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WEAKLY CLOSED DIRECT FACTORS OF SYLOW SUBGROUPS

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In many finite classical linear groups and permutation groups, certain Sylow subgroups have weakly closed direct factors. In this paper we establish a sufficient condition for this to occur in arbitrary finite groups.

The purpose of this paper is to prove the following result:

THEOREM A. *Let p be an odd prime, and let P be a Sylow p-subgroup of a finite group G. Suppose Q and R are subgroups of G such that $P = Q \times R$. Assume that no indecomposable factor of R is isomorphic to a subgroup of Q. Then P contains a weakly closed direct factor that is isomorphic to R.*

Our notation is taken from [3]. In addition, for every finite p-group P, we let

$$d(P) = \max. \{|A| \mid A \text{ is an Abelian subgroup of } P\}$$

and

$$J(P) = \langle A \mid A \text{ an Abelian subgroup of } P \text{ and } |A| = d(P)\rangle .$$

The following lemma is a special case of a result of Wielandt (Satz 6 of [9]).

LEMMA 1. *Let A and B be subgroups of a finite group G such that $G = AB$. Suppose p is a prime, A_p is a normal p-subgroup of A, and B_p is a normal p-subgroup of B. Then $\langle A_p, B_p\rangle$ is a p-group.*

Proof. By Sylow's Theorem, $\langle (A_p)^g, B_p\rangle$ is a p-group for some $g \in G$. Take $a \in A$ and $b \in B$ such that $ab = g$. Then $(A_p)^g = ((A_p)^a)^b = (A_p)^b$. Also, $(B_p)^{b^{-1}} = B_p$. Thus

$$\langle A_p, B_p\rangle = \langle (A_p)^a, (B_p)^{b^{-1}}\rangle = \langle (A_p)^g, B_p\rangle^{b^{-1}} ,$$

which is a p-group.

An automorphism α of a group G is said to be *central* if $g^\alpha g^{-1} \in Z(G)$ for all $g \in G$. We say that an element (or a subgroup) of Aut G *fixes* a subgroup H of G if it (or its elements) map H onto H.

THEOREM 1. *Let π be a set of primes and G be a finite π-group.*

Suppose $G = H \times K$ and no indecomposable factor of H is isomorphic to an indecomposable factor of K. Let $A = \operatorname{Aut} G$ and let C be the group of central automorphisms of G. Then G has the following properties:

(a) *If $H^* \cong H$, $K^* \cong K$, and $G = H^* \times K^*$, then $G = H^* \times K = H \times K^*$.*

(b) *The groups $H \times Z(K)$, $Z(H) \times K$, H', and K' are characteristic subgroups of G.*

(c) *There exists a normal, nilpotent π-subgroup D of A that is contained in C and permutes transitively the pairs (H^*, K^*) such that*

$$H^* \cong H,\ K^* \cong K,\ \text{and}\ G = H^* \times K^* .$$

(d) *If B is a π'-subgroup of A then there exists a pair (H^*, K^*) such that*

$$H^* \cong H, K^* \cong K, G = H^* \times K^* ,$$

and B fixes H^* and K^*. Moreover, if B fixes H, we may take $H^* = H$.

Proof. (a) Represent H and K as products of indecomposable factors, say, $H = H_1 \times \cdots \times H_r$ and $K = K_1 \times \cdots \times K_s$. Then $G = H \times K = H_1 \times \cdots \times H_r \times K_1 \times \cdots \times K_s$. Since $H^* \cong H$ and $K^* \cong K$, we have a similar representation

$$G = H^* \times K^* = H_1^* \times \cdots \times H_r^* \times K_1^* \times \cdots \times K_s^* .$$

Obviously, there exists a one-to-one correspondence ϕ between the factors F of the first representation and those of the second representation. By the Krull-Schmidt Theorem [7, p. 81], ϕ may be chosen to have the properties that $\phi(F) \cong F$ for each F and

$$G = \phi(H_1) \times \cdots \times \phi(H_r) \times K_1 \times \cdots \times K_s .$$

Clearly, for every H_i, $\phi(H_i)$ is some H_j^*. Hence $G = H^* \times K$. By symmetry, $G = H \times K^*$.

(b) Let $\alpha \in A$. Then $G = H^\alpha \times K^\alpha$. By (a), $G = H^\alpha \times K$. Thus

$$(C(K))^\alpha = (H \times Z(K))^\alpha \subseteq H^\alpha Z(G) \subseteq C(K) .$$

Hence $H \times Z(K)$ is a characteristic subgroup of G. Since $H' = (H \times Z(K))'$, H' is also a characteristic subgroup of G. By symmetry, $Z(H) \times K$ and K' are characteristic in G.

(c) For each $\alpha \in C$, define $\alpha - 1$ by $g^{\alpha-1} = g^{-1}g^\alpha$ for all $g \in G$. Since $\alpha \in C$, $\alpha - 1$ is an endomorphism of G and $G^{\alpha-1} \subseteq Z(G)$. Thus $g^{\alpha-1} = g^\alpha g^{-1}$ for all $g \in G$.

Let D_H be the group of all $\alpha \in C$ for which $g^\alpha = g$ for all $g \in H$ and $g^{\alpha-1} \in \boldsymbol{Z}(H)$ for all $g \in G$. Then

$$(1) \qquad H^{\alpha-1} = 1 \quad \text{and} \quad G^{\alpha-1} \subseteqq \boldsymbol{Z}(H), \quad \text{for} \quad \alpha \in D_H .$$

Define D_K similarly.

Suppose $\alpha \in D_H$. Let $\eta = \alpha - 1$. Take $g \in G$, and let $h = g^\eta$. By (1), it is clear by induction that

$$g^{\alpha^i} = gh^i \quad \text{for} \quad i = 1, 2, 3, \cdots .$$

Thus
(2) the order of α, the exponent of $G^{\alpha-1}$, and the exponent of $G/\mathrm{Ker}\,(\alpha - 1)$ are equal.
We also observe from (1) that if $\alpha, \beta \in D_H$, then $\alpha\beta = \beta\alpha$. Thus
(3) D_H is an Abelian π-group.

Suppose $\alpha \in D_H$, $\beta \in D_K$, and α and β have relatively prime orders. Let $g \in G$, and let $h = g^{\alpha-1}$ and $k = g^{\beta-1}$. Then $h \in \boldsymbol{Z}(H)$ and $k \in \boldsymbol{Z}(K)$. By (2), the order of h divides the order of α. Since an analogue of (2) also holds for elements of D_K, $h \in \mathrm{Ker}\,(\beta-1)$. Similarly, $k \in \mathrm{Ker}\,(\alpha-1)$. Hence

$$g^{\alpha\beta} = (g^\alpha)^\beta = (gh)^\beta = g^\beta h^\beta = g^\beta h = gkh = ghk$$

and

$$g^{\beta\alpha} = (g^\beta)^\alpha = (gk)^\alpha = g^\alpha k^\alpha = g^\alpha k = ghk = g^{\alpha\beta} .$$

Thus $\alpha\beta = \beta\alpha$. In particular, if p and q are distinct primes,
(4) the Sylow p-subgroup of D_H centralizes the Sylow q-subgroup of D_K.

Suppose $H^* \cong H$, $K^* \cong K$, and $G = H^* \times K^*$. By (a),

$$G = H \times K = H \times K^* = H^* \times K .$$

Define a mapping $\eta: G \to G$ as follows: For each $k \in K$, take $h' \in H$ and $k^* \in K^*$ such that $k = h'k^*$. Let $k^\eta = h'$. For $h \in H$ and $k \in K$, let

$$(hk)^\eta = k^\eta .$$

Then η is an endomorphism of G. Since K and K^* centralize H, $G^\eta = K^\eta \subseteqq \boldsymbol{Z}(H) \subseteqq \boldsymbol{Z}(G)$. Hence the mapping $\alpha: G \to G$ given by $g^\alpha = (g^\eta)^{-1}g$ is an endomorphism of G. Since $H^\alpha = H$ and $K^\alpha = K^*$, α is an automorphism of G. Clearly, $\alpha \in D_H$. Thus D_H permutes transitively all the direct factors of G that are isomorphic to K. Similarly D_K permutes transitively all the direct factors of G that are isomorphic to H.

Let A_H be the set of all $\alpha \in A$ such that $H^\alpha = H$. Define A_K similarly. Then

(5) $$D_H \lhd A_H \quad \text{and} \quad D_K \lhd A_K \,.$$

Let $\alpha \in A$. Then $H^\alpha \cong H$, $K^\alpha \cong K$, and $G = H^\alpha \times K^\alpha$. Hence there exists $\beta \in D_H$ such that $K^\beta = K^\alpha$. Therefore $K^{\alpha\beta^{-1}} = K$, and $\alpha\beta^{-1} \in A_K$. Thus $\alpha \in A_K A_H$. So

(6) $$A = A_K A_H = A_H A_K \,.$$

Let $I = A_H \cap A_K$, and take $\alpha \in A_H$. As in the previous paragraph, there exists $\beta \in D_H$ such that $K^\alpha = K^\beta$. Thus $\alpha\beta^{-1} \in A_H \cap A_K = I$. So $A_H = ID_H = D_H I$. Similarly, $A_K = ID_K = D_K I$.

Let p be a prime. By (5), $O_p(D_H)$ is a normal subgroup of A_H and $O_p(D_K)$ is a normal subgroup of A_K. Let $D_p = \langle O_p(D_H), O_p(D_K)\rangle$. By (5), (6), and Lemma 1, D_p is a p-group. By (3) and (4), every p'-element in D_H or D_K centralizes D_p. Since D_p normalizes itself, D_H and D_K normalize D_p. Since I normalizes D_H and D_K, I normalizes D_p. Hence

$$N(D_p) \supseteq \langle D_H, D_K, I\rangle = \langle D_H I, D_K I\rangle = A_H A_K = A \,.$$

Let D be the subgroup of C generated by the groups D_p for all primes p. Then $D_H \subseteq D$ and $D_K \subseteq D$, by (3). Suppose $H^* \cong H$, $K^* \cong K$, and $G = H^* \times K^*$. Then there exists $\alpha \in D_K$ and $\beta \in D_H$ such that $H^{*\alpha} = H$ and $((K^*)^\alpha)^\beta = K$. Now $\alpha\beta \in D$, $H^{*\alpha\beta} = H$, and $K^{*\alpha\beta} = K$. This completes the proof of (c).

(d) Retain the notation of (c). Then $I = A_H \cap A_K$ and $A = ID$. Since $D \subseteq BD \subseteq A = ID$, $BD = (BD \cap I)D$. Note that D is nilpotent and $|B|$ and $|D|$ are relatively prime. By Schur's Theorem [10, p. 162], $BD \cap I$ splits over $D \cap I$. Let B^* be a complement of $D \cap I$ in $BD \cap I$. Thus B^* is a complement of D in BD. By the Schur-Zassenhaus Theorem [10, p. 162], B^* is conjugate to B in BD. Take $\alpha \in BD$ such that $B = \alpha^{-1}B^*\alpha$. Since $B^* \subseteq A_H \cap A_K$, B fixes H^α and K^α.

If B fixes H, then $B \subseteq A_H = ID_H$. An argument similar to the previous one shows that $\alpha B \alpha^{-1} \subseteq I$ for some $\alpha \in BD_H$. Then B fixes H^α and K^α, and $H^\alpha = H$. This completes the proof of Theorem 1.

LEMMA 2. *Let p be a prime and P be a p-subgroup of a finite group G. Suppose H is a p'-subgroup of G that normalizes P. Then:*

(a) $P = [P, H]C_P(H)$;

(b) $[[P, H], H] = [P, H]$; *and*

(c) *if P is Abelian, then $P = [P, H] \times C_P(H)$.*

Proof. This result is well known. Parts (a) and (b) appear as Corollary 3 of Theorem 1 of [4]. Part (c) follows directly from part (a) and from the lemma on page 172 of [10].

LEMMA 3. *Let p be a prime and P be a p-subgroup of a finite group G. Suppose H is a p'-subgroup that normalizes P. Assume that*

(a) *P is Abelian and H centralizes $\Omega_1(P)$*
or that

(b) *P has no Abelian direct factors and H centralizes $P/\boldsymbol{Z}(P)$. Then H centralizes P.*

Proof. (a) By Lemma 2, $P=[P,H]\times C_P(H)$. Hence $\Omega_1([P,H])=1$. Therefore, $[P,H]=1$, i.e., H centralizes P.

(b) Let $Q=[P,H]$. Then $Q\subseteqq \boldsymbol{Z}(P)$, so Q is Abelian. By Lemma 2, $P=QC_P(H)$, $Q=[Q,H]$, and $Q\cap C_P(H)=[Q,H]\cap C_Q(H)=1$. Since $Q\subseteqq \boldsymbol{Z}(P)$, $C_P(H)\lhd P$. Hence $P=Q\times C_P(H)$. By (b), $Q=1$.

LEMMA 4. *Let P and Q be normal Abelian p-subgroups of a finite group G. Suppose that $Q\subseteqq P$ and that some Sylow p-subgroup of G normalizes some complement of Q in P. Then G normalizes some complement R of Q in P.*

Proof. By constructing a semi-direct product if necessary, we may assume that G is a splitting extension of P by a group E that is isomorphic to $G/C(P)$. Let S be a Sylow p-subgroup of E. Then S normalizes some complement R^* of O in P. Now, SP is a Sylow p-subgroup of G and SR^* is a complement of Q in SP. Thus SP splits over Q. By a theorem of Gaschütz [6, p. 246], G splits over Q. Let C be a complement of Q in G, and let $R=C\cap P$.

The following result is a special case of a theorem of Wielandt (Satz 12, page 193, of [8]).

LEMMA 5. *Suppose p is a prime and P is a Sylow p-subgroup of a finite group G. Let $n=|N(P)/P|$. Let V be the transfer of G into P/P'.*

(a) *If $a\in P\cap \boldsymbol{Z}(N(P))$ and $a^p=1$, then $V(a)=a^nP'$.*

Furthermore, suppose $P'\subseteqq Q\subseteqq P$ and suppose W is the transfer of G into P/Q. Then:

(b) *If $A\subseteqq P\cap \boldsymbol{Z}(N(P))$ and $A\cap Q=1$, then $A\cap G'=A\cap \operatorname{Ker} W=1$.*

(c) *If $Q\lhd N(P)$, then $\Omega_1(Q\cap \boldsymbol{Z}(P))\subseteqq \operatorname{Ker} W$.*

Proof. (a) Let $r=|G:P|$, and let Px_i, $i=1,2,\cdots,r$, be the distinct cosets of P in G. We may assume that

$$x_1,\cdots,x_n\in N(P);\ Px_ia=Px_i(1\leqq i\leqq s);$$
$$Px_ia\neq Px_i(s+1\leqq i\leqq r)\ ,$$

where $s \geqq n$. Since $a^p = 1$, Lemma 14.4.1, page 206, of [6] yields

$$V(a) = P' \prod_{i \leqq i \leqq s} x_i a x_i^{-1} .$$

Since $a \in \boldsymbol{Z}(\boldsymbol{N}(P))$,

$$V(a) = P' a^n \prod_{n < i \leqq s} x_i a x_i^{-1} . \tag{7}$$

Suppose $x \in P$ and $n < i \leqq s$. Then $(Px_i)x = Px_j$ for some j. Since

$$Px_j a = Px_i x a = Px_i a x = Px_i x = Px_j$$

and since $x_i \notin \boldsymbol{N}(P)$, $n < j \leqq s$. Thus P permutes the cosets Px_i, $n < i \leqq s$, by right multiplication. We may assume that $Px_{n+1}, \cdots, Px_t$ are representatives of the distinct orbits of P. For $i = n+1, \cdots, t$, let P_i be the subgroup of P fixing Px_i, and let $y_{i1}, \cdots, y_{im_i}$ be representatives of the distinct left cosets of P_i in P. Then the orbit of Px_i is $Px_i y_{ij}$, $1 \leqq j \leqq m_i$.

Suppose $n + 1 \leqq i \leqq t$. Since $x_i \notin \boldsymbol{N}(P)$, $Px_i P \neq Px_i$ Thus $P_i \subset P$ and

$$m_i \equiv |P : P_i| \equiv 0, \quad \text{modulo} \quad p . \tag{8}$$

We may assume that, for $k = n+1, \cdots, s$, every x_k has the form $x_i y_{ij}$ for some (unique) i and j. By (7) and (8),

$$\begin{aligned} V(a) &= P' a^n \prod_{n < i \leqq t} \prod_{1 \leqq j \leqq m_i} x_i y_{ij} a y_{ij}^{-1} x_i^{-1} \\ &= P' a^n \prod_{n < i \leqq t} (x_i a x_i^{-1})^{m_i} = P' a^n , \end{aligned}$$

as desired.

(b) Suppose $a \in A$ and $a^p = 1$. Now, W is simply the composition of V with the natural mapping of P/P' into P/Q. Hence $W(a) = a^n Q$, by (a). Since p does not divide n and since $a \notin Q$, $W(a) \neq Q$. Thus $A \cap \operatorname{Ker} W$ has no elements of order p, so $A \cap \operatorname{Ker} W = 1$. Since $G' \subseteqq \operatorname{Ker} W$, $A \cap G' = 1$.

(c) Let $B = \Omega_1(Q \cap \boldsymbol{Z}(P))$ and $N = \boldsymbol{N}(P)$. Since $N/\boldsymbol{C}_N(B)$ is a p'-group,

$$B = [B, N] \times \boldsymbol{C}_B(N) ,$$

by Lemma 2. Obviously, $[B, N] \subseteqq G' \subseteqq \operatorname{Ker} W$. Let $a \in \boldsymbol{C}_B(N)$. From (a),

$$W(a) = (a^n P')Q = a^n Q = Q ,$$

so $a \in \operatorname{Ker} W$. Thus $B \subseteqq \operatorname{Ker} W$. This completes the proof of Lemma 5.

We now require the following proposition, which is the main result of [5]:

THEOREM 2. *Let p be an odd prime, and let P be a Sylow p-subgroup of a finite group G. Suppose $x \in P \cap \boldsymbol{Z}(\boldsymbol{N}(\boldsymbol{J}(P)))$. Then $g^{-1}xg = x$ whenever $g \in G$ and $g^{-1}xg \in P$.*

THEOREM 3. *Let p be a prime, and let P be a Sylow p-subgroup of a finite group G. Suppose Q and R are normal subgroups of $\boldsymbol{N}(P)$ and $P = Q \times R$. Assume that $R \subseteqq \boldsymbol{O}_p(G)$ and that no indecomposable direct factor of R is isomorphic to a subgroup of Q. Then R' is a normal subgroup of G, and there exists a normal subgroup R^* of G such that $P = Q \times R^*$. Moreover, if p is odd and R/R' is a normal subgroup of $\boldsymbol{N}_{G/R'}(\boldsymbol{J}(P/R'))$, we may take $R^* = R$.*

Proof. Let $Q_1 = \boldsymbol{O}_p(G) \cap Q$. Since $R \subseteqq \boldsymbol{O}_p(G) \subseteqq P = R \times Q$, $\boldsymbol{O}_p(G) = R \times Q_1$. Now, no indecomposable factor of R is isomorphic to an indecomposable factor of Q_1. By Theorem 1, $R\boldsymbol{Z}(Q_1)$ and R' are characteristic subgroups of $\boldsymbol{O}_p(G)$ and are therefore normal subgroups of G.

Let $T = R\boldsymbol{Z}(Q_1) = \boldsymbol{Z}(Q_1) \times R$. Represent R as a direct product of an Abelian subgroup R_a and a subgroup R_b having no Abelian direct factors. By Theorem 1, we may assume that R_a and R_b are normalized by a complement of P in $\boldsymbol{N}(P)$ and are therefore normal in $\boldsymbol{N}(P)$. If $R_a \neq 1$, let p^e be the minimum of the exponents of the indecomposable factors of R_a. If $R_a = 1$, let $p^e = p|T|$. Then let

$$T_0 = \langle x^{p^{e-1}} \mid x \in T \rangle .$$

Now $T_0 \lhd G$ and

$$\Omega_1(R_a) \subseteqq T_0 \subseteqq R . \tag{9}$$

Since Q centralizes R, Q centralizes T_0 and $T/\boldsymbol{Z}(T)$. Let

$$C = \boldsymbol{C}_G(T/\boldsymbol{Z}(T)) \cap \boldsymbol{C}_G(T_0) \quad \text{and} \quad H = CT .$$

Then C and H are normal in G and $P = QR \subseteqq CT = H$.

Let K be a complement of P in $\boldsymbol{N}_H(P)$. Since $H/C \cong T/(C \cap T)$, $K \subseteqq C$. Thus $[T, K] \subseteqq \boldsymbol{Z}(T)$ and K centralizes T_0. Therefore $[R_b, K] \subseteqq \boldsymbol{Z}(R_b)$ and, by (9), K centralizes $\Omega_1(R_a)$. By Lemma 3, K centralizes R_a and R_b. So K centralizes R.

Let $\bar{H} = H/R'$, $\bar{R} = R/R'$, $\bar{K} = KR/\bar{R}'$, and so forth. Then $\bar{R} \subseteqq \boldsymbol{Z}(\bar{P})$ and $\boldsymbol{N}_{\bar{H}}(\bar{P}) = \bar{P}\bar{K}$, so

$$\boldsymbol{N}_{\bar{H}}(\bar{P}) \quad \text{centralizes } \bar{R} . \tag{10}$$

Let W be the transfer of $\bar{H}$ into $\bar{P}/\bar{Q}$. By Lemma 5(b),

$$\bar{R} \cap \bar{H}' \subseteqq \bar{R} \cap \operatorname{Ker} W = 1 . \tag{11}$$

By the Frattini argument,

$$G = HN(P) . \tag{12}$$

Suppose p is odd and $\bar{R} \lhd N_{\bar{G}}(J(\bar{P}))$. Then by (11)

$$[\bar{R}, N_{\bar{H}}(J(\bar{P}))] \subseteqq \bar{R} \cap \bar{H}' = 1 .$$

Thus by Theorem 2 no element of $\bar{R}$ is conjugate to any other element of $\bar{P}$. Since $\bar{R} \subseteqq O_p(\bar{G}) \subseteqq \bar{P}$, we must have $\bar{R} \subseteqq Z(\bar{H})$. Therefore, $R \lhd H$. By (12) R is normal in G, as claimed.

Let us return to the general case. Now, $\bar{P} = \bar{Q} \times \bar{R}$. By (11), $\bar{R} \cap \operatorname{Ker} W = 1$. Since

$$|\operatorname{Image}(W)| \leqq |\bar{P}/\bar{Q}| = |\bar{R}| ,$$

$\bar{R}$ is a complement to $\operatorname{Ker} W$ in $\bar{H}$. Hence $\bar{R}$ is a complement to $\bar{T} \cap \operatorname{Ker} W$ in $\bar{T}$. Since W depends only on $\bar{H}$ and $\bar{Q}$ and since $N(P)$ normalizes H and Q, $N(P)$ normalizes $\operatorname{Ker} W$. By (12), $\bar{G}$ normalizes $\operatorname{Ker} W$. Hence $\bar{T} \cap \operatorname{Ker} W \lhd \bar{G}$. Now $\bar{T}' = \bar{R}' = 1$ and $\bar{P}$ normalizes $\bar{R}$. By Lemma 4, there exists a complement $\bar{R}^*$ of $\bar{T} \cap \operatorname{Ker} W$ in $\bar{T}$ such that $\bar{R}^* \lhd \bar{G}$. Let R^* be the subgroup of T that contains R' and satisfies $R^*/R' = \bar{R}^*$.

By Lemma 5, $\Omega_1(Z(\bar{Q})) \subseteqq \operatorname{Ker} W$. Since $\Omega_1(Z(Q))R'/R' \subseteqq \Omega_1(Z(\bar{Q}))$, (11) yields

$$\Omega_1(Z(Q)) \cap R^* \subseteqq \Omega_1(Z(Q)) \cap R' \subseteqq Q \cap R = 1 .$$

Hence $Q \cap R^*$ is normal in Q but intersects $Z(Q)$ in 1, so $Q \cap R^* = 1$. Consequently, $|QR^*| = |Q||R^*| = |Q||R| = |P|$. Since $Q, R^* \lhd P$, $P = Q \times R^*$. This completes the proof of Theorem 3.

We now require the following concepts and results of Alperin and Gorenstein (§ 2 of [2] and § 5 of [1]):

DEFINITION. Let G be a finite group and p be a prime. Let $\mathscr{H}$ be the set of all nonidentity p-subgroups of G. A *conjugacy functor* W on $\mathscr{H}$ is a mapping from $\mathscr{H}$ into $\mathscr{H}$ that satisfies the following two conditions for each H in $\mathscr{H}$:

(a) $W(H) \subseteqq H$;

(b) $W(H^x) = W(H)^x$ for all $x \in G$.

THEOREM 4. *Let p be a prime and P be a nonidentity Sylow p-subgroup of a finite group G. Let W be a conjugacy functor on the set of nonidentity p-subgroups of G. Then there exists a class* *of nonidentity subgroups of P, called well-placed subgroups, having the following properties*:

(1) *If H is a well-placed subgroup then $N(H) \cap P$ is a Sylow p-subgroup of $N(H)$, and $W(N(H) \cap P)$ is a well-placed subgroup.*

(2) *Suppose $R \subseteqq P, g \in G$, and $R^g \subseteqq P$. Then there exists a sequence of well-placed subgroups $H_1, \cdots, H_n$ and elements $x_1, \cdots, x_n$ of G such that*

(a) $g = x_1 \cdots x_n$,

(b) $x_i \in N(H_i), 1 \leqq i \leqq n$, *and*

(c) $R \subseteqq H_1$ *and* $R^{x_1 \cdots x_i} \subseteqq H_{i+1}, 1 \leqq i \leqq n-1$.

Theorem 4 easily yields the following result:

COROLLARY. *Let p be a prime and P be a Sylow p-subgroup of a finite group G. Suppose $Q \subseteqq P$ and Q is not weakly closed in P with respect to G. Then there exists $H \subseteqq P$ and $g \in N(H)$ such that H is well-placed, $Q \subseteqq H$, and $Q^g \neq Q$.*

THEOREM 5. *Let p be a prime, and let P be a Sylow p-subgroup of a finite group G. Suppose $P = Q \times R$ and no indecomposable direct factor of R is isomorphic to a subgroup of Q. Let J be the subgroup of P that contains R' and satisfies $J/R' = J(P/R')$. Then*

(a) *There exists $R^* \lhd N(J)$ such that $P = Q \times R^*$.*

(b) *If p is odd and R^* satisfies* (a), *R^* is weakly closed in P with respect to G.*

Proof. (a) Let K be a complement of P in $N(P)$. By Theorem 1, we may assume that K normalizes Q and R. Hence $Q, R \lhd N(P)$. Since $R/R' \subseteqq Z(P/R')$,

$$R \subseteqq J \subseteqq O_p(N(J)) .$$

Thus, (a) follows from Theorem 3.

(b) Assume p is odd and R^* satisfies (a) but is not weakly closed in P. We may assume that $R = R^*$. By a theorem of Burnside [6, p. 46], there exists a subgroup P_0 of P such that $P_0 \supseteqq R$ and $R \not\lhd N(P_0)$. Since

$$R \subseteqq P_0 \subseteqq P = R \times Q, \qquad P_0 = R \times (P_0 \cap Q) .$$

By Theorem 1 and our hypothesis on Q and on R, $R' \lhd N(P_0)$. Therefore, R is not weakly closed in P with respect to $N(R')$. Since $P \subseteqq N(J) \subseteqq N(R')$, we may assume that $R' \lhd G$.

We define a conjugacy functor W on the set of nonidentity subgroups H of G as follows:

$$W(H) = H, \quad \text{if} \quad R' \not\subseteq H\,;$$

and

$$R' \subseteq W(H) \quad \text{and} \quad W(H)/R' = J(H/R'), \quad \text{if} \quad R' \subseteq H\,.$$

By the Corollary of Theorem 4, there exists a well-placed subgroup H of G having the properties that $H \supseteq R$ and $R \ntriangleleft N(H)$. Choose H such that $P \cap N(H)$ has maximal order subject to these conditions. Let $P_1 = P \cap N(H)$. Since H is well-placed, P_1 is a Sylow p-subgroup of $N(H)$. By Theorem 3, $R/R' \ntriangleleft N_{G/R'}(J(P_1/R'))$. Hence $P_1 \subset P$ by (a). But $J(P_1/R') = W(P_1)/R'$. Thus $R \subseteq P_1$ and $R \ntriangleleft N(W(P_1))$. Since H is well placed and $P_1 \subset P$, $W(P_1)$ is well placed and

$$P_1 \subset P \cap N(P_1) \subseteq P \cap N(W(P_1))\,.$$

But this contradicts the choice of H. Thus we have proved Theorem 5. Theorem A obviously follows from Theorem 5.

REMARK. Let A^n and S^n be the alternating and symmetric groups of degree n, for $n = 4, 6$. Since Theorem 2 holds for $p = 2$ when S^4 is not involved in G [5], Theorem A holds for $p = 2$ when S^4 is not involved in $N(R')/R'$.

Let $H = A^6$, and let R be an indecomposable 2-group of order greater than eight. Take a transposition τ in S^6 and a subgroup R_0 of index two in R. Consider R as an operator group on H by defining $h^r = h$ when $r \in R_0$ and $h^r = \tau^{-1}h\tau$ when $r \in R$ and $r \notin R_0$. Let G be the semi-direct product of H by R, and embed H and R in G in the natural manner. Then $C_H(R)$ contains a Sylow 2-subgroup Q of H. Let $P = Q \times R$. Then P is a Sylow 2-subgroup of G and R is not isomorphic to any subgroup of Q, but P has no weakly closed direct factor isomorphic to R.

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