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NONCONSTANT LOCALLY RECURRENT FUNCTIONS

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The purpose of this paper is to develop a new method of using the Baire Category Theorem to obtain counterexamples in analysis. The method is used to show that a certain class of nonconstant locally recurrent functions is of second category in a suitable metric space of continuous functions. In § 1 an explicit example is given of a nonconstant locally recurrent function. This example is included because it clarifies the category argument in § 4.

1. A simple example of a nonconstant locally recurrent function.

DEFINITION 1. A real-valued continuous function f of a real variable is said to be locally recurrent if for any x in its domain of definition and any neighborhood N of x, there exists $y \neq x$ in N such that $f(x) = f(y)$.

K. A. Bush [2] has given an example of a nonconstant locally recurrent function. The author believes that the example given below is simpler.

A sequence $\{f_n\}_{n=0}^{\infty}$ of functions on $[0, 1]$ will be defined. These functions are continuous and piecewise linear. Further, f_m is linear in any interval of the form $[n/9^m, (n+1)/9^m]$ where n and m are non-negative integers such that $0 \leqq n < 9^m$. Thus the function f_m is described completely if we give the values of $f_m(n/9^m)$, $(0 \leqq n \leqq 9^m)$. These functions will be defined inductively. We start with $f_0(x) = x$. Now suppose f_k is defined for some k. We define f_{k+1} as follows:

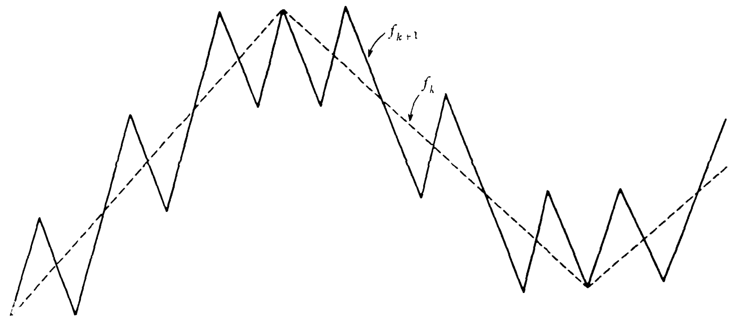
(a) $f_{k+1}(3m/9^{k+1}) = f_k(3m/9^{k+1})$, $0 \leqq 3m \leqq 9^{k+1}$.

(b) $f_{k+1}((3m+1)/9^{k+1}) = f_k((3m+3)/9^{k+1})$, $0 \leqq 3m+1, 3m+3 \leqq 9^{k+1}$.

(c) $f_{k+1}((3m+2)/9^{k+1}) = f_k(3m/9^{k+1})$, $0 \leqq 3m, 3m+2 \leqq 9^{k+1}$.

The figure shows a portion of the graphs of f_k and f_{k+1}.

An important feature of these functions is the relation $f_k(n/9^k) = f_l(n/9^k)$ for $l \geqq k$ and $0 \leqq n \leqq 9^k$. Also notice that on any interval of the form $[n/9^k, (n+1)/9^k]$, the values of f_m, $m \geqq k$ must lie between $f_k(n/9^k)$ and $f_k((n+1)/9^k)$. It is not hard to see that the f_n converge uniformly and thus the limit function $f(x) = \lim_{n\to\infty} f_n(x)$ is continuous. It is obviously locally recurrent at points of the form $n/9^k$. That f is locally recurrent at any point x in $[0, 1]$ follows from an application of the intermediate value theorem for continuous functions. In fact, for any $k \geqq 0$, x must lie in an interval of the form $I = [3m/9^k,$



FIGURE

$(3m+3)/9^k]$, an interval of length $3/9^k$. We have $f((3m+1)/9^k) = f((3m+3)/9^k) = m_1$ and $f((3m+2)/9^k) = f(3m/9^k) = m_2$. Also, either $\sup_{x \in I} f(x) = m_1$ and $\inf_{x \in I} f(x) = m_2$, or $\sup_{x \in I} f(x) = m_2$ and $\inf_{x \in I} f(x) = m_1$. The intermediate value theorem shows that every value of the function in I, except m_1 and m_2, must occur at least three times in I. Of course, m_1 and m_2 occur at least twice.

2. A class of complete metric spaces. In this section certain abstract tools will be developed for the purpose of showing the existence of various functions with pathological properties. Using these tools one may show that the set of functions with a certain pathology, for example, the pathology of the function of the previous section, is of the second category in a suitable complete metric space. An early result of this type was obtained by Banach [1] and Mazurkiewicz [3] who showed that the nowhere differentiable functions are of second category in the space of continuous functions with the uniform metric. The space and the metric considered here will be different.

Let $\mathscr{F}$ be the set of all functions which map a set T into a set S. We require that T is the union of nonempty, disjoint subsets T_i, $i = 1, 2, \cdots$. We provide $\mathscr{F}$ with a metric d as follows. If f and g belong to $\mathscr{F}$, we define

$$d(f, g) = 1/k$$

where k is the smallest integer such that $f(x) \neq g(x)$ for some x in T_k. If $f(x) = g(x)$ for all $x \in T$, we put $d(f, g) = 0$. It is easy to see that d is a metric.

Let $\mathscr{G}$ be the set of functions which, for some positive integer k, maps $\bigcup_{j=1}^{k} T_j$ into S. Further, let us put $\mathscr{F}^* = \mathscr{F} \cup \mathscr{G}$.

DEFINITION 2. A subset $\mathscr{H}$ of $\mathscr{F}^*$ is said to be *hereditary* if f belongs to $\mathscr{H}$ if and only if every restriction of f which is in $\mathscr{F}^*$ also belongs to $\mathscr{H}$. (A function g is a restriction of f if the domain of definition of g is a *proper* subset of the domain of definition of f and if $f(x)=g(x)$ for any x in the domain of definition of g.)

DEFINITION 3. A subset $\mathscr{A}$ of $\mathscr{H}(\mathscr{H} \subset \mathscr{F}^*)$ is said to be *absorbing* with respect to $\mathscr{H}$ if the following two conditions hold.

1. If $g \in \mathscr{A}$ and g is a restriction of $f \in \mathscr{H}$, then $f \in \mathscr{A}$.
2. For any $g \in \mathscr{G} \cap \mathscr{H}$, there exists f in $\mathscr{G} \cap \mathscr{A}$ such that g is a restriction of f.

THEOREM 1. *Let $\mathscr{H}$ be a hereditary subset of $\mathscr{F}^*$. Then $\mathscr{H} \cap \mathscr{F}$ is complete with respect to the metric d.*

Proof. Let $\{f_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $\mathscr{H} \cap \mathscr{F}$. Then $f_n(x)$ is constant for all sufficiently large values of n. This constant value will be denoted $f(x)$. Thus we have $\lim_{n\to\infty} f_n(x) = f(x)$ for all x in T. It remains to show $f \in \mathscr{H}$. In fact, any restriction g of $f\,(g \in \mathscr{F}^*)$ is also the restriction of some f_n; and since each f_n belongs to $\mathscr{H}$ and since $\mathscr{H}$ is hereditary, we have that g belongs to $\mathscr{H}$. Again, since $\mathscr{H}$ is hereditary, and since an arbitrary restriction in $\mathscr{F}^*$ of f is in $\mathscr{H}$, it follows that f belongs to $\mathscr{H}$.

THEOREM 2. *Let $\mathscr{A}$ be absorbing with respect to a hereditary set $\mathscr{H}$. Then $(\mathscr{H} - \mathscr{A}) \cap \mathscr{F}$ is nowhere dense in $\mathscr{H} \cap \mathscr{F}$ with respect to the metric d.*

Proof. Let f be an arbitrary element of $\mathscr{H} \cap \mathscr{F}$. We shall show that, given any positive integer k, there exists g in $\mathscr{A} \cap \mathscr{F}$ and a positive integer l such that

$$(1) \qquad \{h \in \mathscr{H} \cap \mathscr{F} : d(h, f) < 1/k\} \supset \{h \in \mathscr{H} \cap \mathscr{F} : d(h, g) < 1/l\}$$

and

$$(2) \qquad \{h \in \mathscr{H} \cap \mathscr{F} : d(h, g) < 1/l\} \subset \mathscr{A} \cap \mathscr{F}.$$

This will show that $(\mathscr{H} - \mathscr{A}) \cap \mathscr{F}$ is nowhere dense in $\mathscr{H} \cap \mathscr{F}$.

In fact, let f_k be the restriction of f to the set $\bigcup_{j=1}^{k-1} T_j$. Because $\mathscr{A}$ is absorbing, there exists h_l in $\mathscr{G} \cap \mathscr{A}$ such that f_k is a restriction of h_l. Let the domain of definition of h_l be $\bigcup_{j=1}^{l} T_j$. Because $\mathscr{A}$ is absorbing, h_l can be extended to h_{l+1} on $\bigcap_{j=1}^{l+1} T_j (h_{l+1} \in \mathscr{A})$, and we proceed inductively to define the successive extensions $h_k \in \mathscr{A}$ $k = l, l+1, \cdots$. The totality of these extensions defines a function g defined on all of T. Since $h_k \in \mathscr{A} \subset \mathscr{H}$ and $\mathscr{H}$ is hereditary, g belongs to $\mathscr{H}$, and therefore also to $\mathscr{A}$ since $\mathscr{A}$ is absorbing. The function g and the integer l have the properties required in (1) and (2).

3. A lemma uniformly continuous functions. The following well-known result is needed later. The proof is routine and will be omitted.

LEMMA 1. *Suppose that the real function f is defined on a set S of real numbers and is uniformly continuous there. Then f can be extended in exactly one way to a uniformly continuous function on the closure of S.*

4. Application of methods of § 2 to nonconstant locally recurrent functions. Let T_i consist of all rational numbers of the form $n/9^{i-1}$ where n and i are nonnegative integers such that $0 \leqq n \leqq 9^{i-1}$, $i > 0$, and n is not divisible by 9. Let $T = \bigcup_{i=0}^{\infty} T_i$. Let S be the set of all real numbers, and $\mathscr{F}$ the set of real valued functions on T. As in § 2, $\mathscr{G}$ consists of all real functions defined on sets $\bigcup_{i=0}^{N} T_i$, and $\mathscr{F}^* = \mathscr{F} \cup \mathscr{G}$.

Let $\mathscr{H}$ consist of elements f of $\mathscr{F}^*$ which satisfy the following conditions:

Condition 1. Suppose the domain of definition of f contains $r/3^k$ and $(r+1)/3^k$. Let

$$m = \min\,(f(r/3^k), f((r+1)/3^k)) \tag{3}$$

and

$$M = \max\,(f(r/3^k), f((r+1)/3^k))\,. \tag{4}$$

Then $M \neq m$ and for all x in the domain of definition of f such that $r/3^k < x < (r+1)/3^k$ we have $m \leqq f(x) \leqq M$.

Condition 2. If the domain of definition of f contains $r/3^k$ and $(r+1)/3^k$, then

$$|\,f(r/3^k) - f((r+1)/3^k)\,| \leqq 3^{k/2}\,.$$

Let $\mathscr{A}_n$ $(n = 0, 1, \cdots)$ be the subset of $\mathscr{H}$ consisting of functions f which satisfy, in addition to the foregoing two conditions, the following:

Condition 3. There exists $i \geqq n$ such that T_i belongs to the domain of definition of f, and for any r such that $0 \leqq 3r < 9^{i-1}$ we have

$$\text{a.}\quad f(3r/9^{i-1}) = f((3r+2)/9^{i-1}) \tag{6}$$

$$\text{b.}\quad f((3r+1)/9^{i-1}) = f((3r+3)/9^{i-1})\,. \tag{7}$$

THEOREM 3. *$\mathscr{H}$ is hereditary, and for each n $(n = 0, 1, \cdots)$ $\mathscr{A}_n$ is absorbing.*

Proof. It is easy to verify that $\mathscr{H}$ is hereditary. The essential thing is that Conditions 1 and 2 involve only universal quantifiers, and no existential quantifiers. If there Conditions hold for all points in the domain of f, and if g is a restriction of f, then they also hold, *a fortiori*, for all points in the domain of g, because the latter set is a subset of the former.

Now we shall show that $\mathscr{A}_n$ is absorbing. It is clear that part 1 of Definition 3 is satisfied. To show part 2, suppose $g \in \mathscr{G} \cap \mathscr{H}$. Let the domain of definition of g be $\bigcup_{i=1}^{N} T_i = R_N$. Let us define f on $\bigcup_{i=1}^{N+1} T_i$ so that if $x \in R_N$, then $f(x) = g(x)$.

We define f at points of the form $(9r + 3)/9^N$ and $(9r + 6)/9^N$ by linear interpolation, i.e.,

$$f((9r + 3)/9^N) = \tfrac{2}{3} f(r/9^{N-1}) + \tfrac{1}{3} f((r + 1)/9^{N-1})$$
$$f((9r + 6)/9^N) = \tfrac{1}{3} f(r/9^{N-1}) + \tfrac{2}{3} f((r + 1)/9^{N-1}) .$$

On the remaining points of T_{N+1}, f is now uniquely determined by imposing requirements a and b above for $\mathscr{A}_n$ with n replaced by $N + 1$:

$$f(3r/9^N) = f((3r + 2)/9^N)$$
$$f((3r + 1)/9^N) = f((3r + 3)/9^N) .$$

The author hopes that the geometry of this construction is made clear by the figure. The example of §1 is based on this construction. (However, the functions illustrated in the figure are defined for all x in $[0, 1]$ whereas the function f above is defined only at finitely many points.)

It is rather clear from the construction that f satisfies the condition on maxima and minima in Condition 1 above. We now show that Condition 2 holds. We have given $g \in \mathscr{H}$ and hence the condition holds for pairs of points $r, (r + 1)/3^k$ in case k is less than $2N-1$. We must show that the condition holds for pairs of points of the type $3r/9^N, 3r + 3/9^N$, and then for pairs of points of the type $s/9^N, (s + 1)/9^N$. For a pair of the former type we have

$$\begin{aligned} &| f(3r/9^N) - f((3r + 3)/9^N) | \\ &\quad = \tfrac{1}{3} | f([3r/9]/9^{N-1}) - f([3r/9]/9^{N-1}) | \\ &\quad \leqq 3^{-1} 9^{-(n-k)/2} < (3/9^N)^{1/2} . \end{aligned}$$

For a pair of the latter type we have

$$\begin{aligned}&|f(s/9^N) - f((s+1)/9^N)|\\ &\quad = |f(3[s|/3]/9^N) - f(3([s/3]+1)/9^N)|\\ &\quad = \tfrac{1}{3}|f([s/9]/9^{N-1}) - f(([s/9]+1)/9^{N-1})|\\ &\quad \leqq 3^{-1}9^{-(N-1)/2} = 1/9^{N/2}.\end{aligned}$$

Now, from Theorem 2, $(\mathscr{H} - \mathscr{A}_n) \cap \mathscr{F}$ is nowhere dense in $\mathscr{H} \cap \mathscr{F}$, for each n, with respect to the metric d. It follows that $(\mathscr{H} - \bigcup_{n=0}^{\infty} \mathscr{A}_n) \cap \mathscr{F}$ is of the first category, and thus by the Category Theorem we have the following.

THEOREM 4. *The set of functions f on T such that for any n there exist $i \geqq n$ such that Condition* 3 *is satisfied is of the second category with respect to the metric d in the space of functions of T which satisfy Conditions* 1 *and* 2.

Now we wish to extend our functions, using Lemma 1, to functions defined on the whole interval $[0, 1]$.

THEOREM 5. *The functions in $\mathscr{H} \cap \mathscr{F}$ are uniformly continuous. In fact, they satisfy a Hölder condition with the exponent* 1/2.

Proof. Let x and y $(x < y)$ be arbitrary points in T. The interval $[x, y]$ can be expressed uniquely as a countable union of intervals. $I_n = [a_n, b_n]$ disjoint except possibly for end points, such that the following conditions are satisfied.

1. Each I_n is of the form $[r/3^k, (r+1)/3^k]$.
2. No I_n is a subset of an interval J of the above form such that J is a subset of $[x, y]$.

It is clear that for any k, there can be among the I_n at most four intervals of length $1/3^k$. Let L_n denote the length of I_n. We have for any f in $\mathscr{H} \cap \mathscr{F}$,

$$\begin{aligned}|f(y) - f(x)| &\leqq \sum |f(b_n) - f(a_n)|\\ &\leqq \sum L_n^{1/2} < 4\sum_{k=0}^{\infty}\left(\frac{y-x}{3_r+1}\right)^{1/2}\\ &< \frac{4\sqrt{3}}{\sqrt{3}-1}(y-x)^{1/2}.\end{aligned}$$

Now by Lemma 1 the functions in $\mathscr{H} \cap \mathscr{F}$ can be extended to all of $[0, 1]$ in a unique way. Further, it is clear that these functions satisfy a Hölder condition with the exponent 1/2.

THEOREM 6. *The continuous extensions of the functions in $(\bigcap_{n=0}^{\infty} \mathscr{A}_n) \cap \mathscr{F}$ are nonconstant and everywhere locally recurrent.*

Proof. Let f be in $(\bigcap_{n=0}^{\infty} \mathscr{A}_n) \cap \mathscr{F}$ and let x be any point in $[0, 1]$. For any $\varepsilon > 0$, x belongs to an interval of the form $I = [3r/9^k, (3r+1)/9^k]$ of length less than ε, such that

$$f(3r/9^k) = f((3r+2)/9^k)$$

and

$$f((3r+1)/9^k) = f((3r+3)/9^k)\,.$$

By the intermediate value theorem for continuous functions, there exists $y \in I$, $x \neq y$, such that $f(x) = f(y)$. This proves local recurrence. The functions are nonconstant because of Condition 1.

5. Other applications. It is hoped that the methods of § 2 will be useful as a tool for the construction of various counterexamples.

The methods of § 2 can be used in a routine way to give the following results.

PROPOSITION 1. The set of real functions which are not convex in any interval is of second category with respect to the metric d in the space of monotone functions.

PROPOSITION 2. The set of real functions f such that for any interval $I \subset [0, 1]$ one has

$$\limsup_{x\ y \in I} \frac{f(y) - f(x)}{y - x} = 1$$

and

$$\liminf_{x, y \in I} \frac{f(y) - f(x)}{y - x} = -1$$

is of second category with respect to the metric d in the space of functions which satisfy the Lipschitz condition $|f(y) - f(x)| \leqq |x - y|$ for all x and y in $[0, 1]$.

References

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