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A CHARACTERIZATION OF THE LINEAR SETS SATISFYING HERZ'S CRITERION

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Let E be a closed subset of T, the circle group, which we identify with the real numbers modulo 1. E is said to satisfy Herz's criterion (briefly, E satisfies (H)), if there exists an infinite set of positive integers N, such that

$(*)$ for all integers j with $0 \leqq j < N$, each of the numbers j/N either belongs to E or is distant by at least $1/N$ from E.

The main theorem proved here, is that E satisfies (H) if and only if there exists a sequence of sets $F_1, F_2, \cdots$ with $E = \bigcap_{i=1}^{\infty} F_i$ and positive integers $N_1 < N_2 < \cdots$ satisfying the following properties for all i:

(1) N_i divides N_{i+1} and $F_i \supset F_{i+1}$.

(2) F_i is a finite union of disjoint closed intervals each of whose end points is of the form j/N_i for some integer j.

(3) If for some integer j, $j/N_i \in F_i$, then $j/N_i \in F_{i+1}$.

The motivation for studying sets E satisfying (H) is the result of Herz (c.f. [1]) that all such sets satisfy spectral synthesis, and of course that the Cantor set is an example. (See also [2], Chapter IX).

Now suppose that $E = \bigcap_{i=1}^{\infty} F_i$, with F_i and N_i satisfying (1)–(3) for all i. It is then evident that E satisfies (H), since the numbers N_i will satisfy $(*)$ for all i. Moreover, E is obtained by a sort of disection procedure. Indeed, F_{i+1} may be obtained from F_i by removing from certain of the closed intervals $[j/N_i, (j+1)/N_i]$ included in F_i, one or more open intervals of the form

$$\left(\frac{l}{N_{i+1}}, \frac{q}{N_{i+1}}\right)$$

where $j/N_i \leqq l/N_{i+1} < q/N_{i+1} \leqq (j+1)/N_i$.

The "only if" part of our main result is demonstrated following the proof of Theorem 4 below. The latter result is somewhat stronger than our main theorem, and enables us to show that certain sets fail to satisfy (H) (in particular, the symmetric sets of ratio ξ, where ξ is a rational number with $1/\xi$ unequal to an integer. (C.f. [2], pp. 13–15 for the definition of these sets).

§1. *Preliminaries*. We identify the points of $\boldsymbol{T}$ with $[0, 1)$, where addition and subtraction are taken modulo 1. If x and y belong to $\boldsymbol{T}$, then the distance between them, $\rho(x, y)$, is defined to be the distance from $x-y$ to the closest integer on the real line. If E

is a subset of $\boldsymbol{T}$, then $\rho(x, E)$ is defined as $\inf_{f \in E} \rho(x, f)$.

Throughout this paper, E shall refer to a closed proper nonempty subset of $\boldsymbol{T}$ and $\mathscr{N}$ shall denote the set of all positive integers N satisfying $(*)$. (Thus if E satisfies (H), $\mathscr{N}$ is an infinite set (and conversely)). Every variable "N", with or without sub or superscripts, refers to a member of $\mathscr{N}$, and every variable "j" refers to an integer.

If L and M are positive integers, we write $L \mid M$ if there is an integer q with $Lq = M$.

Given a set S, "$\sim S$" denotes its complement.

Let $[x]$ be the greatest integer less than or equal to x. We remind the reader that if U is a proper connected open subset of $\boldsymbol{T}$, there will exist unique real numbers $a < b \leqq a + 1$, such that $0 \leqq b < 1$, and such that $U = \{x - [x]: a < x < b\}$. We then define the length of U to be $b-a$, with the left and right end points of U being a $- [a]$ and b respectively.

DEFINITION. Let x be a member of E for which there exists a j with $0 \leqq j < N$, such that $x = j/N$.

x is called N-initial if $(j - 1)/N \notin E$.

x is called N-terminal if $(j + 1)N \notin E$.

x is called an N-end if x is N-initial or N-terminal.

We note that if x is N-initial (N-terminal) then x is a right (left) end point of a component of $\sim E$ of length at least $2/N$. Indeed, if x is N-initial, we may close a j so that $x - (1/N) = j/N$, and $j/N \notin E$. Hence the open interval $((j/N) - (1/N), (j/N) + (1/N))$ cannot contain any points of E, and of course $x = (j + 1)/N$ belongs to E.

2. Our first result shows that if E satisfies (H), then the boundary points of components of $\sim E$ must be rational numbers.

LEMMA 1. *Let U be a component of $\sim E$, of length l. Then if $N > 1/l$, the end points of U are N-ends.*

Proof. Let x be the left end point of U. Then $x \in E$. Suppose it were false that $x = j/N$ for some j. There would then exist a $0 \leqq j < N$ such that $x \in (j/N, (j + 1)/N)$. Since $(1/N) < l$, we would have that $((i + 1)/N \in U)$, so $(j + 1)/N \notin E$. But

$$\rho\Big(\frac{j+1}{N}, E\Big) \leqq \rho\Big(\frac{j+1}{N}, x\Big) < \frac{1}{N},$$

a contradiction. Thus, there exists a $j, 0 \leqq j < N$, with $x = j/N$. But then $(j + 1)/N \notin E$, since the length of $(j/N, (j + 1)/N)$ is $1/N < l$, hence $(j + 1)/N \in U$. Thus, x is N-terminal. The proof that the right end point of U is N-initial is similar.

Our next task is to define certain sets that are finite unions of disjoint closed intervals, that approximate E. First, we note that if x is N-initial, then x is associated with a unique N-terminal number (possibly equal to x), as follows: let k be the smallest integer l, with $0 \leqq l < N$, such that $x + (l+1)/N \notin E$. (Note that $l = N-2$ is such an integer.) Then $x + k/N$ is the uniquely determined N-terminal number.

Wed efine $I_x = [x, x + (k/N)]$ and $E_N = \bigcup\{I_x\colon x$ is N-initial$\}$. If there do not exist any N-ends, set $E_N = \mathrm{T}$. Let l_1 be the maximum of the lengths of components of $\sim E$.

Then if $N > 1/l_1$, there will exist N-ends by Lemma 1 and hence E_N will be a proper subset of T. Of course, $I_x \cap I_{x'} = \phi$ for x and x' different N-ends; so E_N is a disjoint union of intervals with end points all of the form j/N.

LEMMA 2. *For all N and N', $N' < N$ implies $E_N \subset E_{N'}$.*

Proof. Let $N' < N$ be fixed, and let x be a fixed N-initial number. It follows directly from the definitions that $E \subset E_{N'}$; thus since $x \in E$, there is a (unique) N'-end y, such that $x \in I'_y$, where $I'_y = [y, z]$, with z the unique N'-terminal number associated with y.

Now choose an integer l with $0 \leqq l < N$ such that

$$z \in \left[\frac{l}{N}, \frac{l+1}{N}\right) .$$

Then $(l+1)/N \notin E$, since $(l+1)/N \in (z, z + 1/N)$. Thus we must have that $z = l/N$, or else $\rho(l/N, E) \leqq \rho(l/N, z) < 1/N$. Hence z is N-terminal, and so it follows from the definition of I_x that $I_x \subset I'_y$.

Thus $E_N \subset \bigcup\{I'_y\colon y$ is N'-initial$\} = E_{N'}$.

Our last lemma enables us to obtain certain canonical members of N crucial for the proof of Theorem 4 (whose proof also shows that the number N/d below equals q_i, where $l_{i+1} \leqq \dfrac{1}{N} < l_i$ and q_i, l_i are defined directly preceeding the statement of Theorem 4).

LEMMA 3. *Let $S_N = \{0 \leqq j < N\colon j/N$ is an N-end$\}$.*
Let d be a positive integer such that $d \mid N$ and $d \mid j$ for all $j \in S_N$. Then $(N/d) \in \mathscr{N}$.

Proof. We may and shall assume that $d > 1$. Put $M = N/d$, and let l be an integer with $0 \leqq l < M$, such that $l/M \notin E$. It remains for us to show that $\rho(l/M, E) \geqq 1/M$. If this is not the case, then either $\{(l-1)/M, l/M)$ or $\{l/M, (l+1)/M)$ contains a point of E. Suppose the first possibility; then

$$\left(\frac{l-1}{M}, \frac{l}{M}\right) = \left(\frac{d(l-1)}{N}, \frac{dl}{N}\right)$$

contains an N-end.

Indeed there is, in the first place, an integer r, $d(l-1) < r < dl$, such that $r/N \in E$. For if

$$x \in \left(\frac{d(l-1)}{N}, \frac{dl}{N}\right)$$

belongs to E, we can certainly find such an r with $\rho(x, r/N) < 1/N$. Then $r/N \in E$ since $N \in \mathscr{N}$ is always assumed. Now let k be the least integer greater than or equal to r such that $(k+1)/N \notin E$. Evidently $k \leqq dl - 1$ since $l/M = dl/N \notin E$, and k/N is an N-end.

Hence there is a $j \in S_N$ such that $k/N = j/N \pmod 1$. Since $d \mid N$ and $d \mid j$, it follows that $d \mid k$. But $d(l-1) < k < dl$, hence

$$l - 1 < \frac{k}{d} < l\ ,$$

a contradiction.

The argument for the case when $((l/M), (l+1)/M)$ contains a point of E, is practically identical to this.

The next result implies our main theorem, and is useful in determining if a given set fails (H). We shall need the following assumptions and notation:

Assume that $\sim E$ has infinitely many components, all with rational end points.

Let $l_1, l_2, \cdots$ be an enumeration of their lengths, with $l_i > l_{i+1} > 0$ for all i. Evidently $\sum_{i=1}^{\infty} l_i \leqq 1$, so $l_i \to 0$ as $i \to \infty$.

Let U_i be the union of all the components of $\sim E$ of lengths greater than or equal to l_i, K_i the set of end points of these components, and q_i the least common multiple of the denominators of the members of K_i, expressed in the lowest form.

THEOREM 4. *If E satisfies (H), then for infinitely many integers i, the following three conditions must hold simultaneously*:

(a) $l_{i+1} \leqq \dfrac{1}{q_i}$.

(b) $2l_{i+1} < l_i$.

(c) *For each integer j with $0 \leqq j < q_i$,* if *$j/q_i \notin E$, then $j/q_i \in U_i$.*

REMARK. If E is a set for which condition (c) holds for infinitely many i, then E satisfies (H). Indeed, the boundary points of U_i are all of the form j/q_i; thus if i satisfies (c), $N = q_i$ satisfies $(*)$. Moreover, $\{q_i : i \text{ satisfies (c)}\}$ will then be an infinite set. Indeed, $(1/q_i) \leqq l_i$ for all i. Thus fixing i, if we choose $k > i$ such that $l_k < (1/q_i)$, we have that $(1/q_k) < (1/q_i)$, so there are at most finitely many j's such that $q_j = q_i$.

Proof of Theorem 4. Assume that E satisfies (H), and fix $N \in \mathscr{N}$ with $N > 1/l_1$.

Then there is a unique i such that $l_{i+1} \leqq (1/N) < l_i$. By Lemma 1, each member of K_i is an N-end. Letting E_N be as defined before the proof of Lemma 2, we thus have $U_i \subset \sim E_N$. Moreover, every component of $\sim E_N$ is a component of $\sim E$, of length greater than or equal to $2/N$, by the definition of E_N. Thus, every component of $\sim E_N$ is of length greater than l_{i+1}, whence $\sim E_N \subset U_i$, and every N-end is a member of K_i, since it is an end point of a component of $\sim E$ of length greater than or equal to l_i. Thus $E_N = \sim U_i$ and the set of N-ends equals K_i. So every element in K_i is of the form j/N, whence $q_i \mid N$, so $q_i \leqq N$, and thus (a) follows. Since $2/N$ is less than or equal to the lengths of all the components of $\sim E_N = U_i$, it follows that $2/N \leqq l_i$, whence (b) holds. Finally, it follows from the definition of q_i, that if d is the greatest common divisor of $S_N \cup \{N\}$, then $q_i = N/d$ (where S_N is defined in Lemma 3). Thus by Lemma 3, $q_i \in \mathscr{N}$, whence since $q_i \leqq N$, $E_{q_i} \supset E_N$ by Lemma 2. So suppose that $j/q_i \notin E$. Then

$$\frac{j}{q_i} \notin E_{q_i}$$

by the latter's definition, so $j/q_i \notin E_N$, whence $j/q_i \in U_i$, so (c) holds.

Finally since $\mathscr{N}$ is infinite, there must be infinitely many i's for which there exists an $N \in \mathscr{N}$ with $l_{i+1} \leqq 1/N < l_i$, and consequently for which (a), (b), and (c) all hold.

Proof of the main theorem. Let E satisfy (H), and assume first that $\sim E$ has infinitely many components. Then by Lemma 1, the end points of these components are all rational numbers, so Theorem 4 is applicable; thus condition (c) of that result holds for infinitely many integers i. Now fixing i for which (c) holds, if $N > q_i$, then $q_i \mid N$; indeed, since $q_i \geqq 1/l_i$, we obtain by Lemma 1 that every element of K_i is an N-end, and thus expressable in the form j/N. Moreover, since the boundary points of U_i are all of the form j/q_i, we obtain that $q_i \in \mathscr{N}$.

Thus simply let $j_1, j_2, \cdots$ be an enumeration of a subset of the

i's satisfying (c), such that $q_{i_r} < q_{j_{r'}}$ for all $r < r'$. Then if we put $F_i = \sim U_{j_i}$ and $N_i = q_{j_i}$ for all i, $E = \bigcap_{i=1}^{\infty} F_i$ and (1)–(3) are satisfied for all i. We have also established that *when E satisfies (H) and its complement, has infinitely many components then there exist $N_1 < N_2 < \cdots$ such that for all i and N, if $N \geqq N_i$ then $N_i \mid N$.*

Now if E satisfies (H) and $\sim E$ has only finitely many components, then by Lemma 1, the boundary points of E are all rational numbers. Let M be the least common multiple of the denominators of these numbers expressed in the lowest form; then setting $N_i = 2^{i-1}M$ and $F_i = E$ for all i, it is easily verified that (1)–(3) hold. We remark finally that if $\sim E$ has finitely many components with rational boundary points, then E satisfies (H), and in fact letting M be as above, then for all $L \geqq M$, $L \in \mathscr{N}$ if and only if $M \mid L$. (Thus the statement ending the preceeding paragraph fails for E's such that $\sim E$ has finitely many components.)

We wish finitely to give some examples of sets which fail to satisfy (H). If ξ is a real number with $0 < \xi < 1/2$, S_ξ, the symmetric set of ratio ξ, consists of all numbers x in $\boldsymbol{T}$ such that

$$x = (1 - \xi) \sum_{j=0}^{\infty} \varepsilon_i \, \xi^j$$

where $\varepsilon_j = 0$ or 1, all j. (See pages 13–15 of [2].)

Now ξ is an end point of a component of $\sim S_\xi$, namely $(\xi, 1 - \xi)$.

Hence if ξ is irrational, then S_ξ fails (H) by Lemma 1. If $\xi = 1/L$ for some integer L, then it is well known that S_ξ satisfies (H). We shall show that if $\xi = p/q$, where p and q are relatively prime integers with $p > 1$, then S_ξ fails (H).

Defining l_i and q_i for $E = S_\xi$, we have that $l_i = (1 - 2\xi)\xi^{i-1}$ and $q_i = q^i$ for $i = 1, 2, \cdots$. (It follows from page 14 of [2] that all the end points of components of U_i are of the form l/q^i for some integer l; but p^i/q^i is such an end point, and p^i and q^i are relatively prime.) Now if $l_{i+1} \leqq 1/q_i$, then $(1 - 2(p/q))(p/q)^i \leqq 1/q_i$, or $p^i \leqq q/(q - 2p)$; thus condition (a) of Theorem 4 will be violated for all i sufficiently large.

REFERENCES

1. C.S. Herz, *Spectral synthesis for the Cantor set*, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 42–43.

2. J.P. Kahane, and R. Salem, *Ensembles parfaits et series trigonometriques*, Hermann, Paris, 1963.

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