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A MAXIMUM PRINCIPLE AND GEOMETRIC PROPERTIES OF LEVEL SETS

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There are many results in function theory which relate the behavior of a function in the interior of a domain to its behavior on the boundary. A well known result of this sort is the theorem of study: if the map of the unit disc under a univalent analytic function $f(z)$ is convex, then the map of every concentric disc contained therein is also convex. This theorem has been generalized in many different directions including more general properties of univalent functions, and the convex and star-shaped properties for level surfaces of harmonic functions in E^3. The results for univalent functions depend basically upon Schwarz's lemma, while the results for level surfaces of harmonic functions have been shown previously by means of rather complicated forms of the maximum principle.

In § 1, we give a simple and direct proof of a very general theorem, depending upon a form of the maximum principle, which is then shown in § 2 to easily give the known results as well as several new ones. Some related new problems are discussed in § 3.

1. Main result.

THEOREM 1. *Let C_{0j} and $C_{1j}, j = 0, 1, \cdots, n$, be closed subsets of Euclidean m-space, E^m, such that $C_{0j} \cap C_{1j} = \emptyset$, and define*:

$$\begin{aligned} &p = (P_1, \cdots, P_n), P_j \in E^m \ , \\ &A_j = \partial C_{0j}, B_j = \partial C_{1j}, D_j = E^m - (C_{0j} \cup C_{1j}) \ , \\ &C_0 = C_{01} \times \cdots \times C_{0n} \qquad C_1 = C_{11} \times \cdots \times C_{1n} \ . \end{aligned}$$

Suppose continuous functions $f_j(P_j)\colon E^m \rightarrow E^1$ and $T(p)\colon E^{mn} \rightarrow E^m$ are given such that the following conditions are satisfied for $j = 1, \cdots, n$:

(i) *The sets C_{0j} and C_{1j} are level sets of $f_j(P_j)$ such that*

$$f_j(P_j) = \begin{cases} 0 \ \text{for} \ P_j \in C_{0j} \\ K \ \text{for} \ P_j \in C_{1j} \end{cases} \qquad \begin{aligned} &j = 0, \cdots, n \\ &K \in (0, \infty] \ . \end{aligned}$$

$0 < f_j(P_j) < K$ *for all* $P_j \in D_j, j = 0, 1, \cdots, n$

(ii) $H_j(p) \equiv f_j(P_j) - f_0(T(p))$ *takes its maximum over all*

$$p \in N_j \equiv \{p\colon P_j \in D_j, \ T(p) \in D_0, f_j(P_j) \leqq f_i(P_i), \ i \neq j\}$$

in the set $\partial M_j \cap \partial N_j$, where $M_j \equiv \{p\colon P_j \in D_j, \ T(p) \in D_0\}$.

(iii) *Define* $D(k) = \text{interior } \{p: f_j(P_j) \geqq k, j = 1, \cdots, n\}$

$$D_0(k) = \text{interior } \{P_0: f_0(P_0) \geqq k\}$$
$$D(0) = \{p: f_j(P_j) > 0, j = 1, \cdots, n\}$$
$$D_0(0) = \{P_0: f_0(P_0) > 0\}$$

for $k \in (0, K]$. *We assume that* $T(p) \in D_0(0)$ *whenever* $p \in D(0)$ *and* $T(p) \in D_0(K)$ *whenever* $p \in D(K)$.

(iv) *If* $K = \infty$ *we assume that* $H_j(p)$ *approaches a nonpositive limit at every point of* $\partial D(K)$.

Conclusion: $T(p) \in D_0(k)$ *whenever* $p \in D(k)$ *for* $k \in [0, K]$.

Proof. The conclusion is assumed to be true when $k = 0$ or $k = K$ in hypothesis (iii). If $k \in (0, K)$, suppose $p \in D(k)$. Let j be such that $f_j(P_j) \leqq f_i(P_i)$ for $i = 1, \cdots, n$. We know that $p \in D(k) \Rightarrow p \in D(0) \Rightarrow T(p) \in D_0(0)$. If $T(p) \in C_{10}$ then obviously $H_j(p) \leqq 0$ because $0 < f_j(P_j) < K$. If $P_j \in C_{1j}$ then by our choice of j we have $f_i(P_i) \geqq f_j(P_j) = K$ and so $p \in C_1$, therefore $T(p) \in D_0(K) = C_{10}$, and $H_j(p) = 0$. Thus $p \in N_j$ is the only case left to consider and we argue as follows. Hypothesis (ii) then says that $H_j(p) \leqq H_j(Q)$ for some $Q \in \partial M_j \cap \partial N_j$. The point Q must be such that either $Q_j \in A_j$ or $Q_j \in B_j$ as we have eliminated above the other possibilities. In the former case $H_j(Q) \leqq 0$ holds because $f_j(Q_j) = 0$ and $f_0(P_0) \geqq 0$. In the latter case we know that $f_i(Q_i) \geqq f_j(Q_j) = K$, and thus $Q \in \overline{D(K)}$. This implies that $T(Q) \in \overline{D_0(K)}$ and therefore $H_j(Q) \leqq 0$. When $K = \infty$, hypothesis (iv) allows us to draw this conclusion. Thus $H_j(p) \leqq 0$, which means that $T(p) \in D_0(k)$ as required.

THEOREM 2. *Hypothesis* (ii) *in Theorem* 1. *is implied by the following conditions*:

(ii)′ *Let* $J = \{1, \cdots, n\}, S \subset J, I = J - S = \{i1, \cdots, ih\}$

$$p = (P_{i1}, \cdots, P_{ih}), q = (Q_{i1}, \cdots, Q_{ih}) .$$

For every $j \in J$, S *such that* $j \notin S$, *and* Q *such that* $f_s(Q_s) = f_j(Q_j)$ *for* $s \in S$, *we assume that there exists a function* $t(p) = (t_0(p), \cdots, t_n(p))$ *such that*:

$$t_i(p) = P_i \text{ for } i \in I$$
$$t_s(q) = Q_s \text{ and } f_s(t_s(p)) = f_j(P_j) \text{ for } s \in S$$

$H_j(t(p)) = f_j(P_j) - f(T(t(p)))$ *takes its maximum over all*

$$p \in N_j(S) = \{p: P_j \in D_j, T(t(p)) \in D_0, f_j(P_j) < f_i(P_i), i \in I, i \neq j\}$$

in the set $\partial N_j(S)$.

Comment. This condition gives us a means of eliminating those boundary points of N_j for which $f_j(P_j) = f_i(P_i)$ for some $i \neq j$. In our applications the functions $t(p)$ will be rotations of level surfaces of the functions f_j.

Proof. When $n = 1$, there is only one $f_j(P_j)$ and the theorem is clearly true; no functions $t(p)$ are needed. For $n > 1$, the argument proceeds by induction on n; the assumed existence of functions $t(p)$ is exactly what is needed at each step in the induction.

2. **Applications.** The first of the following two theorems is due to Beckenbach and Graham [2], whose proof depends upon Schwarz's lemma. The second is a doubly connected version of the first and is closely allied with the Hadamard Three Circles Theorem.

THEOREM 3. *Let the analytic function $f(z)$ with $f(0) = 0$, regular in the unit circle $|z| < 1$, map $|z| < 1$ in a one-to-one way on a plane domain, and let the map of $|z| < r, 0 < r \leqq 1$, be denoted by $D(r)$. Let the constants $r_j, j = 0, 1, \cdots, n$, satisfy $0 < r_j \leqq 1$, w denote the vector $(w_1, \cdots, w_n)$, $w_j = f(z_j)$, and $T(w)$ be a regular analytic function of the n complex variables $w_1, \cdots, w_n$ with $D(r_j)$ as range of the variable w_j, and $T(0) = 0$. Suppose that for each vector w with w_j in $D(r_j)$, the point $T(w) = w_0$ is in $D(r_0)$. Then for each w with w_j in $D(r_j r)$, $0 < r \leqq 1$, the point $T(w) = w_0$ is in $D(r_0 r)$, whenever r_0 satisfies $r_0 \geqq \sum_1^n a_i r_i$, where a_i is the coefficient of w_i in the expansion of $T(w)$.*

THEOREM 4. *Let the analytic function $f(z)$, regular in the annulus $\rho < |z| < 1$, map $\rho < |z| < 1$ in a one-to-one way on a plane domain bounded by two simple closed curves such that the inner curve B is the map of $|z| = \rho$, the outer curve A is the map of $|z| = 1$, and the orientation is preserved. Let $D(r)$ denote the interior of the image of $|z| = r$, the constants $r_j, j = 0, 1, \cdots, n$, satisfy $\rho \leqq r_j \leqq 1$, and $T(w)$ be a regular analytic function of the n complex variables $w_1, \cdots, w_n$, with $D(r_j)$ as range of the variable w_j. Suppose that for each w with w_j in $D(r_j)$, the point $T(w) = w_0$ is in $D(r_0)$, and for each w with w_j in $D(\rho)$ the point $T(w)$ is in $D(\rho)$. Then for each w with w_j in $D(r_j(\rho/r_j)^\sigma)$, $0 \leqq \sigma \leqq 1$, the point $T(w) = w_0$ is in $D(r_0(\rho/r_0)^\sigma)$.*

Proof of Theorem 3. We define the functions

$$f_j(w_j) = -\log|f^{-1}(w_j)| + \log r_j, j = 0, 1, \cdots, n$$
$$A_j = f\{z: |z| = r_j\}, B_j = \{f(0)\} = \{0\}, \text{ and } K = \infty .$$

Hypothesis (i) of Theorem 1 is clearly satisfied and (iii) is assumed to be true. We shall give explicitly a function $t(p)$ satisfying (ii)$'$ of Theorem 2. Let $p = (w_1, \cdots, w_n)$, $q = (w_{01}, \cdots, w_{0h})$ and define the constants α_s by $z_{0s} = \exp(i\alpha_s)z_{0j}$. The function $t(p) = (t_1(w_j), \cdots, t_n(w_j))$, where $t_s(w_j) \equiv f(f^{-1}(w_j)\exp(i\alpha_s))$, satisfies the requirements in Theorem 2 because the $t_s(w_j)$ are analytic and thus $T(t(w_j))$ satisfies the principle of the maximum. It remains to show that hypothesis (iv) holds. We observe that by the construction of the domain N_j we have $f_j(w_j) \leqq f_i(w_i)$ and thus $r_i^{-1}|f^{-1}(w_i)| \leqq r_j^{-1}|f^{-1}(w_j)|$, from which

$$\begin{aligned} H_j(t(w_j)) &\equiv f_j(w_j) - f_0(T(t(w_j))) \\ &= \log\left|\frac{r_j f^{-1}(T(t(w_j)))}{r_0 f^{-1}(w_j)}\right| \leqq 0 \end{aligned}$$

whenever w_j is small enough and $r_0 \geqq \sum_1^n a_i r_i$, since $T(0) = 0$ and $t(0) = 0$.

Proof of Theorem 4. Let

$$f_j(w_j) = (\log|f^{-1}(w_j)| - \log r_j)/(\log\rho - \log r_j)$$

and $K = 1$. Then hypothesis (iv) is unnecessary and the other hypotheses follow just as in the proof of Theorem 3, with the same functions $t(w_j)$.

Theorem 3 contains a result of Ford [4] as a special case. A domain is said to be star-shaped if it satisfies hypothesis (iii) with $T(P_1) = \lambda P_1$, and convex if $T(P_1, P_2) = \lambda P_1 + (1-\lambda)P_2$ for $0 \leqq \lambda \leqq 1$. Beckenbach and Graham [2] derive many well known (and some not well known) results from Theorem 3. These include the theorem of Study [24] and Radó [16] for convex domains, and the similar results of Takahashi [25], Seidel [22], and Nabetani [13] for star-shaped domains. The analogous results for doubly connected domain were given by Komatu [12], who used very precise estimates obtained from the theory of elliptic functions in the proof. His theorems (Theorem 5) follow directly from Theorem 4.

THEOREM 5. *If $D(1)$ and $D(\rho)$ are both star-shaped with respect to a point inside the map of $|z| = \rho$ then $D(r)$, $\rho \leqq r \leqq 1$ is also star-shaped with respect to that point. If $D(1)$ and $D(\rho)$ are both convex, then $D(r)$ is convex.*

A different approach has been used by Walsh [26, 27, 28] to study the shape of level curves of the Green's function of a function of a complex variable in terms of its curvature.

We now prove a theorem, concerned with functions which satisfy elliptic partial differential equations and the star-shaped and convex properties in E^3, which contain the previously known results in harmonic functions.

THEOREM 6. *Let $f(P)$ be defined on a doubly connected domain $D \subset E^3$, and suppose that $f(P)$ and $f(\lambda P)$ satisfy the same elliptic partial differential equation there. Let A and B be the outer and inner boundary surfaces of D and suppose*

$$f(P) = \begin{matrix} 0 \ \ for \ P \in A \\ K \ for \ P \in B \end{matrix} \quad where \ K \in (0, \infty] .$$

We allow the case $K = \infty$ only when $B = \{0\}$, and assume that f is sufficiently well behaved at 0 that $f(\lambda P) \geqq f(P)$ for $\lambda \leqq 1$ and $|P|$ sufficiently small in this case.

Proof. We first observe that if $K = \infty$, then hypothesis (iv) of Theorem 1 is satisfied because

$$|\lambda P| \leqq \lambda |P| \leqq |P| \text{ and } |\lambda P_1 + (1 - \lambda)P_2| \leqq \max(|P_1|, |P_2|) .$$

Hypotheses (i) and (iii) are satisfied by assumption so we turn our attention to the verification of condition (ii)′. First of all, solutions of elliptic partial differential equations satisfy the maximum and minimum principles (see e.g., Hopf [9]) and $H(P) = f(P) - f(\lambda P)$ satisfies an elliptic equation by hypothesis. Thus the proof is completed for the star-shaped case because no functions $t(p)$ are needed. It remains to be shown that, in the convex case, there is a function $t(p)$ satisfying (ii)′. But since $D(0)$ and $D(K)$ are convex, they are also star-shaped with respect to any point $R \in D(K)$. We can define $t(P_j)$ as the unique point obtained by rotating E^3 about R such that Q_j goes into Q_s (here $j = 1$ or 2 and $s \neq j$) followed by a projection along the ray through R upon the level surface determined by P_j, because every level surface is star-shaped by the first part of this theorem. We now note that the function $H_j(t(P_j))$ satisfies the maximum principle because the function $t(P_j)$ is one-to-one in a neighborhood of P_j and a maximum of $H_j(t(P_j))$ at P_j would thus imply a maximum of $H_j(P)$ at $P = t(P_j)$, contradicting the theorem of Hopf. Therefore (ii)′ is satisfied and the proof is complete.

We list some special cases of Theorem 6 which have occurred in the literature. For the Greens function of a star-shaped domain: Gergen [7] and Warschawski [29]; for the Greens function of a convex domain: Gabriel [5, 6]; for convex logarithmic potential surfaces: Nikliborc [14]; for harmonic functions and doubly connected domains when the two given level surfaces are star-shaped: Brunner [3]; and for harmonic functions on doubly connected domains when the two given level surfaces are convex: Stoddart [23].

Theorem 6 can be easily extended to many other classes of functions. $f_1(P), \cdots, f_n(P)$ could be subfunctions and $f_0(P)$ could be a superfunction of such a solution of an elliptic equation. Solutions of certain parabolic equations also are known to satisfy a maximum principle (see e.g., Nirenberg [15]), as well as various types of hyperbolic and mixed elliptic-hyperbolic equations (Germain and Bader [8]; Agmon, Nirenberg and Protter [1]), from which mutations of Theorem 6 may be formed. The condition that $f(P)$ and $f(\lambda P)$ satisfy the same equation can be relaxed; all that is needed is that $H(P) = f(P) - f(\lambda P)$ be a solution of an elliptic partial differential equation in the star-shaped case. A similar comment applies to the convex case.

3. Some general problems. We have investigated the geometric behavior of a real valued function $f(P)$ in domains $D(k)$ bounded by level sets (curves, surfaces, etc.) on which $f(P)$ has a constant value $k > 0$. If $D(K)$ and $D(0)$ have a property, then what can be said about $D(k)$, $0 \leqq k \leqq K$? In general, very little can be said about $D(k)$, but we have shown in Theorem 1, under certain conditions on the function and geometric property, that $D(k)$ also has the same property.

We shall discuss the situation in the plane, where the level sets are closed curves. The analogous problems for level sets in higher dimensional space, in particular surfaces in E^3, have been investigated only to the extent given in this paper.

For certain geometric properties (those characterized by an analytic function $T(w)$) Theorem 3 allows us to conclude that if $D(1)$ has property T, then $D(r)$ has property T for $0 \leqq r \leqq 1$. We might ask for the largest r, $0 \leqq r \leqq 1$, such that $D(r)$ has property S when $D(1)$ has property T, and call this r the radius of property S. There are many results of this type in the theory of univalent functions; most of them are concerned with the star-shaped and convex properties (e.g., radius of convexity when $D(1)$ is star-shaped) and variants of them (typically real, close to star, close to convex, starlike of order α, etc.). Recently Hummel [10, 11] and Robertson [17, 18, 19] have used the variational techniques of Schiffer [20, 21] to develop a systematic approach to these problems, in terms of the variation of functions of positive real part. Extensions of the results of this paper (Theorems 1 and 3) to radius of property S have not as yet been accomplished, and provide interesting future research possibilities.

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