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REPRESENTATION OF REAL NUMBERS BY GENERALIZED GEOMETRIC SERIES

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We shall say that the series of real numbers, $\sum_{i=0}^{\infty} 1/a_i$, is a generalized geometric series (g.g.s.) if and only if $a_i^2 \leqq a_{i+1}a_{i-1}$ for all $i \geqq 1$. (Note that the series is geometric if and only if equality holds.) In this paper we investigate the representation of positive real numbers less than or equal to one by generalized geometric series of the form $\sum_{i=0}^{\infty} x^i/c_i$ where the c_i are positive integers and $x \geqq 1$.

1. Preliminary results.

LEMMA 1. *If $\sum_{i=0}^{\infty} 1/a_i$ is a g.g.s. and*

$$|a_k| < |a_{k+1}|\ , \quad \textit{then} \quad \sum_{i=k+1}^{\infty}\left|\frac{1}{a_i}\right| \leqq \frac{1}{|a_{k+1}| - |a_k|}\ .$$

Proof. Since $|a_k/a_{k+t}| \leqq |a_k/a_{k+1}|^t$ for all $t \geqq 1$, we have

$$\sum_{i=k+1}^{\infty}\frac{1}{|a_i|} \leqq \frac{1}{|a_k|}\sum_{i=1}^{\infty}\left|\frac{a_k}{a_{k+1}}\right|^t = \frac{1}{|a_{k+1}| - |a_k|}\ .$$

The following theorem readily follows from Lemma 1.

THEOREM 1. *The* g.g.s. $\sum_{i=0}^{\infty} 1/a_i$ *converges if and only if there exists k such that $|a_k| < |a_{k+1}|$.*

THEOREM 2. *Let $\sum_{i=0}^{\infty} 1/a_i$ be a* g.g.s. *with $0 < a_0 < a_1$. Let $\alpha = \sum_{i=1}^{\infty} 1/a_i$, $S_k = \sum_{i=0}^{k} 1/a_i$ and $t_{k+1} = a_{k+1}/a_k - 1$. Then*

(i) *the sequence of half-open intervals $\{(S_k, S_k + 1/(a_{k+1} - a_k)]\}$ is a sequence of nested intervals whose intersection is α,*

(ii) $t_k \leqq t_{k+1} \leqq 1/a_k(\alpha - S_k)$.

Proof. Since the series is a g.g.s., we have

$$\frac{1}{a_{k+1} - a_k} \geqq \frac{1}{a_{k+1}} + \frac{1}{a_{k+2} - a_{k+1}}\ .$$

Hence the sequence of intervals in (i) above is nested. Also $a_k < a_{k+1}$ for all $k \geqq 0$. Thus, using Lemma 1,

$$(1) \qquad S_k < \alpha \leqq S_k + \frac{1}{a_{k+1} - a_k} = S_k + \frac{1}{a_k t_{k+1}}\ .$$

Since $a_k/a_{k+1} \leqq a_0/a_1$ for all $k \geqq 0$, we have

$$0 \leqq \lim_{k\to\infty} \frac{1}{a_{k+1} - a_k} \leqq \lim_{k\to\infty} \frac{1}{a_k\left(\dfrac{a_1}{a_0} - 1\right)} = 0 ,$$

and it follows from (1) that the intervals converge to α.

Inequalities (ii) are obtained from (1) and the definition of t_k.

COROLLARY. *Let* $x > 0$ *and let* $\alpha = \sum_{i=0}^{\infty} x_i/c_i$ *be a* g.g.s. *with* $0 < c_0 x < c_1$; *let* $S_k = \sum_{i=0}^{k} x^i/c_i$ *and* $s_{k+1} = c_{k+1}/c_k - 1$. *Then*

(i) *the sequence of half-open intervals* $\{(S_k, S_k + x^{k+1}/(c_{k+1} - xc_k)]\}$ *is a sequence of nested intervals whose intersection is* α,

(ii) $s_k \leqq s_{k+1} \leqq x^{k+1}/(c_k(\alpha - S_k)) + (x - 1)$.

Proof. Apply the theorem with $a_k = c_k/x^k$ observing that in this case

$$t_{k+1} = \frac{c_{k+1}}{xc_k} - 1 = \frac{s_{k+1} + 1}{x} - 1 .$$

3. The representation of reals. The corollary to Theorem 2 suggests an algorithm for constructing a g.g.s. of the form $\sum_{i=0}^{\infty} x^i/c_i$, where the c_i are integers and $x \geqq 1$, which converges to a given positive real number $\beta \leqq 1$.

To obtain such a series let $\{s_0, s_1, s_2 \cdots\}$ be any sequence of positive integers such that $s_0 = [1/\beta]$ and, for $k \geqq 0$,

$$\max\left\{\frac{x^{k+1}}{c_k(\beta - S_k)} - 1,\ s_k - 1\right\} < s_{k+1} \leqq \frac{x^{k+1}}{c_k(\beta - S_k)} + (x - 1)$$

where $c_k = \prod_{j=0}^{k} (s_j + 1)$ and $S_k = \sum_{j=0}^{k} x^j/c_j$.

Such a sequence of integers exists since

$$\begin{aligned} c_k(\beta - S_k) &= c_{k-1}(s_k + 1)(\beta - S_{k-1}) - x^k \\ &\leqq c_{k-1}\left(\frac{x^k}{c_{k-1}(\beta - S_{k-1})} + x\right)(\beta - S_{k-1}) - x^k = xc_{k-1}(\beta - S_{k-1}) \end{aligned}$$

and hence

$$s_k \leqq \frac{x^k}{c_{k-1}(\beta - S_{k-1})} + (x - 1) \leqq \frac{x^{k+1}}{c_k(\beta - S_k)} + (x - 1) .$$

The resulting series, $\sum_{i=0}^{\infty} x^i/c_i$ where $c_i = \prod_{j=0}^{i} (s_j + 1)$, is a g.g.s. since $s_{k+1} \geqq s_k$. Also since $\beta \leqq 1$,

$$c_1 = (s_1 + 1)c_0 > \frac{c_0 x}{c_0\beta - 1} = \frac{c_0 x}{\left(\left[\frac{1}{\beta}\right] + 1\right)\beta - 1} \geqq xc_0 .$$

Thus the series satisfies the hypotheses of the corollary to Theorem 2. Now from the manner in which the sequence $\{s_k\}$ has been obtained, we have

$$\frac{x^{k+1}}{c_k(\beta - S_k)} - 1 < s_{k+1} \leqq \frac{x^{k+1}}{c_k(\beta - S_k)} + (x - 1)$$

and thus

$$\begin{aligned} S_{k+1} = S_k + \frac{x^{k+1}}{c_k(s_{k+1} + 1)} &< \beta \leqq S_k + \frac{x^{k+1}}{c_k(s_{k+1} + 1 - x)} \\ &= S_k + \frac{x^{k+1}}{c_{k+1} - xc_k} . \end{aligned}$$

Therefore, by (i) of the corollary, $\beta = \sum_{i=0}^{\infty} x^i/c_i$.

If $x \neq 1$, the sequence $\{s_k\}$ obtained by the above process is not unique. For example, if $\beta = 1$ and $x = 2$, we have $s_0 = 1$,

$$\begin{aligned} &\max\left\{\frac{x}{c_0\beta - 1} - 1, s_0 - 1\right\} \\ &\qquad = \max\{1, 0\} = 1 \quad \text{and} \quad \frac{x}{c_0 - 1} + x - 1 = 3 . \end{aligned}$$

Thus there are two possible values for s_1. To obtain uniqueness, we must further restrict the s_k. One restriction that leads to a unique representation is to require that $s_k \geqq xs_{k-1}$. We now turn our attention to series which satisfy this condition.

THEOREM 3. *Let* $s > 0$, $x \geqq 1$. *For* $k \geqq 0$, *let* $s_k = x^k s$, $c_k = \prod_{i=0}^{k}(s_i + 1)$. *Then* $\sum_{i=0}^{\infty} x^i/c_i = 1/s$; *that is*

$$\frac{1}{s} = \frac{1}{s+1} + \frac{x^2}{(s+1)(xs+1)} + \frac{x^2}{(s+1)(xs+1)(x^2s+1)} + \cdots .$$

Proof. Let $S_k = \sum_{i=0}^{k} x_i/c_i$. We shall show by induction that $S_k + 1/c_k s = 1/s$ for all $k \geqq 0$. For $k = 0$, we have

$$s_0 + \frac{1}{c_0 s} = \frac{1}{c_0}\left(1 + \frac{1}{s}\right) = \frac{1}{s+1}\left(\frac{s+1}{s}\right) = \frac{1}{s} .$$

If $S_k + 1/c_k s = 1/s$, then

$$S_{k+1} + \frac{1}{c_{k+1}s} = S_k + \frac{x^{k+1}}{c_{k+1}} + \frac{1}{c_{k+1}s}$$
$$= \frac{1}{s} - \frac{1}{c_k s} + \frac{1}{c_k}\left(\frac{sx^{k+1}+1}{s(sx^{k+1}+1)}\right) = \frac{1}{s} .$$

It also follows by induction that $c_k > (s+1)^k$ and hence, since $s + 1 > 1$, $\lim_{k\to\infty} 1/c_k s = 0$. Thus $\lim_{k\to\infty} S_k = \lim_{k\to\infty}(1/s + 1/c_k s) = 1/s$.

THEOREM 4. *Let $x \geqq 1$. Let $\{s_0, s_1, s_2, \cdots\}$ be a sequence of positive integers such that $s_k \geqq xs_{k-1}$ for all $k \geqq 0$ and let $c_k = \prod_{i=0}^{k}(s_i + 1)$. Then $\sum_{i=0}^{\infty} x^i/c_i$ is a convergent* g.g.s. *Furthermore if $\alpha = \sum_{i=0}^{\infty} x^i/c_i$ and $S_k = \sum_{i=0}^{k} x_i/c_i$ then*

(i) *the sequence of half-open intervals $\{(S_k, S_k + x^{k+1}/(c_{k+1} - c_k)]\}$ is a sequence of nested intervals whose intersection is α,*

(ii) $s_{k+1} = [x^{k+1}/c_k(\alpha - S_k)]$ *for all $k \geqq 0$, $s_0 = [1/\alpha]$,*

(iii) *if x is rational, then α is rational if and only if $sk = xs_{k-1}$ for all k sufficiently large.*

Proof. Since $c_i/c_{i-1} = s_i + 1 \leqq s_{i+1} + 1 = c_{i+1}/c_i$, it follows that

$$\left(\frac{c_i}{x^i}\right)^2 \leqq \left(\frac{c_{i-1}}{x^{i-1}}\right)\left(\frac{c_{i+1}}{x^{i+1}}\right)$$

and hence $\sum_{i=0}^{\infty} x^i/c_i$ is a g.g.s. The series converges by Theorem 1 since $c_0 < c_1/x$.

To establish (i), we first observe that $s_{k+j+1} \geqq x^j s_{k+1}$ for all $j \geqq 0$. Thus, using Theorem 3, we have

$$\begin{aligned}
S_k < \alpha &= S_k + \sum_{i=0}^{\infty} \frac{x^{k+j+1}}{c_{k+j+1}} \\
&= S_k + \frac{x^{k+1}}{c_k}\left(\frac{1}{s_{k+1}+1} + \frac{x}{(s_{k+1}+1)(s_{k+2}+1)}\right. \\
&\quad \left. + \frac{x^2}{(s_{k+1}+1)(s_{k+2}+1)(s_{k+3}+1)} + \cdots\right) \\
&\leqq S_k + \frac{x^{k+1}}{c_k}\left(\frac{1}{s_{k+1}+1} + \frac{x}{(s_{k+1}+1)(xs_{k+1}+1)}\right. \\
&\quad \left. + \frac{x^2}{(s_{k+1}+1)(xs_{k+1}+1)(x^2 s_{k+1}+1)} + \cdots\right) \\
&= S_k + \frac{x^{k+1}}{c_k} \cdot \frac{1}{s_{k+1}} = S_k + \frac{x^{k+1}}{c_{k+1} - c_k} .
\end{aligned}$$

Furthermore, since $s_{k+2} - xs_{k+1} \geqq 0$, we have

$$\frac{x^{k+1}}{c_k s_{k+1}} - \frac{x^{k+2}}{c_{k+1}s_{k+2}} = \frac{x^{k+1}}{c_{k+1}}\left(\frac{s^{k+1}+1}{s_{k+1}} - \frac{x}{s_{k+2}}\right)$$
$$= \frac{x^{k+1}}{c_{k+1}}\left(1 + \frac{1}{s_{k+1}} - \frac{x}{s_{k+2}}\right) \geqq \frac{x^{k+1}}{c_{k+1}} .$$

Thus

$$S_{k+1} + \frac{x^{k+2}}{c_{k+1}s_{k+2}} = S_k + \frac{x^{k+1}}{c_{k+1}} + \frac{x^{k+2}}{c_{k+1}s_{k+2}} \leqq S_k + \frac{x^{k+1}}{c_k s_{k+1}}$$

and the sequence of intervals in (i) is nested. Since

$$S_k + \frac{x^{k+1}}{c_{k+1} - c_k} \leqq S_k + \frac{x^{k+1}}{c_{k+1} - xc_k} ,$$

by part (i) of the corollary to Theorem 2, the intersection of the intervals is α.

To establish (ii), we have from (i) that

$$S_k + \frac{x^{k+1}}{c_k(s_{k+1}+1)} = S_{k+1} < \alpha \leqq S_k + \frac{x^{k+1}}{c_k s_{k+1}} .$$

Solving these inequalities for s_{k+1}, we have

$$\frac{x^{k+1}}{c_k(\alpha - S_k)} - 1 < s_{k+1} \leqq \frac{x^{k+1}}{c_k(\alpha - S_k)} . \tag{2}$$

Also, since $s_k \geqq x^k s_0$, using Theorem 3, we have

$$\frac{1}{s_0+1} = S_0 < \alpha \leqq \frac{1}{s_0+1} + \frac{1}{(s_0+1)(xs_0+1)}$$
$$+ \frac{x^2}{(s_0+1)(xs_0+1)(x^2s_0+1)} + \cdots = \frac{1}{s_0}$$

and hence $s_0 = [1/\alpha]$.

We turn now to the proof of (iii). Suppose $s_k = xs_{k-1}$ for all $k > k_0$. Then $s_{k_0+j} = x^j s_{k_0}$ for all $j \geqq 0$. Thus, again using Theorem 3, we have

$$\alpha = S_{k_0-1} + \frac{x^{k_0}}{c_{k_0-1}}\left(\frac{1}{s_{k_0+1}}\right.$$
$$\left. + \frac{x}{(s_{k_0+1})(xs_{k_0}+1)} + \frac{x^2}{(s_{k_0+1})(xs_{k_0+1})(x^2 s_{k_0+1})} + \cdots\right)$$
$$= S_{k_0-1} + \frac{x^{k_0}}{c_{k_0-1}} \cdot \frac{1}{s_{k_0}}$$

which is rational if x is rational.

Conversely, suppose α is rational. From (2) we have

$$\frac{x^{k+1}}{s_{k+1}+1} < C_k(\alpha - S_k) \leqq \frac{x^{k+1}}{s_{k+1}} .$$

Thus

$$\begin{aligned} 0 < c_{k+1}(\alpha - S_{k+1}) &= c_{k+1}\left(\alpha - S_k - \frac{x^{k+1}}{c_{k+1}}\right) \\ &= c_k(s_{k+1}+1)(\alpha - S_k) - x^{k+1} \\ &\leqq c_k\left(\frac{x^{k+1}}{c_k(\alpha - S_k)} + 1\right)(\alpha - S_k) - x^{k+1} = c_k(\alpha - S_k) . \end{aligned} \tag{3}$$

Hence, if $\alpha = p/q$, for all k we have

$$0 < c_{k+1}(p - S_{k+1}q) \leqq c_k(p - S_k q) .$$

Therefore, noting that $c_k S_k$ is an integer, $\{c_k(p - S_k q)\}$ is a nonincreasing sequence of positive integers and thus for k sufficiently large, say $k > k_0$, the terms of the sequence become constant. Hence, for $k > k_0$, $c_{k+1}(\alpha - S_{k+1}) = c_k(\alpha - S_k)$ and thus equality must hold in (3). Therefore $s_{k+1} = x^{k+1}/c_k(\alpha - S_k)$ for $k > k_0$ and, for k sufficiently large,

$$s_{k+1} = \frac{x^{k+1}}{c_k(\alpha - S_k)} = x \cdot \frac{x^k}{c_{k-1}(\alpha - S_{k-1})} = x s_k .$$

THEOREM 5. *Let $0 < \beta \leqq 1$ and let x be a positive integer. Then there exists a unique sequence of positive integers $\{s_0, s_1, s_2 \cdots\}$ such that $s_k \geqq x s_{k-1}$ for $k \geqq 1$ and $\beta = \sum_{i=0}^{\infty} x^i/c_i$ where $c_i = \prod_{i=0}^{k}(s_i + 1)$.*

Proof. Define

$$s_0 = \left[\frac{1}{\beta}\right], \qquad c_0 = s_0 + 1 , \qquad S_0 = \frac{1}{c_0}$$

and, for $k > 0$,

$$s_{k+1} = \left[\frac{x^{k+1}}{c_k(\beta - S_k)}\right], \qquad c_{k+1} = (s_{k+1}+1)c_k , \qquad S_{k+1} = S_k + \frac{x^{k+1}}{c_{k+1}} .$$

Then, in the same manner as inequality (3) was obtained, we have

$$c_{k+1}(\beta - S_{k+1}) \leqq c_k(\beta - S_k) .$$

Thus

$$s_{k+1} > \frac{x^{k+1}}{c_k(\beta - S_k)} - 1 \geqq \frac{x^{k+1}}{c_{k-1}(\beta - S_{k-1})} - 1$$
$$= x\left(\frac{x^k}{c_{k-1}(\beta - S_{k-1})}\right) - 1 \geqq xs_k - 1 \, .$$

Since s_{k+1} and $xs_k - 1$ are integers, it follows that $s_{k+1} \geqq xs_k$. Also from the definition of s_{k+1},

$$\frac{x^{k+1}}{c_k(\beta - S_k)} - 1 < s_{k+1} \leqq \frac{x^{k+1}}{c_k(\beta - S_k)} \, .$$

Therefore

$$S_k < S_{k+1} = S_k + \frac{x^{k+1}}{c_{k+1}} < \beta \leqq S_k + \frac{x^{k+1}}{c_k(\beta - S_k)} \, .$$

Thus from Theorem 4 (i), $\beta = \sum_{i=0}^{\infty} x^i/c_i$. The uniqueness of the sequence $\{s_k\}$ follows from Theorem 4 (ii).

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