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REMARKS ON SCHWARZ'S LEMMA

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If $f(z)$ is an analytic function, regular in $|z| \leqq 1$, $|f(z)| \leqq 1$ for $|z| = 1$ and $f(0) = 0$, then by Schwarz's lemma

$$|f(re^{i\theta})| \leqq r\,, \qquad (0 \leqq r \leqq 1)\,.$$

More generally, if $f(z)$ is regular inside and on the unit circle, $|f(z)| \leqq 1$ on the circle and $f(a) = 0$, where $|a| < 1$, then

$$|f(z) \leqq |(z-a)/(1-\bar{a}z)|\,, \tag{1}$$

inside the circle. In other words,

$$|f(z)/(z-a)| < 1/|1-\bar{a}z|\,, \tag{2}$$

for $|z| \leqq 1$. For a fixed a on the unit circle, let C_a denote the class of functions $f(z)$ which are regular in $|z| \leqq 1$, vanish at the point $z = a$, and for which

$$\max_{|z|=1} |f(z)| = 1\,.$$

Any positive number A being given, it is clearly possible to construct a function $f(z)$ of the class C_a for which

$$\mathscr{M}_f(1) = \max_{|z|=1} |f(z)/(z-a)| > A\,,$$

i.e. $\mathscr{M}_f(1)$ is not uniformly bounded for $f \in C_a$. If $f(z)$ is restricted to a subclass of C_a there may exist a uniform bound for $\mathscr{M}_f(1)$ as $f(z)$ varies within the subclass. It is clear that such is the case for the important subclass consisting of all polynomials of degree at most n vanishing at $z = a$. The problem is to find the uniform bound.

We prove:

THEOREM. *If $p(z)$ is a polynomial of degree n such that $|p(z)| \leqq 1$ on the unit circle, and $p(1) = 0$, then for $|z| \leqq 1$.*

$$|p(z)/(z-1)| \leqq n/2\,. \tag{3}$$

The example $(z^n - 1)/2$ shows that the bound in (3) is precise. The following corollary is immediate.

COROLLARY 1. *If $p(z)$ is a polynomial of degree n satisfying the conditions of the theorem, then $|p'(1)| \leqq n/2$.*

This is interesting in view of the fact that if $p(1) \neq 0$ all we can say [2, p. 357] is that $|p'(1)| \leqq n$.

If $p(z)$ is a polynomial of degree n and $|p(z))\leqq 1$ on the unit circle then the polynomial

$$\frac{p(z)-p(1)}{1+|p(1)|}$$

satisfies the hypotheses of our theorem. Consequently

$$\left|\frac{p(z)-p(1)}{z-1}\right|\leqq\frac{n}{2}(1+|p(1)|)\,,$$

or

$$|p(z)-p(1)|\leqq\frac{n}{2}(1+|p(1)|)\,|z-1|\,.$$

Thus we have:

COROLLARY 2. *If $p(z)$ is a polynomial of degree n and $|p(z)|<1$ on the unit circle, then for $|z|\leqq 1$,*

$$|p(z)|\leqq|p(1)|+\frac{n}{2}(1+|p(1)|)\,|z-1|\,.$$

Proof of the theorem. Set $p(z)/(z-1)=\tilde{p}(z)$ and let the maximum of $|\tilde{p}(e^{i\theta})|$ for $-\pi\leqq\theta\leqq\pi$ occur when $\theta=2\theta_0$. We may suppose that $|e^{2i\theta_0}-1|<2/n$, otherwise there is nothing to prove. Let

$$t(\theta)=e^{-i(n-1)\theta}\tilde{p}(e^{2i\theta})\,,$$

and choose γ such that $e^{i\gamma}t(\theta_0)$ is real. Consider the real trigonometric polynomial

$$T(\theta)\equiv\operatorname{Re}\{e^{i\gamma}t(\theta)\}\,.$$

Since $|e^{i\gamma}t(\theta)|$ has its maximum at θ_0, the real trigonometric polynomial $T(\theta)$ has its maximum modulus at θ_0 where it is actually a local maximum, i.e. $T'(\theta_0)=0$. The function $2\sin\theta\,T(\theta)$ is a real trigonometric polynomial of degree n such that

$$|2\sin\theta\,T(\theta)|=|e^{-i\theta}(e^{2i\theta}-1)\,T(\theta)|\leqq|p(e^{2i\theta})|\leqq 1$$

for $-\pi\leqq\theta\leqq\pi$.

A result of van der Corput and Schaake [1] states that if $F(\theta)$ is a real trigonometric polynomial of degree n and $|F(\theta)|\leqq 1$ for real θ, then

$$n^2F(\theta)^2+(F'(\theta))^2\leqq n^2\,. \tag{4}$$

Applying this result to the trigonometric polynomial $2\sin\theta\,T(\theta)$ we get

$$n^2 \sin^2\theta T^2(\theta) + \{\cos\theta T(\theta) + \sin\theta T'(\theta)\}^2 \leqq \frac{n^2}{4}$$

for $-\pi \leqq \theta \leqq \pi$. Setting $\theta = \theta_0$ we get

$$\{1 + (n^2 - 1)\sin^2\theta_0\} T^2(\theta_0) \leqq \frac{n^2}{4},$$

since $T'(\theta_0) = 0$. Hence

$$|T(\theta_0)| \leqq (n/2)(1 + (n^2 - 1)\sin^2\theta_0)^{-1/2},$$

and the result follows.

REMARK 1. From the method of proof it is clear that if $p(z)$ is a polynomial of degree n, such that $|p(z)| \leqq 1$ on the unit circle and $p(\pm 1) = 0$, then for $|z| \leqq 1$,

$$|p(z)/(z^2 - 1)| \leqq n/4. \tag{5}$$

Or, more generally, if $p(z) = 0$ whenever z is a root of $z^k - 1 = 0$, and $|p(z)| \leqq 1$ for $|z| \leqq 1$, then

$$\max_{|z| \leqq 1} |p(z)/(z^k - 1)| \leqq n/(2k). \tag{6}$$

REMARK 2. For every $\varepsilon > 0$ and a lying inside the unit circle we can construct a polynomial $p(z)$ which vanishes at $z = a$, $|p(z) \leqq 1$ for $|z| \leqq 1$ but

$$\max_{|z| \leqq 1} |p(z)/(z - a)| > \frac{1}{1 - |a|} - \varepsilon.$$

In fact, let

$$\frac{1}{1 - \bar{a}z} = \sum_{\nu=0}^{\infty} a_\nu z^\nu$$

for $|z| < 1/|a|$, and choose N so large that

$$\left|\frac{1}{1 - \bar{a}z} - \sum_{\nu=0}^{N} a_\nu z^\nu\right| < \varepsilon' = \frac{(1 - |a|)\varepsilon}{2 - \varepsilon(1 - |a|^2)}$$

for $|z| \leqq 1$. With this choice of N,

$$\left|(z - a)\sum_{\nu=0}^{N} a_\nu z^\nu\right| < 1 + (1 + |a|)\varepsilon'$$

for $|z| \leqq 1$ and

$$\frac{1}{1+(1+|a|)\varepsilon'}(z-a)\sum_{\nu=0}^{N} a_\nu z^\nu$$

is therefore a polynomial of the type we wanted to construct.

However, if we restrict ourselves to polynomials of degree at most $n(>1)$ then we can easily prove that for $|z| \leqq 1$

$$|p(z)/(z-a)| < n . \tag{7}$$

For $a = 0$ the result is included in Schwarz's lemma. If $a \neq 0$ we observe that

$$p(z) = \int_a^z p'(w)dw$$

and therefore

$$|p(z)| \leqq |z-a| \max_{|w| \leqq 1} |p'(w)| < n|z-a|$$

by an inequality due to M. Riesz [2, p. 357] wherein equality holds only when $p(z)$ is a constant multiple of z^n. This is the same as (7).

References

1. J. G. van der Corput and G. Schaake, *Ungleichunden für Polynome und trigonometrische Polynome*, Compositio Mathematica **2** (1935), 321-361. *Berichtigung zu: Ungleichungen für Polynome und trigonometrische Polynome*, Compositio Mathematica **3** (1936), 128.

2. M. Riesz, *Eine trigonometrische Interpolationformel und einige Ungleichungen für Polynome*, Jahresbericht der Deutschen Mathematiker-Vereinigung **23** (1914), 354-368.

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