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# A TRANSPLANTATION THEOREM FOR JACOBI COEFFICIENTS

RICHARD ASKEY

**Let $f(\theta)$ be integrable on $(0, \pi)$ and define**

$$a_n^{\alpha,\beta} = t_n^{\alpha,\beta} \int_0^\pi f(\theta) P_n^{(\alpha,\beta)}(\cos\theta) \left(\sin\frac{\theta}{2}\right)^{\alpha+(1/2)} \left(\cos\frac{\theta}{2}\right)^{\beta+(1/2)} d\theta$$

**where $P_n^{(\alpha,\beta)}(x)$ is the Jacobi polynomial of degree $n$, order $(\alpha, \beta)$ and**

$$[t_n^{\alpha,\beta}]^2 = \frac{(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)} .$$

**Then if $\alpha, \beta, \gamma, \delta \geqq -1/2$ we have**

$$\sum_{n=0}^\infty | a_n^{(\gamma,\delta)} |^p (n+1)^\sigma \leqq A \sum_{n=0}^\infty | a_n^{(\alpha,\beta)} |^p (n+1)^\sigma$$

**for $1 < p < \infty$, $-1 < \sigma < p-1$ whenever the right hand side is finite.**

**From this result any norm inequality for Fourier coefficients can be transplanted to give a corresponding norm inequality for Fourier-Jacobi coefficients.**

Let $P_n^{(\alpha,\beta)}(x)$ be defined by $(-1)^n 2^n n!(1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x) = (d/dx)^n\{(1-x)^{n+\alpha}(1+x)^{n+\beta}\}$, $\alpha, \beta > -1$. The functions $P_n^{(\alpha,\beta)}(\cos\theta)$ are orthogonal on $(0, \pi)$ with respect to the measure

$$\left(\sin\frac{\theta}{2}\right)^{2\alpha+1} \left(\cos\frac{\theta}{2}\right)^{2\beta+1} d\theta$$

and

$$\begin{aligned} &\int_0^\pi [P_n^{(\alpha,\beta)}(\cos\theta)]^2 \left(\sin\frac{\theta}{2}\right)^{2\alpha+1} \left(\cos\frac{\theta}{2}\right)^{2\beta+1} d\theta \\ &\quad = \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)\Gamma(n+1)} = [t_n^{\alpha,\beta}]^{-2} . \end{aligned} \tag{1}$$

Observe that $t_n^{(\alpha,\beta)} = An^{1/2} + O(n^{-1/2})$ where $A$ is a constant whose numerical value is of no interest to us. For simplicity we set $R_n^{\alpha,\beta}(\theta) = t_n^{\alpha,\beta} P_n^{(\alpha,\beta)}(\cos\theta)[\sin(\theta/2)]^{\alpha+(1/2)}[\cos(\theta/2)]^{\beta+(1/2)}$. The functions $\{R_n^{\alpha,\beta}(\theta)\}_{n=0}^\infty$ form a complete orthonormal sequence of functions on $(0, \pi)$. Also $R_n^{-1/2,-1/2}(\theta) = A\cos n\theta$ and $R_n^{1/2,1/2}(\theta) = A\sin(n+1)\theta$.

If $f(\theta) \in L^1(0, \pi)$ we define its Fourier-Jacobi coefficients by

$$a_n^{\alpha,\beta} = \int_0^\pi f(\theta) R_n^{\alpha,\beta}(\theta) d\theta . \tag{2}$$

We define $l^{p,\sigma}$ to be the space of sequences $\{a_n\}$ such that $\| a_n \|_{p,\sigma} =$

$\left[\sum_{n=0}^{\infty} | a_n |^p (n+1)^\sigma\right]^{1/p}$ is finite. Our main theorem follows.

THEOREM 1. *Let $\alpha, \beta, \gamma, \delta \geqq -1/2$ and $f(\theta) \in L^1(0, \pi)$. Let $a_n^{\alpha,\beta}$ and $a_n^{\gamma,\delta}$ be defined by* (2). *Then if $1 < p < \infty$, $-1 < \sigma < p-1$ and if either $\| a_n^{\alpha,\beta} \|_{p,\sigma}$ or $\| a_n^{\gamma,\delta} \|_{p,\sigma}$ is finite so is the other and*

$$A \leqq \| a_n^{\alpha,\beta} \|_{p,\sigma} / \| a_n^{\gamma,\delta} \|_{p,\sigma} \leqq A \tag{3}$$

*where $A$ is independent of $f$ and thus of $a_n^{\alpha,\beta}$ and $a_n^{\gamma,\delta}$.*

For $\alpha = \beta$, $\gamma = \delta$ this theorem was proved in [1]. The last section of [1] gives two applications of this theorem. They can be carried over word for word to Jacobi coefficients. If all of the formulas for ultraspherical polynomials that were used in [1] were known for Jacobi polynomials, the proof of Theorem 1 could be exactly the same as the proof of the special case of it in [1]. While it is undoubtedly true that the relevant facts stated in [1] do generalize they are at present unknown. An example of such a fact is the following. Consider $P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x)$. This is a polynomial of degree $n+m$ and so

$$P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x) = \sum_{k=0}^{n+m} \alpha_k P_k^{(\alpha,\beta)}(x) .$$

If $\alpha \geqq \beta$ the conjecture is that $\alpha_k \geqq 0$. This is true for $\alpha = \beta$ and was used in [1]. The limiting result $\alpha \to \infty$ is also true and is stated in [4] as a result for Laguerre polynomials. For $\alpha = \beta + 1$ it was proven in [6].

2. In this section we give various results that we need to prove Theorem 1.

For $0 < \theta < \pi/2$, $\alpha \geqq -1/2$, we have the following two inequalities

$$| P_n^{\alpha,\beta}(\cos\theta) | = O(n^\alpha) , \tag{4}$$

$$| R_n^{\alpha,\beta}(\theta) | = \left| t_n^{\alpha,\beta} \left(\sin\frac{\theta}{2}\right)^{\alpha+(1/2)} \left(\cos\frac{\theta}{2}\right)^{\beta+(1/2)} P_n^{(\alpha,\beta)}(\cos\theta) \right| \leqq A . \tag{5}$$

See [7, (7.32.6)]. In (5) the power of $\cos\theta/2$ can be changed at will since $\cos\theta/2$ is bounded away from zero for $0 \leqq \theta \leqq \pi/2$.

$$\frac{d}{dx} P_n^{(\alpha,\beta)}(x) = \frac{1}{2}(n+\alpha+\beta+1) P_{n-1}^{(\alpha+1,\beta+1)}(x) . \tag{6}$$

See [7, (4.21.7)].

The asymptotic formula we need is an easy consequence of two known results which we now state.

If $\alpha > -1$, $\beta$ real and $0 < \theta \leqq \pi - \varepsilon$, $\varepsilon > 0$, then

$$
\begin{aligned}
&\left(\sin\frac{\theta}{2}\right)^{\alpha}\left(\cos\frac{\theta}{2}\right)^{\beta} P_n^{(\alpha,\beta)}(\cos\theta) \\
(7)\qquad &\qquad = N^{-\alpha}\frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}(\theta/\sin\theta)^{1/2}J_\alpha(N\theta) + R_n(\theta)
\end{aligned}
$$

where $N = n + (\alpha + \beta + 1)/2$ and

$$
R_n(\theta) = \begin{cases} \theta^{1/2}O(n^{-3/2}) & n^{-1} \leqq \theta \leqq \pi - \varepsilon \\ \theta^{\alpha+2}O(n^{\alpha}) & 0 < \theta < n^{-1}\,. \end{cases}
$$

$J_\alpha(x)$ is the Bessel function of the first kind of order $\alpha$. See [7, (8. 21. 17)]. We also need a known asymptotic formula for $J_\alpha(x)$.

$$
\begin{aligned}
(8)\qquad x^{1/2}J_\alpha(x) &= A\cos(x - \alpha\pi/2 - \pi/4)[1 + O(x^{-2})] \\
&\quad + A\sin(x - \alpha\pi/2 - \pi 4)[Ax^{-1} + O(x^{-3})]\,, \quad x \to \infty\,.
\end{aligned}
$$

See [7, (1. 71. 8)]. Combining (7), (8) and the asymptotic formula for $t_n^{\alpha,\beta}$ we get

$$
\begin{aligned}
(9)\qquad R_n^{\alpha,\beta}(\theta) &= A\cos(N\theta - \alpha\pi/2 - \pi/4) + A\sin(N\theta - \alpha\pi/2 - \pi/4)/(N\theta) \\
&\quad + O(N^{-1}) + O(N^{-2}\theta^{-2})\,, \qquad 0 < c/n \leqq \theta \leqq \pi/2\,.
\end{aligned}
$$

Finally we need a simple estimate for an integral.

$$
(10)\qquad \int_N^\infty \frac{\cos y}{y}\,dy = O(N^{-1})\,, \qquad N \to \infty\,.
$$

This follows on integrating by parts.

3. We assume that $f(\theta)$ is smooth enough, say $C^2$ and vanishing near 0 and $\pi$, so that the series $\sum a_n R_n^{\alpha,\beta}(\theta)$ converges uniformly on $[0, \pi]$. These conditions are sufficient for $a_n = O(n^{-2})$, integrate by parts twice, and $|R_n^{\alpha,\beta}(\theta)| \leqq A$. We remove this condition after the following argument.

$$
a_n^{\gamma,\delta} = \int_0^\pi f(\theta)R_n^{\gamma,\delta}(\theta)d\theta = \sum_{k=0}^\infty a_k^{\alpha,\beta}\int_0^\pi R_k^{\alpha,\beta}(\theta)R_n^{\gamma,\delta}(\theta)d\theta = \sum_{k=0}^\infty a_k^{\alpha,\beta}R(k, n)\,.
$$

Since $P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x)$, [7, (4. 1. 3)] it is sufficient to estimate $S(k, n) = \int_0^{\pi/2} R_k^{\alpha,\beta}(\theta)R_n^{\gamma,\delta}(\theta)d\theta$. Also because we have made no assumptions about the relationships among $\alpha, \beta, \gamma, \delta$ it is sufficient to consider the case $k \geqq n$. We do this in two stages, $n \leqq k \leqq 2n$ and $k \geqq 2n$. For $n \leqq k \leqq 2n$ the method is the same as in [1]. We repeat it here for convenience and because the other estimate is handled by a refinement of this argument.

$$S(k, n) = \int_0^{\pi/2} = \int_0^{1/k} + \int_{1/k}^{\pi/2} .$$

The first integral is $O(k^{-1})$ since $R_k^{\alpha,\beta}(\theta) = O(1)$, see (5). In the second integral we use (9),

$$R_n^{\alpha,\beta}(\theta) = A \cos\left(N\theta - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) + A \sin\left(N\theta - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right)\Big/ N\theta$$
$$+ O(N^{-1}) + O(N^{-2}\theta^{-2})$$

to get

$$S(k, n) = A \int_{1/k}^{\pi/2} \cos\left(K\theta - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) \cos\left(N\theta - \frac{\gamma\pi}{2} - \frac{\pi}{4}\right) d\theta$$
$$+ \frac{A}{K} \int_{1/k}^{\pi/2} \sin\left(K\theta - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) \cos\left(N\theta - \frac{\gamma\pi}{2} - \frac{\pi}{4}\right) \frac{d\theta}{\theta}$$
$$+ \frac{A}{N} \int_{1/k}^{\pi/2} \cos\left(K\theta - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) \sin\left(N\theta - \frac{\gamma\pi}{2} - \frac{\pi}{4}\right) \frac{d\theta}{\theta} + O(K^{-1}) .$$

The first integral is $A/(K - N) + O(K^{-1})$, the second is

$$\frac{A}{K} \log \frac{K}{K - N} + O(K^{-1}) ,$$

the third is $(A/N) \log N/(K - N) + O(K^{-1})$ by a simple computation. The details are in [1]. The one time this argument breaks down is when $K = N$. In this case $S(k, n) = O(1)$ by (5).

Now we consider the case $k > 2n$. This time we need not be so careful, i.e., all our estimates may be $O$ estimates, but the details turn out to be harder than in the above case. This probably isn't necessary but we have not found a simple proof of the following estimates. There is one case, $\gamma = \alpha + 2$, $\beta = \delta$, in which it is possible to give easy estimates as we will show later. But this is a very singular case.

As before $S(k, n) = \int_0^{1/k} + \int_{1/k}^{\pi/2}$. The first integral is $O(K^{-1})$ by (5). Next we show that in the second integral we may replace $R_K(\theta)$ by $\cos\{K\theta - (\alpha\pi/2) - (\pi/4)\}$. Using (9) we see it is sufficient to show that

$$\frac{n^{1/2}}{K} \int_{1/k}^{\pi/2} \sin\left(K\theta - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) P_n^{(\gamma,\delta)}(\cos\theta) \left(\sin\frac{\theta}{2}\right)^{\gamma+(1/2)} \left(\cos\frac{\theta}{2}\right)^{\delta+(1/2)} \theta^{-1} d\theta$$
$$= O(K^{-1}) .$$

Integrating by parts and estimating we have

$$\frac{n^{1/2}}{K^2}\int_{1/k}^{\pi/2} P_n^{(\gamma,\delta)}(\cos\theta)\Big(\sin\frac{\theta}{2}\Big)^{\gamma+(1/2)}\Big(\cos\frac{\theta}{2}\Big)^{\delta+(1/2)}\theta^{-1}d\cos\Big(K\theta-\frac{\alpha\pi}{2}-\frac{\pi}{4}\Big)$$
$$= O\Big(\frac{n^{1/2}}{K^2}P_n^{(\gamma,\delta)}(\cos\theta)\theta^{\gamma-(1/2)}\Big)\Big|_{1/k}^{\pi/2} + O\Big[\frac{n^{3/2}}{K^2}\int_{1/k}^{\pi/2}\cos\Big(K\theta-\frac{\alpha\pi}{2}-\frac{\pi}{4}\Big)$$
$$\cdot P_{n-1}^{(\gamma+1,\delta+1)}(\cos\theta)\Big(\sin\frac{\theta}{2}\Big)^{\gamma+(3/2)}\Big(\cos\frac{\theta}{2}\Big)^{\delta+(3/2)}\theta^{-1}d\theta\Big]$$
$$+ O\Big(\frac{n^{1/2}}{K^2}\int_{1/k}^{\pi/2}\cos\Big(K\theta-\frac{\alpha\pi}{2}-\frac{\pi}{4}\Big)P_n^{(\gamma,\delta)}(\cos\theta)\Big(\sin\frac{\theta}{2}\Big)^{\gamma-(1/2)}$$
$$\cdot\Big(\cos\frac{\theta}{2}\Big)^{\delta+(3/2)}\theta^{-1}d\theta\Big)$$
$$+ \text{similar terms.}$$

The integrated term is $O(K^{-1})$ by (5). The second integral is

$$O\Big(K^{-2}\int_{1/k}^{\pi/2}\theta^{-2}d\theta\Big) = O(K^{-1})\,.$$

by (5). The first integral we write as $\int_{1/k}^{1/n} + \int_{1/n}^{\pi/2}$. Using (4) in $\int_{1/k}^{1/n}$ we have the bound $(n^{3/2}/K^2)\int_{1/k}^{1/n} n^{\gamma+1}\theta^{\gamma+(1/2)}d\theta = O(K^{-1})$. In $\int_{1/n}^{\pi/2}$ we use (9) to get

$$\frac{An}{K^2}\int_{1/n}^{\pi/2}\cos\Big(K\theta-\frac{\alpha\pi}{2}-\frac{\pi}{2}\Big)\Big[\cos\Big(N\theta-\frac{\gamma\pi}{2}-\frac{\pi}{4}\Big)$$
$$+\frac{A\sin\Big(N\theta-\frac{\gamma\pi}{2}-\frac{\pi}{4}\Big)}{N\theta} + O(N^{-1}) + O(N^{-2}\theta^{-2})\Big]\theta^{-1}d\theta$$
$$= \frac{An}{K^2}\int_{1/n}^{\pi/2}\frac{\cos(K-N)\theta}{\theta}d\theta + O(K^{-1})$$
$$+ \text{terms similar to the first.}$$

Changing variables we get $\int_{(k-n)/n}^{(\pi/2)(k-n)}(\cos y/y)dy$. Since $k > 2n$ we have $(k-n)/n \geqq 1$. Using (10) we get an estimate for the first term of the form $(An/K^2)\{n/(K-n)\} = O(K^{-1})$. Thus it is sufficient to consider

$$t_n^{\gamma,\delta}\int_{1/k}^{\pi/2}\cos\Big(K\theta-\frac{\alpha\pi}{2}-\frac{\pi}{4}\Big)P_n^{(\gamma,\delta)}(\cos\theta)\Big(\sin\frac{\theta}{2}\Big)^{\gamma+(1/2)}\Big(\cos\frac{\theta}{2}\Big)^{\delta+(1/2)}d\theta\,.$$

As above we break this integral into two parts $\int_{1/k}^{1/n} + \int_{1/n}^{\pi/2}$. We treat the first of these first. Integrating by parts and estimating we have

$$\int_{1/k}^{1/n} = O(K^{-1}) + \frac{t_n^{\gamma,\delta}}{K}\int_{1/k}^{1/n} \sin\left(K\theta - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right)\frac{d}{d\theta}\Big[P_n^{(\gamma,\delta)}(\cos\theta)$$
$$\cdot\left(\sin\frac{\theta}{2}\right)^{\gamma+(1/2)}\left(\cos\frac{\theta}{2}\right)^{\delta+(1/2)}\Big]d\theta\,.$$

Using (6) and (4) we see that this integral is $O(K^{-1})$.

In our one remaining integral we may use (9). However to get an estimate of the form $O(K^{-1})$ we must first integrate by parts. Then we get

$$\frac{t_n^{\gamma,\delta}}{2K}\left(\gamma + \frac{1}{2}\right)\int_{1/n}^{\pi/2} \sin\left(K\theta - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right)P_n^{(\gamma,\delta)}(\cos\theta)$$
$$\cdot\left(\sin\frac{\theta}{2}\right)^{\gamma-(1/2)}\left(\cos\frac{\theta}{2}\right)^{\delta+(3/2)}d\theta$$
$$+ \frac{t_n^{\gamma,\delta}}{K}[n+\gamma+\delta+1]\int_{1/n}^{\pi/2} \sin\left(K\theta - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right)P_{n-1}^{(\gamma+1,\delta+1)}(\cos\theta)$$
$$\cdot\left(\sin\frac{\theta}{2}\right)^{\gamma+(3/2)}\left(\cos\frac{\theta}{2}\right)^{\delta+(3/2)}d\theta + O(K^{-1}) \text{ (by (5))}.$$

For the first integral we have the estimate

$$O\Big[\frac{1}{K}\int_{1/n}^{\pi/2} \sin\left(K\theta - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right)\Big[\cos\left(N\theta - \frac{\gamma\pi}{2} - \frac{\pi}{4}\right)$$
$$+ A(N\theta)^{-1}\sin\left(N\theta - \frac{\gamma\pi}{2} - \frac{\pi}{4}\right) + O(N^{-1}) + O(\theta^{-2}N^{-2})\Big]\left(\sin\frac{\theta}{2}\right)^{-1}d\theta\Big]$$
$$= O\Big[\frac{1}{K}\int_{1/n}^{\pi/2}\frac{\sin(K-N)\theta}{\theta}\,d\theta\Big] + O(K^{-1})$$

by (9) and the fact that $(1/\sin\theta) - (1/\theta)$ is bounded. As above this leads to the estimate $O(n/K^2) + O(K^{-1}) = O(K^{-1})$.

A simple computation shows that $t_n^{\gamma,\delta} = At_{n-1}^{\gamma+1,\delta+1}[1 + O(n^{-1})]$ so the second integral may also be estimated by using (9). The estimate is

$$O\Big(\frac{n}{K}\int_{1/n}^{\pi/2} \sin\left(K\theta - \frac{\gamma\pi}{2} - \frac{\pi}{4}\right)\Big[\cos\left(N\theta - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right)$$
$$+ \frac{A\sin\left(N\theta - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right)}{N\theta} + O(N^{-1}) + O(N^{-2}\theta^{-2})\Big]d\theta\Big)$$
$$= O\left(\frac{n}{K}\int_{1/n}^{\pi/2}\sin(K-N)\theta\,d\theta\right) + O\left(\frac{n}{K}\int_{1/n}^{\pi/2}\frac{\cos(K-N)\theta}{N\theta}\,d\theta\right)$$
$$+ \text{similar terms} + O(K^{-1})$$
$$= O\left(\frac{n}{K(K-N)}\right) + O(K^{-1}) = O(K^{-1})$$

by the same type of arguments that have been used often above. Combining all of the above estimates we see that

$$
\begin{aligned}
a_n^{\gamma,\delta} = {} & O\left[\frac{1}{n}\sum_{k=0}^{[n/2]} | a_k^{\alpha,\beta} |\right] + A \sum_{k=[n/2]}^{2n} \frac{a_k^{\alpha,\beta}}{K-N} \\
(11) \qquad & + \frac{A}{n}\sum_{k=[n/2]}^{2n} a_k^{\alpha,\beta} \log\left|\frac{N}{K-N}\right| + A \sum_{k=[n/2]}^{2n} \frac{a_k^{\alpha,\beta}}{K} \log\left|\frac{K}{K-N}\right| \\
& + \left[\sum_{k=2n}^{\infty} \frac{| a_k^{\alpha,\beta} |}{K}\right].
\end{aligned}
$$

As in [1] all of the terms on the right are bounded operators in $l^{p,\sigma}$, $1 < p < \infty$, $-1 < \sigma < p-1$. Thus $\| a_n^{\gamma,\delta} \|_{p,\sigma} \leqq A \| a_n^{\alpha,\beta} \|_{p,\sigma}$ which is (3).

Let $g(\theta) \in C^2$ and vanish near 0 and $\pi$ and let $f(\theta) \in L^1(0, \pi)$. Define their Fourier-Jacobi coefficients, $b_n^{\alpha,\beta}$ and $a_n^{\alpha,\beta}$ respectively, by (2). Then

$$\sum_{n=0}^{\infty} b_n^{\gamma,\delta} a_n^{\gamma,\delta} = \int_0^\pi f(\theta) g(\theta) d\theta = \sum_{n=0}^{\infty} b_n^{\alpha,\beta} a_n^{\alpha,\beta}$$

and thus {with $(1/p) + (1/q) = 1$}

$$
\begin{aligned}
\| a_n^{\gamma,\delta} \|_{p,\sigma} &= \left[\sum_{n=0}^{\infty} | a_n^{\gamma,\delta} |^p (n+1)^\sigma\right]^{1/p} = \sup \sum a_n^{\gamma,\delta} b_n^{\gamma,\delta} = \sup \sum a_n^{\alpha,\beta} b_n^{\alpha,\beta} \\
&\leqq \sup \| a_n^{\alpha,\beta} \|_{p,\sigma} \| b_n^{\alpha,\beta} \|_{q,-q\sigma/p} \leqq A \| a_n^{\alpha,\beta} \|_{p,\sigma}
\end{aligned}
$$

by (3). Here the sup is taken over the sequences $b_n^{\gamma,\delta}$ with

$$\sum | b_n^{\gamma,\delta} |^q (n+1)^{\sigma(1-q)} \leqq 1 .$$

This completes the proof of Theorem 1.

There is a simple substitute in $l^1$ which follows easily from (11).

THEOREM 2. *Let $\alpha, \beta, \gamma, \delta$ be as in Theorem* 1 *and assume* $\sum | a_n^{\alpha,\beta} | \log(n+2) < \infty$. *Then* $\sum | a_n^{\gamma,\delta} | < \infty$ *where*

$$a_n^{\gamma,\delta} = \int_0^\pi f(\theta) R_n^{\gamma,\delta}(\theta)\, d\theta$$

*ıd* $f(\theta) = \sum a_n^{\alpha,\beta} R_n^{\alpha,\beta}(\theta)$.

The inequalities that are needed to prove Theorem 2 from (11) are in [3], where this result was proven for $\alpha = \beta = -1/2$, $\gamma = \delta = 1/2$. To be pedantic here we must be careful for unless $\alpha = -1/2$, $R_n^{(\alpha,\beta)}(0) = 0$ and so $f(\theta) = \sum a_n^{\alpha,\beta} R_n^{\alpha,\beta}(\theta)$ must vanish at $\theta = 0$. Thus if $\alpha = -1/2$ we must assume $f(0) = 0$ and similarly for $\beta = -1/2$, $\theta = \pi$. Theorem 2 is the one place where the above proof is an improvement over the proof in [1] (even in the case $\alpha = \beta$, $\gamma = \delta$) for using the proof in [1] we must add higher powers to the logarithm if $\alpha$ and $\gamma$ are far apart. Even this can be done away with if we use Theorem 4 which follows in the next section. However this problem would again arise if one tried to prove Theorem 2 for Jacobi (not ultraspherical) coefficients by the method in [1] by say holding $\beta$ fixed first and then varying it with fixed $\alpha$.

4. We conclude this paper with two simple theorems that hold in $l^1$. Since the details are easier we first give a theorem for Laguerre coefficients and then finally we give the corresponding theorem for Jacobi coefficients.

The Laguerre polynomial $L_n^\alpha(x)$ is defined by

$$\sum_{n=0}^{\infty} L_n^\alpha(x) r^n = (1-r)^{-\alpha-1} \exp\left(\frac{-xr}{1-r}\right).$$

These functions satisfy

$$\int_0^\infty L_n^\alpha(x) L_m^\alpha(x) x^\alpha e^{-x} dx = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)} \delta_{n,m}, \qquad \alpha > -1 \tag{12}$$

Let $f(x) \in L^1(0, \infty)$ and define its Fourier-Laguerre coefficient by

$$a_n^\alpha = t_n^\alpha \int_0^\infty f(x) L_n^\alpha(x) x^{\alpha/2} e^{-x/2} dx . \tag{13}$$

where

$$t_n^\alpha = \left[\frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)}\right]^{1/2}.$$

We need one more fact about these functions.

$$L_n^{\alpha+1}(x) = \sum_{j=0}^{n} L_j^\alpha(x) . \tag{14}$$

From this we see that

$$L_n^\alpha(x) = L_n^{\alpha+1}(x) - L_{n-1}^{\alpha+1}(x) . \tag{15}$$

THEOREM 3. *Let $f(x) \in L^1(0, \infty)$ and define $a_n^\alpha$, $a_n^{\alpha+2}$ by* (13). *Then if $\alpha > 0$,*

$$A \leqq [\sum | a_n^\alpha |^p]^{1/p} / [\sum | a_n^{\alpha+2} |^p]^{1/p} \leqq A \tag{16}$$

*for $1 \leqq p < \infty$. If $-1 < \alpha < 0$ then* (16) *holds for $2/(2+\alpha) < p < -2/\alpha$.*

Using (12) and (13) we see that

$$\begin{aligned}
a_n^{\alpha+2} &= t_n^{\alpha+2}\int_0^\infty f(x)L_n^{\alpha+2}(x)x^{(\alpha/2)+1}e^{-x/2}dx \\
&= t_n^{\alpha+2}\sum_{k=0}^{\infty} t_k^\alpha a_k^\alpha \int_0^\infty L_k^\alpha(x)L_n^{\alpha+2}(x)x^{\alpha+1}e^{-x}dx \\
&= t_n^{\alpha+2}\sum_{k=0}^{n+1} t_k^\alpha a_k^\alpha \int_0^\infty L_k^\alpha(x)L_n^{\alpha+2}(x)x^{\alpha+1}e^{-x}dx\,.
\end{aligned}$$

Then using (14) and (15) we have

$$\begin{aligned}
a_n^{\alpha+2} &= t_n^{\alpha+2}t_0^\alpha a_0^\alpha \int_0^\infty L_0^\alpha(x)\sum_{j=0}^{n} L_j^{\alpha+1}(x)x^{\alpha+1}e^{-x}dx \\
&\quad + t_n^{\alpha+2}\sum_{k=1}^{n} t_k^\alpha a_k^\alpha \int_0^\infty [L_k^{\alpha+1}(x) - L_{k-1}^{\alpha+1}(x)]\left[\sum_{j=0}^{n} L_j^{\alpha+1}(x)\right]x^{\alpha+1}e^{-x}dx \\
&\quad + t_n^{\alpha+2}t_{n+1}^\alpha a_{n+1}^\alpha \int_0^\infty - L_n^{\alpha+1}(x)L_n^{\alpha+1}(x)x^{\alpha+1}e^{-x}dx \\
&= t_n^{\alpha+2}t_0^\alpha[t_0^{\alpha+1}]^{-2}a_0^\alpha + t_n^{\alpha+2}\sum_{k=1}^{n} t_k^\alpha a_k^\alpha[(t_k^{\alpha+1})^{-2} - (t_{k-1}^{\alpha+1})^{-2}] \\
&\quad - t_n^{\alpha+2}t_{n+1}^\alpha[t_n^{\alpha+1}]^{-2}a_{n+1}^\alpha\,.
\end{aligned}$$

Thus

$$\begin{aligned}
|\,a_n^{\alpha+2}\,| &\leqq A\,|\,a_0^\alpha\,|\,n^{-(\alpha/2)-1} + A\sum_{k=1}^{n} n^{-(\alpha/2)-1}k^{-\alpha/2}k^\alpha\,|\,a_k^\alpha\,| + A\,|\,a_{n+1}^\alpha\,| \\
&\leqq A\,|\,a_0^\alpha\,| + A\sum_{k=1}^{n}|\,a_k^\alpha\,|\,(k/n)^{\alpha/2}n^{-1} + A\,|\,a_{n+1}^\alpha\,|\,.
\end{aligned}$$

Similarly one can show that

$$|\,a_k^\alpha\,| \leqq A\,|\,a_{k-1}^{\alpha+2}\,| + A\sum_{n=k}^{\infty}|\,a_n^{\alpha+2}\,|\,(k/n)^{\alpha/2}n^{-1}\,.$$

Theorem 3 then follows from problem 346 in [5]. Actually there is one application of Theorem 3 and surprisingly it is for $\alpha$ negative. In a paper which will appear, Wainger and I prove the following theorem.

THEOREM A. *Let $\alpha \geqq 0$, $f \in L^1(0, \infty)$ and define*

$$a_n^\alpha = \int_0^\infty f(x)t_n^\alpha L_n^\alpha(x)x^{\alpha/2}e^{-x/2}dx\,.$$

*Let $t(x)$ be a bounded function which is of bounded variation on $(0, \infty)$, with $\int_0^\infty |\,dt(x)\,| \leqq C$. Define*

$$Ta_n^\alpha = \int_0^\infty t(x)f(x)t_n^\alpha L_n^\alpha(x)x^{\alpha/2}e^{-x/2}dx\,.$$

*Then this operator is bounded in $l^p$, $4/3 < p < 4$, i.e.*

$$[\sum |Ta_n^\alpha|^p]^{1/p} \leqq AC[\sum |a_n^\alpha|^p]^{1/p}$$

*where $A$ is independent of $f(x)$ and of $t(x)$.*

We used asymptotic estimates of Erdélyi which have only been proven for $\alpha \geqq 0$. See [2] where the dual result is proven. We can now extend this result to $\alpha \geqq -1/2$ by using Theorem 3. Similar applications are given in [1] and we will not repeat the details here.

It would be interesting to extend Theorem 3 to get a theorem which corresponds to Theorem 1. The estimates of Erdélyi are probably not sufficient to allow one to prove this but they can probably be extended to give two terms plus an error and this might suffice.

The proof of the following theorem for Jacobi coefficients is exactly the same as the proof of Theorem 3 but the coefficients are not as simple so it looks more complicated.

THEOREM 4. *Let $f(\theta) \in L^1(0, \pi)$ and define $a_n^{\alpha,\beta}$, $a_n^{\alpha+2,\beta}$ by (2). Then if $\alpha > -1/2$, $\beta > -1$, and if either $\sum |a_n^{\alpha,\beta}|$ or $\sum |a_n^{\alpha+2,\beta}|$ converges so does the other and*

$$0 < A \leqq \sum |a_n^{\alpha,\beta}| / \sum |a_n^{\alpha+2,\beta}| \leqq A < \infty$$

*with $A$ independent of $f$.*

$$\begin{aligned}
a_n^{\alpha+2,\beta} &= \int_0^\pi f(\theta) R_n^{\alpha+2,\beta}(\theta)\, d\theta = \sum_{k=0}^\infty a_k^{\alpha,\beta} \int_0^\pi R_n^{\alpha+2,\beta}(\theta) R_k^{\alpha,\beta}(\theta) d\theta \\
&= t_n^{\alpha+2,\beta} \sum_{k=0}^\infty a_k^{\alpha,\beta} t_k^{\alpha,\beta} \int_0^\pi P_n^{(\alpha+2,\beta)}(\cos\theta) P_k^{(\alpha,\beta)}(\cos\theta) \\
&\quad \cdot \left(\sin\frac{\theta}{2}\right)^{2\alpha+3} \left(\cos\frac{\theta}{2}\right)^{2\beta+1} d\theta \\
&= 2^{-\alpha-\beta-2} t_n^{\alpha+2,\beta} \sum_{k=0}^\infty a_k^{\alpha,\beta} t_k^{\alpha,\beta} \int_{-1}^1 P_n^{(\alpha+2,\beta)}(x) P^{(\alpha,\beta)}(x)(1-x)^{\alpha+1}(1+x)^\beta dx \\
&= 2^{-\alpha-\beta-2} t_n^{\alpha+2,\beta} \sum_{k=0}^{n+1} a_k^{\alpha,\beta} t_k^{\alpha,\beta} R(k, n) .
\end{aligned}$$

To estimate $R(k, n)$ we use the following.

$$\begin{aligned}
P_n^{(\alpha+1,\beta)}(x) &= \frac{\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+2)} \\
&\quad \cdot \sum_{j=0}^n \frac{(2j+\alpha+\beta+1)\Gamma(j+\alpha+\beta+1)}{\Gamma(j+\beta+1)} P_j^{(\alpha,\beta)}(x) ,
\end{aligned} \tag{17}$$

$$\begin{aligned}
P_n^{(\alpha,\beta)}(x) &= \frac{n+\alpha+\beta+1}{2n+\alpha+\beta+1} P_n^{(\alpha+1,\beta)}(x) \\
&\quad - \frac{n+\beta}{2n+\alpha+\beta+1} P_{n-1}^{(\alpha+1,\beta)}(x) .
\end{aligned} \tag{18}$$

(18) follows from (17) and (17) is (4. 5. 3) in [7]. Using (17) and (18) we see that

$$R(k, n) = \sum_{j=0}^{n} \frac{\Gamma(n+\beta+1)(2j+\alpha+\beta+2)\Gamma(j+\alpha+\beta+2)}{\Gamma(n+\alpha+\beta+3)\Gamma(j+\beta+1)(2k+\alpha+\beta+1)}$$
$$\cdot \int_{-1}^{1} P_j^{(\alpha+1,\beta)}(x)[(k+\alpha+\beta+1)P_k^{(\alpha+1,\beta)}(x)$$
$$-(k+\beta)P_{k-1}^{(\alpha+1,\beta)}(x)](1-x)^{\alpha+1}(1+x)^{\beta}dx\,.$$

For $1 \leqq k \leqq n$ we have

$$R(k, n) = \frac{\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+3)}\Big[\frac{(2k+\alpha+\beta+2)\Gamma(k+\alpha+\beta+2)}{\Gamma(k+\beta+1)(2k+\alpha+\beta+1)}$$
$$\cdot \frac{(k+\alpha+\beta+1)}{[t_k^{\alpha+1,\beta}]^2} - \frac{(2k+\alpha+\beta)\Gamma(k+\alpha+\beta+1)(k+\beta)}{\Gamma(k+\beta)(2k+\alpha+\beta+1)[t_{k-1}^{\alpha+1,\beta}]^2}\Big]$$
$$= \frac{\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+3)(2k+\alpha+\beta+1)}$$
$$\cdot\Big[\frac{\Gamma(k+\alpha+2)(k+\alpha+\beta+1)}{\Gamma(k+1)} - \frac{\Gamma(k+\alpha+1)(k+\beta)}{\Gamma(k)}\Big]$$
$$= \frac{\Gamma(n+\beta+1)\Gamma(k+\alpha+1)}{\Gamma(n+\alpha+\beta+3)(2k+\alpha+\beta+1)\Gamma(k+1)}$$
$$\cdot[(k+\alpha+1)(k+\alpha+\beta+1) - k(k+\beta)]$$
$$= O(n^{-\alpha-2}k^{\alpha})\,.$$

For $k = 0$, $R(k, n) = O(n^{-\alpha-2})$ follows easily from (17), (18) and (1). For $k = n+1$, $R(k, n) = O(n^{-1})$ also follows easily from these same formulas. Thus we have

$$|\, a_n^{\alpha+2,\beta} \,| \leqq An^{-1} \sum_{k=0}^{n+1} |\, a_k^{\alpha,\beta} \,|\, (k/n)^{\alpha+1/2}$$

and

$$\sum_{n=0}^{\infty} |\, a_n^{\alpha+2,\beta} \,| \leqq A \sum_{n=0}^{\infty} |\, a_n^{\alpha,\beta} \,|$$

follows easily by interchanging the order of summation. The other inequality follows by the same argument.

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