PACIFIC JOURNAL OF MATHEMATICS
Vol. 23, No. 1, 1967

ADDITION THEOREMS FOR SETS OF INTEGERS

CALVIN T. LONG

Let C be a set of integers. Two subsets A and B of C are said to be complementing subsets of C in case every $c \in C$ is uniquely represented in the sum

$$C = A + B = \{x \mid x = a + b, a \in A, b \in B\} .$$

In this paper we characterize all pairs A, B of complementing subsets of

$$N_n = \{0, 1, \cdots, n-1\}$$

for every positive integer n and show some interesting connections between these pairs and pairs of complementing subsets of the set N of all nonnegative integers and the set I of all integers. We also show that the number $C(n)$ of complementing subsets of N_n is the same as the number of ordered nontrivial factorizations of n and that

$$2C(n) = \sum_{d \mid n} C(d) .$$

The structure of complementing pairs A and B has been studied by de Bruijn [1], [2], [3] for the cases $C = I$ and $C = N$ and by A. M. Vaidya [7] who reproduced a fundamental result of de Bruijn for the latter case. In case $C = N$ it is easy to see that $A \cap B = \{0\}$ and that $1 \in A \cup B$. Moreover, if we agree that $1 \in A$, it follows from the work of de Bruijn, that, except in the trivial case $A = N$, $B = \{0\}$, A and B are infinite complementing subsets of N if and only if there exists an infinite sequence of integers $\{m_i\}_{i \geqq 1}$ with $m_i \geqq 2$ for all i, such that A and B are the sets of all finite sums of the form

$$(1) \qquad \begin{aligned} a &= \sum x_{2i} M_{2i} , \\ b &= \sum x_{2i+1} M_{2i+1} \end{aligned}$$

respectively where $0 \leqq x_i < m_{i+1}$ for $i \geqq 0$ and where $M_0 = 1$ and $M_i = \prod_{j=1}^{i} m_j$ for $i \geqq 1$. In the remaining case, when just one of A and B is infinite, the same result holds except that the sequence $\{m_i\}$ is of finite length r and that $x_r \geqq 0$. Similar results can also be obtained in the case of complementing k-tuples of subsets of N for $k > 2$.

The case $C = I$ is much more difficult and, while sufficient conditions are easily given, necessary and sufficient conditions that a pair A, B be complementing subsets of I are not known. As an example of sufficient conditions, we note that if A and B are as in (1) above, then A and $-B$ form a pair of complementing subsets of I. This is an immediate consequence of the fact that every integer n can be represented uniquely in the form

$$(2) \qquad n = \sum_{i=0}^{r} (-1)^i x_i M_i$$

with x_i and M_i as in (1). Incidentally, if B is finite, it is not difficult to see that there exists an integer $r_0 \leqq 0$ such that A and $-B$ form a pair of complementing subsets of the set

$$R = \{r \mid r \in I, r \geqq r_0\}.$$

And if A is finite, there exists an integer $s_0 > 0$ such that A and $-B$ are complementing subsets of the set

$$S = \{s \mid s \in I, s \leqq s_0\}.$$

2. Complementing sets of order n. We now investigate the structure of pairs A, B of complementing subsets of the set

$$N_n = \{0, 1, \cdots, n-1\}$$

for integral values of $n \geqq 1$. Such a pair of sets will be called complementing sets of order n and we will write $(A, B) \sim N_n$.

In case $n = 1$, we have only the trivial pair $A = B = \{0\}$. For $n > 1$, it is easy to see that $A \cap B = \{0\}$ and that $1 \in A \cup B$. We choose our notation so that $1 \in A$ and, if m is the least positive element in B, then we also have that $N_m \subset A$ and that none of $m+1, m+2, \cdots, 2m-1$ appear in either A or B. If B does not contain positive elements, we have only the trivial pair $A = N_n$, $B = \{0\}$.

For the remainder of the paper, we restrict our attention to the case $n > 1$ and we use the notation mS to denote the set of all multiples of elements of a set S by an integer m.

LEMMA 1. *Let A, B, C, and D be subsets of N_n such that, for a fixed integer $m \geqq 2$,*

$$A = mC + N_m \quad \text{and} \quad B = mD.$$

Then $(A, B) \sim N_{mp}$ if and only if $(C, D) \sim N_p$ where $p \geqq 1$.

Proof. Suppose first that $(C, D) \sim N_p$. Then, for any $s \in N_{mp}$, there exist integers $q \in N_p$ and $r \in N_m$ such that $s = mq + r$. Since $(C, D) \sim N_p$, there exist $c \in C$ and $d \in D$ such that $q = c + d$. But then

$$s = m(c + d) + r = (mc + r) + md = a + b$$

with $a = mc + r \in A$ and $b = md \in B$. Moreover, if this representation is not unique, there exist $a' \in A$, $b' \in B$, $c' \in C$, $d' \in D$, and $r' \in N_m$ such that

$$s = a' + b' = (mc' + r') + md' .$$

But then $r = r'$ and

$$c + d = q = c' + d'$$

and this violates the condition that q be uniquely represented in the sum $C + D$.

Conversely, suppose that $(A, B) \sim N_{mp}$. Then, for $s \in N_p$, there exist $a \in A$, $b \in B$, $c \in C$, $d \in D$, and $r \in N_m$ such that

$$sm = a + b = (mc + r) + md .$$

But this implies that $r = 0$ and that $s = c + d$. Also, if this representation of s in $C + D$ is not unique, there exist $c' \in C$ and $d' \in D$ such that $s = c' + d'$. But then

$$sm = cm + dm = c'\mathrm{m} + d'm$$

and this violates the condition that sm be uniquely represented in $A + B$.

The next lemma is an adaptation of a key result of de Bruijn [2, p. 16].

LEMMA 2. *If $(A, B) \sim N_n$, then there exist an integer $m \geqq 2$ such that $m \mid n$ and a complementing pair A', B' of order n/m, with $1 \in A'$ if $B \neq \{0\}$, such that*

$$A = mB' + N_m \quad \text{and} \quad B = mA' . \tag{3}$$

Proof. If $B = \{0\}$, then $A = N_n$ and the desired result follows with $A' = B' = \{0\}$ and $m = n$. If $B \neq \{0\}$, let m be the least positive integer in B. Since $1 \in A$ and $A \cap B = \{0\}$, it follows that $m \geqq 2$. Determine the integer h such that

$$hm \leqq n < (h + 1)m .$$

Now the induction of de Bruijn's proof holds for all nonnegative integers less than h and shows that all elements of B less than hm are multiples of m and that, for each k with $0 \leqq k \leqq h - 1$, the set

$$\{km, km + 1, \cdots, km + m - 1\}$$

is either a subset of A or is disjoint from A. This implies that A' and B' exist such that (1) holds and $1 \in A'$ provided we are able to show that $hm + r \notin A \cup B$ for every integer $r \geqq 0$. Contrariwise, suppose that $hm + r \in A$. Then $hm + r + m \in A + B = N_n$, and this is impossible since $hm + r + m \geqq hm + m > n$. Similarly, if $hm + r \in B$, then $(m - 1) + hm + r \in A + B$ and we have the same contradiction. Thus (3) holds and it follows that m divides n and, by Lemma 1, that $(A', B') \sim N_{n/m}$.

The following theorem, which characterizes all complementing pairs of order $n > 1$, now follows by repeated application of Lemma 2.

THEOREM 1. *Sets A_1 and B_1 form a complementing pair of order $n \geqq 2$ if and only if there exists a sequence $\{m_i\}_{i=1}^r$ of integers not less than two such that*

$$n = \sum_{i=1}^{r} m_i$$

and such that A_1 and B_1 are the sets of all finite sums of the form

$$a = \sum_{i=0}^{[(r-1)/2]} x_{2i} M_{2i} \quad \text{and} \quad b = \sum_{i=0}^{[(r-2)/2]} x_{2i+1} M_{2i+1}$$

respectively with $M_0 = 1$, $M_{i+1} = \prod_{j=1}^{i+1} = m_j$ and $0 \leqq x_i < m_{i+1}$ for $0 \leqq i < r$. If $r = 1$, we interpret the notation to mean that $B_1 = \{0\}$.

It follows from Theorem 1 that there exists a one to one correspondence between the set $\mathscr{C}_n$ of all pairs of complementing sets of order $n > 1$ and the set of all ordered finite sequences $\{m_i\}$ with $m_i \geqq 2$ such that $\prod m_i = n$. Thus, if $C(n)$ denotes the number of elements of $\mathscr{C}_n$, then $C(n)$ is equal to the number $F(n)$ of ordered nontrivial factorizations of n. Curiously, as shown by P. A. MacMahon [4; p. 108], $F(n)$ is in turn equal to the number of perfect partitions of $n - 1$. This last result is also listed by Riordan [6; pp. 123–4]. In a second paper, MacMahon [5; pp. 843–4] shows that

$$C(n) = \sum_{j=1}^{q} \sum_{i=0}^{j-1} (-1)^i \binom{j}{i} \sum_{h=1}^{r} \binom{\alpha_h + j - i - 1}{\alpha_h}$$

where $q = \sum_{h=1}^{r} \alpha_h$ and $n = \prod_{h=1}^{r} p_h^{\alpha_h}$ is the canonical representation of n. However, if one actually wants the values of $C(n)$, they are much more easily computed using the result of the following theorem:

THEOREM 2. *If $n > 1$ is an integer, then*

$$C(n) = \frac{1}{2} \sum_{d \mid n} C(d) = 2 \sum_{d \mid n} \mu(d) C(n/d)$$

where μ denotes the Möbius function.

Proof. It follows from Lemma 2 that to each of the $C(n)$ distinct complementing pairs A, B of order n there corresponds a unique complementing pair A', B' of order d where $d \mid n$ and $1 \leqq d < n$. Hence,

$$C(n) \leqq \sum_{d \mid n, d<n} C(d) .$$

Moreover, from each of the $C(d)$ distinct complementing pairs C, D of order d, with $1 \leqq d < n$ and $1 \in D$ if $d \neq 1$, can be formed precisely one pair A, B of complementing sets of order $dq = n$ by the method of Lemma 1. Since the new pairs formed in this way are clearly distinct, it follows that

$$C(n) \geqq \sum_{d \mid n, d<n} C(d) .$$

Thus, equality holds and this implies that

$$C(n) = \frac{1}{2} \sum_{d \mid n} C(d)$$

as claimed. The other equality is an immediate consequence of the Möbius inversion formula.

Except for Theorem 2, the preceding theorems reveal a striking parallel between the structure of complementing subsets of N and the structure of complementing pairs of order n. The next theorem exhibits an additional interesting connecting between these two classes of pairs. Also, it is clear that a similar theorem holds giving sufficient conditions that A and B form a pair of complementing subsets of I.

THEOREM 3. *Let $\{m_i\}_{i\geqq 1}$ and $\{M_i\}_{i\geqq 0}$ be as defined in* (1) *above and let $(C_i, D_i) \sim N_{m_{i+1}}$ for $i \geqq 0$. If A and B are the sets of all finite sums of the form*

$$a = \sum c_i M_i \quad \textit{and} \quad b = \sum d_i M_i$$

respectively with $c_i \in C_i$ and $d_i \in D_i$ for $i \geqq 0$, then $(A, B) \sim N$.

Proof. Let n be any nonnegative integer. Then n can be represented uniquely in the form

$$n = \sum_{i=0}^{r} e_i M_i$$

with $e_i \in N_{m_{i+1}}$ for all i. Since $(C_i, D_i) \sim N_{m_{i+1}}$, there exist $c_i \in C_i$ and $d_i \in D_i$ such that $e_i = c_i + d_i$ uniquely. Therefore,

$$\begin{aligned} n &= \sum_{i=0}^{r} (c_i + d_i) M_i \\ &= \sum_{i=0}^{r} c_i M_i + \sum_{i=0}^{r} d_i M_i \\ &= a + b \end{aligned}$$

with $a \in A$ and $b \in B$. If this representation of n in $A + B$ is not unique, there exist $a' \in A$ and $b' \in B$ such that

$$n = a' + b'$$

where

$$a' = \sum_{i=0}^{s} c_i' M_i \quad \text{and} \quad b' = \sum_{i=0}^{s} d_i' M_i$$

with $c_i' \in C_i$ and $d_i' \in D_i$ for each i. But then

$$n = \sum_{i=0}^{s} (c_i' + d_i') M_i$$

and $c_i' + d_i' \in N_{m_{i+1}}$ since $(C_i, D_i) \sim N_{m_{i+1}}$ for all i. Since representations of n in this form are unique, it follows that $r = s$ and that

$$c_i + d_i = c_i' + d_i'$$

for each i. And this violates the condition that $(C_i, D_i) \sim N_{m_{i+1}}$. Thus, the representation is unique and $(A, B) \sim N$ as claimed.

Note that if r is fixed and $0 \leqq i < r$ in the sums defining A and B in the preceding theorem, then we conclude in the same way that $(A, B) \sim N_{n^r}$.

The author acknowledges his indebtedness to Professors Ivan Niven and E. A. Maier who made several helpful suggestions concerning the writing of this paper.

References

1. N. G. de Bruijn, *On bases for the set of integers*, Publ. Math. (Debrecen) **1** (1950), 232-242.
2. ———, *On number systems*, Nieuw Arch. Wisk. **4** (1956), 15-17.
3. ———, *Some direct decompositions of the set of integers*, Math. Comp. **18** (1964), 537-546.
4. P. A. MacMahon, *The theory of perfect partitions and the compositions of multipartite numbers*, Mess. Math. **20** (1891), 103-119.
5. ———, *Memoir on the theory of compositions of numbers*, Philos. Trans. Roy. Soc. London (A) **184** (1893), 835-901.
6. J. Riordan, *An Introduction to Combinatorial Analysis*, John Wiley and Sons, New York, 1958.
7. A. M. Vaidya, *On complementing sets of non-negative integers*, Math. Mag. **39** (1966), 43-44.

Received February 15, 1966, and in revised form August 22, 1966.

WASHINGTON STATE UNIVERSITY