PACIFIC JOURNAL OF MATHEMATICS
Vol. 26, No. 1, 1968

ON A PROBLEM OF ILYEFF

ZALMAN RUBINSTEIN

Let $P(z)$ be a polynomial whose zeros $z_1, z_2, \cdots, z_n$ $(n \geqq 2)$ lie in $|z| \leqq 1$. It is shown that $P'(z)$ always has a zero in $|z - z_1| \leqq 1$ if $|z_1| = 1$ or if $|z_1| < 1$ and $n = 3, 4$.

In his book *Research Problems in Function Theory* [2] W. K. Hayman mentions the following problem due to L. Ilyeff (Problem 4.5, p. 25): Let $P(z)$ be a polynomial whose zeros $z_1, z_2, \cdots, z_n$ $(n \geqq 2)$ lie in $|z| \leqq 1$. Is it true that $P'(z)$ always has a zero in $|z - z_1| \leqq 1$?

In this note we answer this question in the affirmative if $|z_1| = 1$ for arbitrary n and if $|z_1| < 1$ for $n = 3, 4$. The case $n = 2$ is trivial.

We also show that the disk $|z - z_1| < 1$ always contains a zero of $P'(z)$ regardless of the location of the zeros if $|P'(z_1)| < n$ and if the polynomial $P(z)$ is normalized to be a monic polynomial.

2. The boundary case.

THEOREM 1. *Let $P(z)$ be a polynomial whose zeros $z_1, z_2, \cdots, z_n$ $(n \geqq 2)$ lie in $|z| \leqq 1$ such that $|z_1| = 1$. Then the disk $|z - z_1| \leqq 1$ always contains a zero of $P'(z)$. Furthermore the disk $|z - z_1| < 1$ always contains a zero of $P'(z)$ except when $P(z) = c(z^n - e^{i\theta})$.*

Proof. Without loss of generality we may assume that $z_1 = 1$, $z_k \neq 1$ for $k = 2, 3, \cdots, n$ and $P'(1) = 1$. We shall show that the polynomial $P'(z + 1)$ has at least one zero in the closed unit disk. If this is not so then the following representation of $P'(z + 1)$ is possible [1] for $|z| < 1$.

$$P'(z + 1) = (1 - zf(z))^{n-1} \tag{1}$$

where $f(z)$ is analytic in the unit disk and less than one in modulus.

From (1) by differentiation we obtain

$$P''(1) = (1 - n)f(0) . \tag{2}$$

The polynomial $Q(z)$ defined by the relation $P(z) = (z - 1)Q(z)$ satisfies $Q(1) = P'(1) = 1$ and $2Q'(1) = P''(1)$. Hence applying (2) we obtain

$$\begin{aligned} Q'(1) = \frac{Q'(1)}{Q(1)} &= \frac{1}{1 - z_2} + \frac{1}{1 - z_3} \\ &\quad + \cdots + \frac{1}{1 - z_n} = \frac{1 - n}{2} f(0) \end{aligned} \tag{3}$$

from which we deduce that $|Q'(1)| < (n-1)/2$. On the other hand since $|z_k| \leqq 1$, $\operatorname{Re} 1/(1-z_k) \geqq 1/2$ and thus $\operatorname{Re} Q'(1) \geqq (n-1)/2$. This contradiction proves the theorem.

To prove the second part of the theorem we observe that $|f(z)| \leqq 1$ even if $P'(z+1) \neq 0$ for $|z| < 1$, so that in this case we also obtain a contradiction unless all the z_k lie on the unit circumference and $f(z)$ is a constant of absolute value one. This implies that $P(z)$ has all its zeros on the unit circumference such that $P'(z)$ has an $(n-1)$ fold zero on the circle $|z-1| = 1$.

3. Third and fourth degree polynomials.

THEOREM 2. *Let $P(z)$ be a polynomial of degree three or four whose zeros lie in the closed unit disk. Then any circle of radius one about a zero of $P(z)$ contains a zero of $P'(z)$.*

Proof. We may assume that $P(z) = (z-x)Q(z)$, where $0 < x < 1$ and the zeros $z_k, k = 1, 2, \cdots, n$ of $Q(z)$ lie in $|z| \leqq 1$. We shall prove that the polynomial $f(z) = P'(z+x)$ has a zero in $|z| < 1$.

Consider the following polynomials

$$f(z) = \sum_{k=0}^{n} (k+1) \frac{Q^{(k)}(x)}{k!} z^k$$

$$g(z) = \sum_{k=0}^{n} \frac{1}{k-1} \binom{n}{k} z^k$$

and

$$h(z) = \sum_{k=0}^{n} \frac{Q^{(k)}(x)}{k!} z^k .$$

By a result due to Szegö [4] every zero γ of $h(z)$ has the form $\gamma = -\alpha\beta$, where β is a zero of $g(z)$ and α is a point belonging to a circular region containing all the zeros of $f(z)$. The zeros of $g(z)$ have the form $\beta = -1 + \sqrt[n+1]{1}$ such that $\beta \neq 0$. For $n = 2, 3$ $|\beta| \geqq \sqrt{2}$. If $f(z) \neq 0$ in $|z| < 1$ we may choose α such that $|\alpha| \geqq 1$. Thus $|\gamma| \geqq \sqrt{2}$. Since $h(z) = Q(z+x)$ and $f(z) = P'(z+x)$ it follows that all the zeros of $Q(z)$ satisfy $|z| \leqq 1$ and $|z-x| \geqq \sqrt{2}$ and no zero of $P'(z)$ lies in $|z-x| < 1$.

Consider now the polynomial $R(z) = P(z-1+x) = (z-1)Q_1(z)$, where $Q_1(z) = Q(z-1+x)$. No zero of $R'(z)$ lies in $|z-1| < 1$. By Theorem 1 we shall obtain a contradiction if we can show that all the zeros of $Q_1(z)$ lie in $|z| < 1$. Indeed the zeros of $Q_1(z)$ satisfy the inequalities $|z-1+x| \leqq 1$ and $|z-1| \geqq \sqrt{2}$. A straightforward calculation shows that if $z = u + iv$ these inequalities imply

$$u^2 + v^2 \leqq 3 - \left(x + \frac{1}{x}\right) < 1$$

for $0 < x < 1$. This completes the proof.

4. A particular class of polynomials.

THEOREM 3. *Let* $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$. *If* $P(z_1) = 0$ *and* $|P'(z_1)| < n$, *then* $P'(z)$ *has a zero in* $|z - z_1| < 1$.

Proof. Write $P(z) = (z - z_1)Q(z)$ and set $f(z) = P'(z + z_1)$ and $f^*(z) = z^{n-1}\bar{f}(1/z)$. We have $f(e^{i\theta}) = f^*(e^{i\theta})$ and

$$f(z) = nz^{n-1} + \cdots + Q(z_1)$$
$$f^*(z) = \overline{Q(z_1)}z^{n-1} + \cdots + n\ .$$

If $Q(z_1) \neq 0$ the polynomial $nf^*(z) - \overline{Q(z_1)}f(z)$ is of degree not exceeding $(n - 2)$ and since $Q(z_1) = P'(z_1)$ it follows by Rouché's theorem that $f^*(z)$ has at most $(n - 2)$ zeros in $|z| < 1$. Therefore $f(z)$ has at least one zero in $|z| < 1$. This means that $P'(z)$ has at least one zero in $|z - x| < 1$. If $Q(z_1) = 0$ then $P'(z_1) = 0$ and the same is true. From Theorem 3 we can deduce that Ilyeff's conjecture is true if all the coefficients of $Q(z)$ are less than one in modulus. This includes in particular the case where the theorem of Enström-Kakeya [3] is applicable, i.e. when the coefficients of $Q(z)$ form a monotonically decreasing sequence of positive numbers.

BIBLIOGRAPHY

1. J. Dieudonné, *Sur quelques applications de la théorie des fonctions bornées aux polynômes dont toutes les racines sont dans un domain circulaire donné*, Actualités Scient. et Industri. **114** (1934), 5-24.
2. W. K. Hayman, *Research Problems in Function Theory*, The Athlone Press, London, 1967.
3. M. Marden, *The Geometry of Polynomials*, Mathematical Surveys No. III, 2nd ed., Amer. Math. Soc., Providence, 1966.
4. G. Szegö, *Bemerkungen zu einem Satz von J. H. Grace über die Wurzeln algebraischer Gleichungen*, Math. Zeit. **13** (1922), 28-55.

Received April 24, 1967. This work was supported in part by the National Science Foundation GP-5221.

CLARK UNIVERSITY