PACIFIC JOURNAL OF MATHEMATICS
Vol. 27, No. 3, 1968

COMMUTATIVE SEMIGROUPS WHICH ARE ALMOST FINITE

BRUCE A. JENSEN AND DONALD W. MILLER

Semigroups satisfying certain finiteness conditions are studied. It is shown that an infinite semigroup S every proper subsemigroup of which is finite is a group; thus in particular if S is commutative then it is isomorphic to the group $Z(p^\infty)$ for some prime p. An infinite commutative semigroup every proper homomorph of which is finite is shown to be imbeddable in an infinite cyclic group with zero element adjoined and its structure is described.

In his monograph on infinite abelian groups Kaplansky [2] includes the following two exercises concerning an infinite abelian group G:

(I) If every proper subgroup of G is finite then, for some prime p, G is isomorphic to $Z(p^\infty)$.

(II) If every proper homomorph of G is finite then G is an infinite cyclic group.

The converse of each of these implications is, of course, also true.

It is natural to ask what conclusions can be drawn for commutative semigroups under analogous hypotheses. In §1 it is shown that if S is an infinite semigroup each of whose proper subsemigroups is finite then S is a group. Thus in particular if S is commutative the conclusion of (II) is obtained.

A semigroup S is said to be homomorphically finite, or, for brevity, *HF*, if S is infinite while each proper homomorph of S is finite. Section 2 is devoted to showing that an infinite commutative semigroup S is *HF* if and only if S is imbeddable in an infinite cyclic group. A description of all such semigroups is given.

The notation and terminology used in this paper follow that of Clifford and Preston [1].

1. Infinite semigroups whose proper subsemigroups are finite.

We begin with a lemma.

LEMMA 1. *Let S be an infinite semigroup having no proper infinite subsemigroups. Then:*

(i) *S is periodic;*

(ii) $S^2 = S$;

(iii) *S is not nil.*

Proof. (i) If $a \in S$ then either $\langle a \rangle$, the subsemigroup of S

generated by a, is finite or $S = \langle a \rangle$ is cyclic. In the latter case, however, S contains the proper infinite subsemigroup $\langle a^2 \rangle$, contrary to hypothesis.

(ii) If $x \in S \backslash S^2$, the complement of S^2 in S, then $S \backslash \{x\}$ is a proper infinite subsemigroup of S. Hence $S = S^2$.

(iii) If S contains no zero element, (iii) holds by default. Hence suppose that 0 is a zero element of S and that S is nil. By (ii) we can choose $a \in S$ such that $aS \neq 0$. Hence either aS is finite or $aS = S$. If $aS = S$ then $S = a^n S$ for every positive integer n so, since S is nil, $S = a^k S = 0S = 0$ for some positive integer k, a contradiction. Hence assume that $aS = \{x_0, x_1, \cdots, x_n\}$, where $n > 0$, $x_0 = 0$ and $x_i \neq x_j$ for $i \neq j$. For $i = 0, 1, \cdots, n$, define

$$S_i = \{y \in S \mid ay = x_i\}$$

and define a binary relation $\leqq$ on the set $\mathscr{S}$ of all S_i by stipulating that $S_i \leqq S_j$ if and only if there exists s in S^1 such that $x_j = x_i s$. Clearly $\leqq$ is reflexive and transitive on $\mathscr{S}$. Moreover suppose $S_i \leqq S_j$ and $S_j \leqq S_i$, say $x_j = x_i s$ and $x_i = x_j t$, where s, $t \in S^1$. Then

$$x_j = x_j (ts)^k, \qquad k = 1, 2, 3, \cdots .$$

Since S is nil this implies that either $x_j = 0$, whence also $x_i = 0$, or $ts \notin S$. In the latter case, $s = t = 1$ so again $x_i = x_j$. Thus $\leqq$ is a partial ordering of $\mathscr{S}$.

Evidently $S_i \leqq S_0$ for $i = 0, 1, \cdots, n$. Moreover there must exist an integer $N, 1 \leqq N \leqq n$, such that

$$(1) \qquad S_i \leqq S_N \text{ implies } i = N, \text{ all } S_i \in \mathscr{S} .$$

Let $y \in S_N$. By (ii) $y = uv$ for some $u, v \in S$. Since $\mathscr{S}$ describes a partition of $S, u \in S_i$ for exactly one $i, 0 \leqq i \leqq n$. Therefore $x_N = ay = auv = x_i v$ so $S_i \leqq S_N$ whence, by (1), $i = N$ and $x_N = x_N v$. Consequently $x_N = x_N v^k$ for $k = 1, 2, 3, \cdots$ so $x_N = 0 = x_0$, contrary to $N > 0$. This establishes (iii).

THEOREM 1. *If S is an infinite semigroup each of whose proper subsemigroups is finite then S is a group.*

Proof. Let $A_1 = \{x \in S \mid xS \text{ is finite}\}$ and $A_2 = \{x \in S \mid xS = S\}$. For $i = 1, 2$, if $A_i \neq \varnothing$ (the null set) then A_i is a subsemigroup of S. Thus, since A_1 and A_2 partition S, either $A_1 = S$ and $A_2 = \varnothing$ or vice versa. An analogous argument on the principal left ideals of S leads to the conclusion that S satisfies exactly one of the following:

(i) xS and Sx are finite, all $x \in S$;

(ii) xS is finite and $Sx = S$, all $x \in S$;

(iii) $xS = S$ and Sx is finite, all $x \in S$;

(iv) $xS = Sx = S$, all $x \in S$.

Denote the set of idempotents of S by E; since S is periodic, $E \neq \emptyset$.

Case (i). If E is finite then S^1ES^1 is a finite ideal of S and the Rees factor semigroup S/S^1ES^1 is an infinite semigroup having only finite proper subsemigroups. However some power of each element of S is idempotent so S/S^1ES^1 is nil. Since this contradicts Lemma 1, E must be infinite.

For each $e \in E$ define the subsemigroups L_e and R_e of S by

$$L_e = \{x \in S \mid xe = e\} , \qquad R_e = \{y \in S \mid ey = e\} .$$

If $L_e = L_f = R_e = R_f = S$ for some $e, f \in E$ then $e = ef = f$. Hence, since E is infinite, there must exist an e in E such that either $L_e \neq S$ or $R_e \neq S$. Assume the former; then L_e is finite. Therefore $E' = E \backslash L_e$ is an infinite subset of S so E' generates S. Consequently there exist elements $f_1, f_2, \cdots, f_k$ of E' such that $f_1 f_2 \cdots f_k = e$. Thus $f_1 e = e$ so $f_1 \in L_e$, contradicting $f_1 \in E'$. The assumption that $R_e \neq S$ leads to a similar contradiction, so case (i) is eliminated.

Cases (ii) and (iii) are left-right duals so only one of them need be considered. Suppose then that for each x in S, Sx is finite and $xS = S$. Then $ex = x$ for all $e \in E$ so E is a right zero subsemigroup of S. Thus if $e \in E$ and $E' = E \backslash \{e\}$ then E' is either empty or a proper subsemigroup of S. In either case we conclude that E is finite.

Suppose $E = \{e\}$. Then, since S is periodic, there corresponds to each element x of S a positive integer $n = n(x)$ such that $x^n = e$. Therefore $xe = x^{n+1} = ex = x$ so $Se = S$, contradicting the finiteness of Se. Hence E has order $k > 1$, say $E = \{e_1, \cdots, e_k\}$. For each i, $1 \leqq i \leqq k$, define $T_i = \{x \in S \mid e_i \in \langle x \rangle\}$. By the periodicity of S, the set $\{T_1, \cdots, T_k\}$ is a partition of S. If $x \in T_i$, say $x^m = e_i$, then, as above, it follows that $xe_i = x$, so that $T_i \subseteqq U_i = \{x \in S \mid xe_i = x\}$, $i = 1, \cdots, k$. Conversely let $x \in U_i$ and suppose $x^n = e_j$ for some $n > 0$ and $j, 1 \leqq j \leqq k$. As above,

$$e_j = x^n = x^n e_i = e_j e_i = e_i .$$

Thus $i = j$ so $U_i \subseteqq T_i$. Therefore $T_i = U_i$ is a subsemigroup of S for $i = 1, \cdots, k$. But since $S = \bigcup_1^k T_i$ and $k > 1$ it follows that at least one of the T_i is an infinite proper subsemigroup of S, a contradiction.

This leaves only case (iv). Thus S is a group.

Combining Theorem 1 with Kaplansky's Exercise (I) we then have the following result.

THEOREM 2. *If S is an infinite commutative semigroup each of* *whose proper subsemigroups is finite then S is isomorphic to the group $Z(p^\infty)$ for some prime p.*

2. **Commutative *HF* semigroups.** An *HF* (or homomorphically finite) semigroup is defined to be an infinite semigroup each of whose noninjective homomorphisms has finite image. An *HF* group is an *HF* semigroup which is also a group, e.g., the infinite cyclic group or any infinite simple group. The following two results are immediate consequences of these definitions.

LEMMA 2. *Every proper nonzero ideal of an HF semigroup S has finite complement in S.*

LEMMA 3. *If S is an infinite semigroup then either all or none of S, S^1, S^0 and $(S^1)^0$ are HF semigroups.*

L. Rédei [3, Satz 82] has given essentially the following characterization of the subsemigroups of the additive semigroup N of positive integers.

LEMMA 4. (*Rédei*). *Let N be the additive semigroup of all positive integers and let $d, r \in N$. Define*

$$I = \{nd \mid n \in N, nd \geqq r\} \tag{2}$$

and let A be any subset of $dN \backslash I$ such that $A + A \subseteqq A \cup I$. Then $S = S(d, r, A)$ is a subsemigroup of N, and every subsemigroup of N is so obtainable. Furthermore, for suitable choice of r' and A', $S(d, r, A) \cong S(1, r', A')$.

Rédei's result can easily be extended to the additive group Z of all integers.

LEMMA 5. *If S is a nonzero subsemigroup of Z then either S is isomorphic to Z or, for suitable $r \in N$ and $A \subseteqq rN$, S is isomorphic to $S(1, r, A)$ with or without an adjoined identity.*

Proof. In view of the isomorphism between the subsemigroups N and $-N = (-1)N$ of Z we need only consider those subsemigroups of Z which contain both a positive and a negative integer. Let S be such a semigroup and let $S_1 = S \cap N$, $S_2 = S \cap (-N)$. For $i = 1, 2$ it follows from Lemma 4 that S_i is isomorphic to $S(d_i, r_i, A_i)$ for suitable $d_i, r_i, \in N$ and $A_i \subseteqq r_i N \backslash I_i$, where $I_i = \{nd_i \mid n \in N, nd_i \geqq r_i\}$. Moreover $ud_1 \in S_1$ and $-vd_2 \in S_2$, and hence $ud_1 - vd_2 \in S$, for all sufficiently large integers u and v. Therefore $(d_1, d_2) \in S$ so $d_1 = d_2$. It then follows that S is the cyclic subgroup of Z generated by d.

Commutative HF semigroups can now be characterized.

THEOREM 3. *Let S be an infinite commutative semigroup. Then S is homomorphically finite if and only if S is imbeddable in an infinite cyclic group with adjoined zero. If this is the case then either S is itself an infinite cyclic group or S is isomorphic to a subsemigroup $\bar{S}$ of the additive semigroup of all nonnegative integers with zero element ∞ adjoined. In the latter event there exist positive integers $a_1, \cdots, a_k$ and r such that*

$$\bar{S} = \{a_1, a_2, \cdots, a_k\} \cup \{n \mid n \geqq r\} ,$$

possibly with adjoined zero element ∞.

Proof. Let S be a subsemigroup of the additive group of integers with adjoined zero ∞. By Lemma 3 there is no loss of generality in assuming that $\infty \notin S$. Hence by Lemma 5 either S is an infinite cyclic group, and thus is homomorphically finite, or S is isomorphic to some semigroup $S(1, r, A)$, so that $S = I \cup A$, where $I = \{n \mid n \in N, n \geqq r)$ is an ideal of S. Assuming the latter, let σ be a nontrivial congruence on S which is not one-to-one, so that $a \,\sigma\, b$ for two distinct elements a, b of S. Then $(na) \,\sigma\, (nb)$ for all $n > 0$ so σ is not one-to-one on I, whence we can assume that $a, b \in I$, with $a < b$. Then $(a + (r + k)) \,\sigma\, (b + (r + k))$ for each $k \geqq 0$.

Define $m = b - a$ and let $x, y \in I$, with $x, y \geqq a + r$ and $x \equiv y (\text{mod } m)$, say $x = a + s, y = a + s + tm$, where $s, t \in N$ and $s \geqq r$. Since $a \,\sigma\, b$ then $(a + s) \,\sigma\, (b + s)$, i.e., $(a + s) \,\sigma\, (a + s + m)$. Thus by induction $(a + s) \,\sigma\, (a + s + tm)$ so $x \,\sigma\, y$. It follows that the factor semigroup I/σ is finite so by the finiteness of A, S/σ is also finite. Therefore S is an HF-semigroup.

Conversely let S be a commutative HF semigroup. For each c in S define the congruence σ_c on S by

$$a \,\sigma_c\, b \text{ if and only if } ac = bc, \text{ all } a, b \in S .$$

If there exists an element c in S such that S/σ_c is finite then the ideal Sc of S would also be finite, which, in the light of Lemma 2, would contradict the assumption that S is infinite unless $Sc = 0$.

Suppose $Sc = 0$ and let $J = \{x \in S \mid Sx = 0\}$. Then J is an ideal of S so either $J = 0$ or S/J is finite. In the latter case, we conclude that S^2 is also finite; thus $S^2 = 0$ since S^2 and S/S^2 cannot both be finite. However it is evident that the condition $S^2 = 0$ cannot hold in an HF semigroup S, so $J = 0$. Hence $Sc = 0$ only if $c = 0$.

Thus S/σ_c is infinite, and σ_c is one-to-one, for all c in $S\backslash 0$ so S is a commutative cancellative semigroup, possibly with an adjoined zero. In any event, S contains no proper zero divisors.

Let $T = S\backslash 0$ or $T = S$ according as S does or does not contain a zero element. Let G denote the group of quotients of T and regard T as a subsemigroup of G, in the usual manner. Suppose σ is a congruence on G which is not one-to-one, say $(a/b)\,\sigma\,(c/d)$, where $a, b, c, d \in S$ and $ad \neq bc$. Then $((a/b)bd)\,\sigma\,((c/d)bd)$, i.e., $(ad)\,\sigma\,(bc)$. Consequently σ', the restriction of σ to S, is not one-to-one on T so T/σ' is finite.

For $x \in T$ let $[x]$ and $[x]'$ denote the σ-class of G and the σ'-class of T, respectively, containing x. Then the homomorphism of T/σ' into G/σ defined by $[x]' \to [x]$, all $x \in T$, is injective. Thus T/σ' is cancellative and hence is a finite abelian group. It is readily verified that the mapping of S/σ' defined by $[x] \to [x]'$, all $x \in S$, is an isomorphism of S/σ' onto G/σ. Therefore G/σ is also finite so G is an abelian HF-group. Thus by Kaplansky's Exercise II, G is cyclic.

An application of Lemma 5 now completes the proof.

References

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Received October 31, 1967. Under the partial support of a N.S.F. Nebraska Cooperative Fellowship.