalized so that the entries on its main diagonal are all 1. V is upper triangular and its entries are, in general, functions of n. Since $AV = V\Lambda$ we conclude that

$$A^k = V\Lambda^k V^{-1} . \tag{2.3}$$

Essentially, the following arguments rest on the observation that Λ^k is known to us and V and its inverse are independent of k.

LEMMA 3. *If* $V^{-1} = (\bar{v}_{ij})$ *then* $\bar{v}_{1j} = 1, j = 1, \cdots, s$.

Proof. Let U be the eigenmatrix of A^T, i.e.,

$$A^T U = U\Lambda .$$

Let U (which is lower triangular) be normalized so that the entries on its main diagonal are all 1. Since $B_n(x^j; 1) = 1$ the column sums of A are all 1 and hence the row sums of A^T are all 1. The first column of U is the eigenvector associated with the eigenvalue $\pi_1 = 1$, and hence consists of all entries 1. Due to the way we have normalized V and U we know that $U^T = V^{-1}$ and the lemma is proved.

LEMMA 4. *If n is fixed*

$$\lim_{k\to\infty} A^k e_s = (1, 0, 0, \cdots, 0)^T .$$

Proof. The entries on the main diagonal of Λ^k are $\pi_1^k, \cdots, \pi_s^k$ and

$$\lim_{k\to\infty} \pi_j^k = 0 , \qquad j = 2, \cdots, s$$
$$\lim_{k\to\infty} \pi_1^k = 1 .$$

Thus, as $k \to \infty$, $V\Lambda^k V^{-1}$ approaches a matrix whose first row consists of all 1's, by Lemma 3, and the rest of whose elements are all 0. Clearly,

$$(1, 0, 0, \cdots, 0)^T = \left(\lim_{k\to\infty} A^k\right) e_s = \lim_{k\to\infty} (A^k e_s) .$$

THEOREM 1. *If n is fixed then*

$$\lim_{j\to\infty} B_n^{(j)}(f; x) = f(0) + (f(1) - f(0))x , \qquad 0 \leqq x \leqq 1 . \tag{2.4}$$

Proof. Let $B_n(f; x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n$, then

$$B_n^{(j)}(f; x) = \alpha_0 + \alpha_1 B_n^{(j-1)}(x; x) + \alpha_2 B_n^{(j-1)}(x^2; x) + \cdots + \alpha_n B_n^{(j-1)}(x^n; x) ;$$

hence, in view of Lemma 1 and Lemma 4, with $s = 1, 2, \cdots, n$,

$$\lim_{j\to\infty} B_n^{(j)}(f; x) = \alpha_0 + (\alpha_1 + \cdots + \alpha_n)x$$
$$= f(0) + (f(1) - f(0))x .$$

REMARK. The convergence in (2.4) is uniform since we have a sequence of polynomials of fixed degree approaching a fixed polynomial of the same degree for all x on a bounded interval. Also we have used the obvious fact that $B_n(1) = 1$, all n.

It is a curious fact that the matrix V has the property that v_{ij} is independent of n, for $j = 1, 2, 3$. We have, when $s = 3$,

$$V = \begin{pmatrix} 1 & -1 & 1/2 \\ 0 & 1 & -3/2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $p_2(x) = -x + x^2$ and $p_3(x) = (1/2)x - (3/2)x^2 + x^3$, then we conclude that,

$$B_n^{(j)}(p_2) = \left(1 - \frac{1}{n}\right)^j p_2 , \qquad j = 0, 1, 2, \cdots$$
$$B_n^{(j)}(p_3) = \left[\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\right]^j p_3 .$$

These results should be contrasted to the well-known remark (Cf., Schoenberg [3]) that the Bernstein operators are "poor reproducers", in that they never reproduce polynomials of degree greater than 1.

3. Limit of the coupled iterates. Suppose $f(x) = x^s$. Theorem 1 tells us that for fixed n, $B_n^{(j)}(x^s) \to x$ as $j \to \infty$, while according to (1.2), $B_n(x^s) \to x^s$ as $n \to \infty$. Thus, it is of interest to "play-off" the upper and lower subscripts in $B_n^{(j)}(x^s)$, by considering $j = g(n)$. To this end we must examine the behavior of the eigenmatrix, V, as $n \to \infty$.

Let the elements of V be $v_{ij}(=v_{ij}(n))$. For $j = 1, \cdots, s$ we have

$$A(v_{1j}, \cdots, v_{sj})^T = \pi_j(v_{1j}, \cdots, v_{sj})^T . \tag{3.1}$$

We examine these linear equations more closely. Since V is upper triangular,

$$v_{ij} = 0 , \qquad i = j + 1, \cdots, s , \tag{3.2}$$

and because of the way we have normalized V

$$v_{jj} = 1 . \tag{3.3}$$

It remains, then, to determine the behavior of $v_{ij}(n)$, $i < j$, as $n \to \infty$.

We consider the relevant linear equations from (3.1) (and write v_i in place of v_{ij} for simplicity)

$$\begin{aligned} a_{j-1,j-1}v_{j-1} + a_{j-1,j} &= \pi_j v_{j-1} \\ a_{j-2,j-2}v_{j-2} + a_{j-2,j-1}v_{j-1} + a_{j-2,j} &= \pi_j v_{j-2} \\ &\vdots \\ a_{11}v_1 + a_{12}v_2 + \cdots + a_{1,j-1}v_{j-1} + a_{1,j} &= \pi_j v_1 . \end{aligned} \tag{3.4}$$

Define $\pi_{ij} = \pi_i - \pi_j$, let P denote the determinant $|p_{ij}|$ such that

$$p_{ij} = \begin{cases} a_{ij} & i < j \\ \pi_{ij} & i = j , \\ 0 & i > j \end{cases}$$

then

$$P = \prod_{k=1}^{j-1} \pi_{kj} .$$

Let $P^{(i)}$ denote the determinant identical to P except that the i-th column of P is replaced by $(-a_{1j}, -a_{2j}, \cdots, -a_{j-1,j})$. Then, if we solve (3.4) for $v_i(=v_{i,j})$ by Cramer's rule, we obtain

$$v_i = \frac{P^{(i)}}{P} . \tag{3.5}$$

If we denote by $P_{pj}^{(i)}$ the minor of $-a_{pj}$ in $P^{(i)}$, then $P_{pj}^{(i)}$ is upper triangular and

$$(-1)^{i+p}P_{pj}^{(i)} = \begin{cases} 0 & p < i \\ P/\pi_{ij} & p = i \\ a_{i,i+1}a_{i+1,i+2} \cdots a_{p-1,p}P\Big/\prod_{k=i}^{p} \pi_{kj} & p > i . \end{cases}$$

Now,

$$(-1)^{i+p+1}a_{pj}P_{pj}^{(i)}/P = \begin{cases} -a_{ij}/\pi_{ij} & p = i \\ \dfrac{(-)^{i+p+1}a_{pj}a_{i,i+1} \cdots a_{p-1,p}}{\prod_{k=i}^{p} \pi_{kj}} & p > i , \end{cases} \tag{3.6}$$

and for $q < j$,

$$\begin{aligned} \pi_{qj} &= \pi_q\Big[1 - (1 - q/n) \cdots \Big(1 - \frac{j-1}{n}\Big)\Big] \\ &= \pi_q\Big\{\frac{1}{n}[q + (q+1) + \cdots + (j-1)] + O(n^{-2})\Big\} \end{aligned} \tag{3.7}$$

as $n \to \infty$. Since $\pi_i \to 1$ as $n \to \infty$, we obtain, in view of (3.6), (3.7), and (2.1),

$$\lim_{n\to\infty} \frac{a_{pj}P_{pj}^{(i)}}{P} = 0 , \qquad p < j - 1 ,$$

while

$$\lim_{n\to\infty} \frac{a_{j-1,j}P_{j-1,j}^{(i)}}{P} = \left\{\prod_{t=i}^{j-1} \left(\frac{j-t}{2}\right)(j+t-1)\right\}^{-1} \sigma_{t+1}^{t} .$$

Thus, we obtain, finally, that

$$\text{(3.8)} \quad \lim_{n\to\infty} v_{ij} = v_{ij}^* = (-1)^{j+i}2^{j-i} \frac{\prod_{t=i}^{j-1}\binom{t+1}{2}}{[(j-i)!]^2\binom{2j-2}{j-1}} , \quad i = 1, \cdots, j-1 .$$

where we have used the fact that (Cf., Jordan [1])

$$\sigma_{t+1}^{t} = \binom{t+1}{2} .$$

(3.2), (3.3), and (3.8) give the limit of V as $n \to \infty$. In an entirely analogous fashion, with A^T in place of A, we may obtain the limit of V^{-1} as $n \to \infty$. We suppress the details, but the result is

$$\text{(3.9)} \qquad \lim_{n\to\infty} \bar{v}_{ij} = \bar{v}_{ij}^* = \begin{cases} 0 , & i > j \\ 1 , & i = j \\ 2^{j-i} \dfrac{\prod_{t=i}^{j-1}\binom{t+1}{2}}{[(j-i)!]^2\binom{i+j-1}{j-i}} , & i < j . \end{cases}$$

Let us put

$$\text{(3.10)} \qquad E_j = \exp\left[-\binom{j}{2}\right] = \lim_{n\to\infty} \pi_j^n .$$

THEOREM 2. *Suppose $g(n)$ is a nonnegative integer for each n, and*

$$\text{(3.11)} \qquad \lim_{n\to\infty} \frac{g(n)}{n} = \alpha ,$$

then we have

$$\lim_{n\to\infty} B_n^{(g(n))}(x^s) = \sum_{i=1}^{s} b_i x^i \tag{3.12}$$

where

$$b_i = \frac{i}{s}\binom{s}{i}^2 \sum_{j=i}^{s} \frac{(-1)^{j+i}\binom{s-i}{j-i}^2}{\binom{2j-2}{j-i}\binom{j+s-1}{s-j}} E_j^\alpha\ , \tag{3.13}$$

$i = 1, \cdots, s$ (where, when $\alpha = \infty$ in (3.11), we have $E_1^\alpha = 1$ and $E_j^\alpha = 0, j > 1$ in (3.13)).

Proof. $A^{g(n)} = V\Lambda^{g(n)}V^{-1}$. Now

$$\lim_{n\to\infty} \Lambda^{g(n)} = \Lambda^*$$

where Λ^* is a diagonal matrix with entries $E_j^\alpha, j = 1, \cdots, s$ on its main diagonal.
Let

$$\lim_{n\to\infty} V = V^*$$

and

$$\lim_{n\to\infty} V^{-1} = (V^{-1})^* = (V^*)^{-1}\ .$$

The entries in V^* and $(V^*)^{-1}$ are given by (3.2), (3.3), (3.8), and (3.9). Thus, we may conclude that

$$V^*\Lambda^*(V^*)^{-1}e_s = \left(\lim_{n\to\infty} A^{g(n)}\right)e_s = \lim_{n\to\infty}(A^{g(n)}e_s)$$

and the existence of the limit in (3.12) is established. In order to verify (3.13), we need only note that

$$(b_1, \cdots, b_s)^T = V^*\Lambda^*(V^*)^{-1}e_s\ , \tag{3.14}$$

so that

$$b_i = \sum_{j=1}^{s} v_{ij}^* \bar{v}_{js}^* E_j^\alpha\ , \qquad i = 1, \cdots, s\ . \tag{3.15}$$

REMARK. If $\alpha = 0$, then $\Lambda^* = I$ and we conclude from (3.14) that $(b_1, \cdots, b_s)^T = e_s$, or $b_j = 0, j = 1, \cdots, s-1, b_s = 1$. In particular, then, if $g(n) \equiv 0$, we have proved (1.2) for the case $f(x) = x^s$. As a curiosity we also note that we have established the seemingly nontrivial identities

$$\sum_{j=i}^{s} \frac{(-1)^{j+i}\binom{s-i}{j-i}^2}{\binom{2j-2}{j-i}\binom{j+s-1}{s-j}} = 0 , \qquad i = 1, \cdots, s-1 . \tag{3.16}$$

With some simplification (3.16) may be written in the equivalent form (3.17) which holds for odd t and n positive

$$\sum_{k=0}^{n} (-1)^k \binom{t+k}{t}\binom{2n+t}{n-k} \frac{2k+t}{k+t} = 0 . \tag{3.17}$$

Additionally, since

$$\sum_{i=1}^{s} a_{ij} = 1 , \qquad j = 1, \cdots, s$$

and

$$\sum_{j=1}^{s} a_{ij} v_{jk} = \pi_k v_{ik}, \qquad i = 1, \cdots, s ; \quad k = 1, \cdots, s ,$$

we obtain, after summing on i on both sides of (3.18) and interchanging the order of summation on the left

$$\sum_{j=1}^{s} v_{jk} = \pi_k \sum_{i=1}^{s} v_{ik} ,$$

from which we conclude that, if δ_{1k} is a Kronecker delta.

$$\sum_{i=1}^{s} v_{ik} = \delta_{1k}$$

and hence also

$$\sum_{i=1}^{s} v_{ik}^* = \delta_{1k} .$$

We thus have the seemingly nontrivial identities:

$$1 + \sum_{i=1}^{j-1} (-1)^{j+i} 2^{j-i} \frac{\prod_{t=i}^{j-1} \binom{t+1}{2}}{[(j-i)!]^2 \binom{2j-2}{j-i}} = 0 , \qquad j = 2, \cdots, s , \tag{3.19}$$

or, equivalently, if $n \geqq 1$,

$$\sum_{k=0}^{n} (-1)^k \binom{n+k}{k}\binom{n}{k} \frac{1}{k+1} = 0 . \tag{3.20}$$

4. Iterates of all orders. If t is any real number, $-\infty < t < \infty$, we are now in a position to define $B_n^{(t)}(f)$, in a manner consistent with our definition when t is a nonnegative integer. We define

$$(4.1)\quad B_n^{(t)}(x^k) = b_1(t)x + b_2(t)x^2 + \cdots + b_k(t)x^k\,, \qquad k = 1, 2, \cdots,$$

where

$$(4.2)\qquad (b_1(t), \cdots, b_k(t))^T = V\Lambda^t V^{-1}e_k\,.$$

In (4.2), Λ^t is defined to be the diagonal $k \times k$ matrix whose entries on the main diagonal are $\pi_1^t, \pi_2^t, \cdots, \pi_k^t$. It now follows that, since $e_1, \cdots, e_s$ is a basis in $E^s(s \leqq n)$, if

$$(4.3)\qquad p = \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_s x^s\,,$$

then

$$(4.4)\qquad B_n^{(t)}(p) = \sum_{i=1}^{s} \alpha_i B_n^{(t)}(x^i)\,.$$

Moreover, if we define

$$(4.5)\qquad B_n^{(t)}(c) = c$$

and

$$(4.6)\qquad B_n^{(t)}(c + p) = c + B_n^{(t)}(p)$$

where c is a constant and p is given by (4.3), then we obtain

$$(4.7)\qquad B_n^{(t)}(p) = \sum_{i=0}^{s} \alpha_i B_n^{(t)}(x^i)$$

when

$$p = \alpha_0 + \alpha_1 x + \cdots + \alpha_s x^s\,.$$

We observe further that if $-\infty < u < \infty$, then

$$\Lambda^{u+t} = \Lambda^u \Lambda^t$$

and so it is easy to see that

$$B_n^{(t+u)}(x^k) = B_n^{(t)}(B_n^{(u)}(x^k)) = B_n^{(u)}(B_n^{(t)}(x^k))\,,$$

and hence

$$B_n^{(t+u)}(p) = B_n^{(t)}(B_n^{(u)}(p)) = B_n^{(u)}(B_n^{(t)}(p))$$

for any polynomial p of degree at most n.

If f is bounded on $[0, 1]$, we can now define

$$(4.8)\qquad B_n^{(t)}(f) = B_n^{t-1}(B_n(f))\,.$$

This definition focuses attention on the case $t = 0$. The polynomial of degree at most n

$$B_n^*(f) = B_n^{(0)}(f) = B_n^{-1}(B_n f)$$

is a kind of surrogate f. How is this polynomial related to f? It is clear that if $f = p$, a polynomial of degree at most n, then

$$B_n^* p = p \ .$$

In particular, let $p = L_n(f)$ be the unique polynomial of degree at most n which agrees with $f(x)$ at $x = j/n, j = 0, \cdots, n$. Then $B_n(f) = B_n(L_n(f))$ and so

$$B_n^*(f) = B_n^*(L_n(f)) = L_n(f) \ .$$

Of course, this result could have been obtained without the apparatus of this paper, but it comes out of our discussion quite naturally.

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