

THE VOLUME OF A TOTALLY-GEODESIC HYPERSURFACE IN A PINCHED MANIFOLD

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We will give necessary conditions for a compact hypersurface to be totally-geodesic in a manifold of strictly positive curvatures. These conditions relate the volume of the hypersurface to the pinching of the manifold. The method consists of obtaining estimates from above and below for the volume of the manifold. Comparison of these estimates gives inequalities for the volume of the hypersurface. The method appears on the surface to apply to totally-geodesic submanifolds of arbitrary codimension with but a little modification. We are grateful to R. L. Bishop for pointing out that below the surface are snags that we have not yet been able to avoid.

The existence theory for closed geodesics, highly developed yet still incomplete, stands in sharp contrast to the existence theory for higher-dimensional critical submanifolds of Riemannian manifolds. It has long been known that every compact Riemannian manifold carries a closed geodesic. Of course, if the fundamental group of the manifold is not trivial, every nontrivial homotopy class of closed curves contains at least one closed geodesic.

If the k^{th} homotopy group of a manifold is not trivial, one might hope analogously for a minimal k -sphere in each nontrivial homotopy class. Aside from the obvious (and classical) difficulties with the behavior of the higher-dimensional volume functionals, there is an even more fundamental difficulty, pointed out by Kervaire and Milnor: a nontrivial homotopy class on a smooth manifold may fail to have an embedded smooth (C^1) representative! Part of the difficulty may be side-stepped by working in a larger category. Thus, Almgren has obtained strong results using his theory of varifolds. However, the extremal objects may fail to be manifolds on sets which are small in some measure-theoretic sense.

On the other hand, some positive results are known in the category of manifolds. For example, a holomorphic embedding of a Kähler manifold into a complex projective space (with the canonical Fubini-Study metric) is minimal. But the positive results are still meager.

1. **Notation and preliminary matters.** All manifolds to be considered will be smooth and all Riemannian metrics will be complete. M will be a manifold of dimension m , carrying a Riemannian metric $\langle \cdot, \cdot \rangle$ and associated norm $\| \cdot \|$. Let N be a compact smoothly-embedded

hypersurface in M with normal bundle N^\perp carrying induced norm on the fibers again denoted by $\|\cdot\|$. Let \exp be the exponential map of the Levi-Civita connection ∇ of $\langle \cdot, \cdot \rangle$ and denote the restriction of \exp to N^\perp by \exp^\perp .

Let $U(\delta) = \{v \in N^\perp \mid \|v\| < \delta\}$ and define

$$\tau_N = \sup \{\delta \mid \exp^\perp \text{ is one-to-one on } U(\delta)\}.$$

We have previously considered the problem of estimating τ_N and obtained the following result:

LEMMA 1. ([3, Theorem 2]) *Let M have positive sectional curvatures $K \leq 1$ and let N be a compact totally-geodesic hypersurface. Then $\tau_N \geq \pi/2$.*

(Remark: The proof given in [3] for the above result can be generalized in an obvious fashion to show that the inequality $\tau_N \geq \pi/2$ persists if N is not necessarily a hypersurface but merely has codimension $< m/2$. Notice that the definitions of $U(\delta)$ and τ_N make sense for submanifolds of arbitrary codimension.)

We will have need for some properties of the index form. Let N be a totally geodesic hypersurface and let σ be an N -geodesic; that is, σ is a geodesic parametrized by arc-length with $\dot{\sigma}(0) \in N^\perp$. If X and Y are vector fields along σ and orthogonal to it, the index form of the pair (X, Y) on $(0, r]$ is given by

$$I(X, Y) = \int_0^r \{\langle X', Y' \rangle - \langle R(\dot{\sigma}, X)\dot{\sigma}, Y \rangle\} |_{\dot{\sigma}} dt,$$

where the prime denotes covariant differentiation along σ and

$$R(U, V) = [\nabla_U, \nabla_V] - \nabla_{[U, V]}$$

is the curvature transformation. A Jacobi field is a vector field Y along σ and orthogonal to it satisfying $Y'' + R(\dot{\sigma}, Y)\dot{\sigma} = 0$. If Y is a Jacobi field, then

$$I(X, Y) = \langle X, Y' \rangle |_{\dot{\sigma}}.$$

We use the minimizing properties of Jacobi fields in the following form.

LEMMA 2. *Let Y be a Jacobi field along σ on $[0, r]$ with $Y'(0) = 0$ and let X be any field orthogonal to σ with $X(r) = Y(r)$. Then $I(X, X) \geq I(Y, Y)$ with equality exactly when $X = Y$.*

Proof. If $X \neq Y$, then $X - Y \neq 0$. I is positive definite, so

$$\begin{aligned}
 0 &< I(X - Y, X - Y) \\
 &= I(X, X) - 2I(X, Y) + I(Y, Y) \\
 &= I(X, X) - 2\langle X, Y' \rangle|_0^r + \langle Y, Y' \rangle|_0^r \\
 &= I(X, X) - 2\langle X(r), Y'(r) \rangle + \langle Y(r), Y'(r) \rangle \\
 &= I(X, X) - \langle Y(r), Y'(r) \rangle \\
 &= I(X, X) - I(Y, Y) .
 \end{aligned}$$

The point $\sigma(r)$ (or r itself) is called focal (resp. conjugate) to $\sigma(0)$ along σ if there is a Jacobi field Y along σ with $Y'(0) = 0$ (resp. $Y(0) = 0$) and $Y(r) = 0$. Well-known theorems of Myers [5] and Berger [1] show that, if the Ricci curvatures of M satisfy $Ri \geq a^2 > 0$, there is a focal (resp. conjugate) point along each N -geodesic at distance at most $\pi/2a$ (resp. π/a) from N .

Finally, let ω_{m-1} denote the volume of the Euclidean unit sphere S^{m-1} and let

$$\Omega_{m-1} = \int_0^\pi \sin^{m-1} r dr .$$

2. Volume estimates. We will prove the following.

THEOREM. *Let N be a totally-geodesic, compact hypersurface of M and suppose $0 < a^2 \leq Ri$ and $0 < K \leq b^2$ on M . Then*

1°. *volume $N \leq \omega_{m-1}b/a^m$;*

2°. *if either*

(i) *$b^2/4 < K \leq b^2$ on M , or*

(ii) *M is orientable, even-dimensional, and $0 < a^2 \leq K$ on M ,*

then

$$\text{volume } N \geq \omega_{m-1}a/b^m .$$

Before giving the proof, it is convenient to state a lemma about \exp^\perp . We recall that R. L. Bishop [cf. 2] has considered the behavior of the volume element under the exponential map \exp_p at a point p in M . Let σ be a geodesic parametrized by arc-length with $\sigma(0) = p$ and $\dot{\sigma}(0) = v$. Suppose that $Ri \geq a^2 > 0$ and $0 < K \leq b^2$ on M . Then Bishop proves that the Jacobian determinant $J(t)$ of \exp_p at $tv \in M_p$ satisfies

$$\left(\frac{\sin bt}{bt} \right)^{m-1} \leq J(t) \leq \left(\frac{\sin at}{at} \right)^{m-1} ,$$

at least out to the first conjugate point of p . Equality holds only in the case of constant curvatures. Dually one can consider an N -geodesic σ with $\dot{\sigma}(0) = v$ and the Jacobian determinant $j(t)$ of \exp^\perp at tv . Bounds on the growth of $j(t)$ are given by the following.

LEMMA 3. Suppose $0 < a^2 \leq Ri$ and $0 < K \leq b^2$ on M . Then

$$\cos^{m-1}bt \leq j(t) \leq \cos^{m-1}at ,$$

at least out to the first focal point of N along σ .

Proof. Pick $p \in N$ and $\{y_1, \dots, y_{m-1}\}$ be an orthonormal basis of N_p . Let σ be an N -geodesic with $\dot{\sigma}(0) \in N^\perp$. Let E_1, \dots, E_{m-1} be parallel extensions of y_1, \dots, y_{m-1} along σ . Suppose p has no focal points along σ for $0 < t \leq r$.

Let Z_i be the Jacobi field along σ with $Z'_i(0) = 0$ and $Z_i(r) = E_i(r)$. (Existence of Z_i is assured since r is nonfocal.) Then

$$j(t) = \frac{\|Z_1(t) \wedge \dots \wedge Z_{m-1}(t)\|}{Z} ,$$

where $Z = \|Z_1(0) \wedge \dots \wedge Z_{m-1}(0)\|$. Notice that $j(0) = 1$. Moreover, the values $\{Z_i(r)\}$ are orthonormal. It follows by direct calculation that

$$\frac{j'(r)}{j(r)} = \sum_{i=1}^{m-1} \langle Z_i(r), Z'_i(r) \rangle .$$

Using the assumptions $0 < K \leq b^2$ and $Z'_i(0) = 0$, we obtain

$$\begin{aligned} \langle Z_i(r), Z'_i(r) \rangle &= \int_0^r \{ \|Z'_i\|^2 - K(\dot{\sigma}, Z_i) \|Z_i\|^2 \} |_{t} dt \\ &\geq \int_0^r \{ \|Z'_i\|^2 - b^2 \|Z_i\|^2 \} |_{t} dt \\ &\geq \langle hE_i, h'E_i \rangle |_{r} , \end{aligned}$$

using Lemma 2. Here, $h(t)E_i(t)$ corresponds to a "Jacobi field" for the sphere of constant curvature b^2 with $h(r) = 1$ and $h'(0) = 0$. Thus, $h(t) = \cos bt / \cos br$. It follows that

$$\langle Z_i(r), Z'_i(r) \rangle \geq -b \tan br .$$

Summing,

$$\frac{j'(r)}{j(r)} \geq -(n-1)b \tan br ,$$

from which (using $j(0) = 1$)

$$j(r) \geq \cos^{n-1} br .$$

Next, using Lemma 2 applied to the Jacobi fields Z_i and the vector field hE_i ,

$$\langle Z_i(r), Z'_i(r) \rangle \geq \int_0^r \{(h')^2 - K(\dot{\sigma}, E_i)h^2\} |_{t_i} dt .$$

Summing and using $Ri \geq a^2 > 0$,

$$\begin{aligned} \frac{j'(r)}{j(r)} &\leq (n - 1) \int_0^r \{(h')^2 - a^2 h^2\} |_{t_i} dt \\ &\leq -(n - 1)a \tan ar , \end{aligned}$$

if $h(t) = \cos at / \cos ar$. Thus, as above,

$$j(r) \leq \cos^{n-1} ar .$$

This completes the proof of the Lemma.

Proof of the Theorem.

1°. By a well-known theorem of S. B. Myers [5], the assumption $Ri \geq a^2 > 0$ implies that the diameter of M is at most π/a . Pick p in M and let $B(\pi/a)$ be the π/a -ball in M_p . Using Bishop's estimates,

$$\begin{aligned} \text{volume } M &\leq \int_{B(\pi/a)} (\text{Jacobian determinant of } \exp_p) dB \\ &= \int_{S^{m-1}} \int_0^{\pi/a} J(r) r^{m-1} dr d\omega_{m-1} \\ &\geq \int_{S^{m-1}} \int_0^{\pi/a} r^{m-1} \left(\frac{\sin ar}{ar} \right)^{m-1} dr d\omega_{m-1} \\ &= \frac{\omega_{m-1} \Omega_{m-1}}{a^m} . \end{aligned}$$

Next, \exp^\perp is a bijection on $U(\pi/2b)$ by Lemma 1, so that from Lemma 3 we get

$$\begin{aligned} \text{volume } M &\geq \text{volume } (\exp^\perp U(\pi/2b)) \\ &= \int_{U(\pi/2b)} (\text{Jacobian determinant of } \exp^\perp) dU \\ &= \int_N \int_{-\pi/2b}^{\pi/2b} j(r) dr dN \\ &\geq \int_N \int_{-\pi/2b}^{\pi/2b} \cos^{m-1} br dr dN \\ &= (\text{volume } N) \frac{\Omega_{m-1}}{b} . \end{aligned}$$

Comparing the two estimates of volume M , we get the first inequality of the theorem.

2°. If either (i) or (ii) holds, the injectivity radius of M is at least π/b [4]. Therefore, as in 1°, we estimate

$$\begin{aligned} \text{volume } M &\geq \int_{S^{m-1}} \int_0^{\pi/b} r^{m-1} \left(\frac{\sin br}{br} \right)^{m-1} dr d\omega_{m-1} \\ &= \frac{\omega_{m-1} \Omega_{m-1}}{b^m}. \end{aligned}$$

On the other hand, by Berger's extension [1] of the Rauch Comparison Theorem, every N -geodesic must carry a focal point of N at distance at most $\pi/2a$. Therefore,

$$\begin{aligned} \text{volume } M &\leq \int_N \int_{-\pi/2a}^{\pi/2a} \cos^{m-1} ar dr dN \\ &= \frac{\omega_{m-1} \Omega_{m-1}}{b}. \end{aligned}$$

Comparing these two estimates for volume N , we get the second inequality of the Theorem, thereby completing the proof.

It may be instructive to point out the difficulty which arises in codimension greater than one. Let N be a smoothly embedded submanifold of M of dimension p . There is again a normal bundle N^\perp and a restriction \exp^\perp of the exponential map to N^\perp . If $p > m/2$, Lemma 1 above is still valid. What is needed is an estimate of the growth of the Jacobian determinant of \exp^\perp . We might try to imitate the above process, choosing an N -geodesic σ , p Jacobi fields Z_1, Z_2, \dots, Z_p along σ with $Z_i(r) = E_i(r)$ and $Z_i'(0) = 0$, and $m - p - 1$ Jacobi fields Z_{p+1}, \dots, Z_{m-1} along σ with $Z_j(0) = 0$ and $Z_j'(r) = E_j(r)$. Unfortunately, one has no control over the interrelations between Jacobi fields of the first type and those of the second. For the calculation to proceed as above, one needs that fields of the first type remain orthogonal to those of the second type and this can not be assured.

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