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ON THE RATE OF DECAY OF SOLUTIONS OF PARABOLIC DIFFERENTIAL EQUATIONS

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In this paper upper and lower bounds are determined for the rate of growth or decay of solutions of parabolic equations for indefinitely increasing time. These bounds are obtained from comparison functions which are constructed systematically with the aid of elliptic extremal operators and in many cases are the best possible for the class of problems which we consider.

The behavior of solutions of parabolic equations for indefinitely increasing time has been investigated by several authors in recent years. We refer in particular to the papers [2], [3] and [9] and the survey article [7]. Most of these articles have been concerned with establishing the approach of solutions to a steady state condition under various assumptions on the coefficients. In the present paper we determine upper and lower bounds for the rate at which solutions decay (or possibly grow) in a half cylinder as $t \to \infty$. These results, presented in Theorems 1 and 2, are obtained for a class of parabolic equations, assuming only uniform parabolicity and boundedness of the coefficients. The main tool is the maximum principle and the use of comparison functions constructed with the aid of elliptic extremal operators [15, 16]. By this device the estimates which we obtain are often the best possible for the class of operators which we consider.

2. Notations and basic hypotheses.

Let Ω be a bounded, open, connected subset of $E^n(n \geqq 1)$ with boundary Γ and closure $\bar{\Omega} = \Omega \cup \Gamma$. Let D denote the $(n+1)$-dimensional half cylinder $D = \Omega \times (0, \infty)$.

In this paper we shall be concerned with the linear differential operator

$$L - k\frac{\partial}{\partial t} \equiv \sum_{i,j=1}^{n} a_{ij}(x, t)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x, t)\frac{\partial}{\partial x_i} + c(x, t) - k(x, t)\frac{\partial}{\partial t} \tag{2.1}$$

with real valued coefficients defined for all $(x, t) \equiv (x_1, \cdots, x_n, t) \in D$ and satisfying there the following hypothesis I:

(1) $\sum_{i=1}^{n} a_{ii}(x, t) \equiv 1$ for all $(x, t) \in D$;

(2) There exists a constant α, $0 < \alpha \leqq 1/n$, such that $\sum_{i,j=1}^{n} a_{ij}(x, t)\xi_i\xi_j \geqq \alpha \sum_{i=1}^{n} \xi_i^2$ for all $(x, t) \in D$ and all real n-vectors $(\xi_1, \cdots, \xi_n)$.

(3) $b_1(x, t), \cdots, b_n(x, t), c(x, t)$ are bounded in D;

(4) There exist constants k_0, k_1 such that $0 < k_0 \leqq k(x, t) \leqq k_1$ in D.

We remark that any uniformly parabolic operator of the form $\sum_{i,j=1}^{n} A_{ij}\,(\partial^2/\partial x_i \partial x_j) + \sum_{i=1}^{n} B_i\,(\partial/\partial x_i) + c - (\partial/\partial t)$ with bounded coefficients can be normalized to the form (2.1) by dividing by $\sum_{i=1}^{n} A_{ii}$.

Note that the conditions $I(1)$ and $I(2)$ imply that $\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \leqq [1-(n-1)\alpha] \sum_{i=1}^{n} \xi_i^2$ in D for all real n-vectors $(\xi_1, \cdots, \xi_n)$ and that in the case $\alpha = 1/n$ the principal part of the operator L reduces to $(1/n)\Delta$, where Δ denotes the n-dimensional Laplacian.

It is well known [4-7, 10, 14, 17-18] that under the hypotheses I the operator (2.1) enjoys the maximum and boundary point principles. Since we wish to permit possible discontinuities in our solutions $u(x, t)$ on the initial surface $\bar{\Omega} \times \{0\}$ we state these principles in the following form: Suppose that $u(x, t)$ is a real valued function, differentiable with respect to t and twice differentiable with respect to x in D, continuous on $\bar{\Omega} \times (0, \infty)$, and satisfying $Lu - k(\partial u/\partial t) \geqq 0$ in D. Suppose that $u \leqq 0$ on $\Gamma \times (0, \infty)$ and that $\overline{\lim}_{(x,t)\to\bar{\Omega}\times\{0\}} u(x, t) \leqq 0$. Then $u(x, t) \leqq 0$ in $\bar{\Omega} \times (0, \infty)$. If $u(x^0, t_0) = 0$ for some point $(x^0, t_0) \in D$ then $u(x, t) \equiv 0$ for all $x \in \bar{\Omega}$, $0 < t \leqq t_0$. If $u(x^0, t_0) = 0$ for some point $(x^0, t_0) \in \Gamma \times (0, \infty)$ and if Γ has the inner sphere property[1] at x^0 then either $u(x, t) \equiv 0$ for all $x \in \bar{\Omega}$, $0 < t \leqq t_0$ or else there exists a *positive* constant m such that $u(x, t_0) \leqq -m\,\|x - x^0\|$ for all x within the prescribed sphere along a fixed line segment from x^0 and sufficiently close to x^0. Here we define

$$\|x\| = \Big(\sum_{i=1}^{n} x_i^2\Big)^{1/2}.$$

With the aid of the maximum principle we establish the following

LEMMA 1. *Suppose that hypotheses I hold. Let $u(x, t)$ be a real valued function, differentiable with respect to t and twice differentiable with respect to x in D, continuous on $\bar{\Omega} \times (0, \infty)$, and satisfying $Lu - k(\partial u/\partial t) \geqq 0$ in D. Suppose that $u \leqq 0$ on $\Gamma \times (0, \infty)$ and that $\overline{\lim}_{(x,t)\to\bar{\Omega}\times\{0\}} u(x, t) \leqq M$. If $u(x^0, t_0) = 0$ for some point $(x^0, t_0) \in \Gamma \times (0, \infty)$ and if Γ has the exterior sphere property[1] at x^0 then there exists a positive constant C such that $u(x, t_0) \leqq C\,\|x - x^0\|$ for all x in $\bar{\Omega}$ and sufficiently close to x^0.*

Proof. Assume first that $c(x, t) \leqq 0$. Without loss of generality we may suppose that the axes have been translated so that the exterior sphere prescribed at x^0 is centered at the origin and has radius $r_0 = \|x^0\|$. Define $r = \|x\|$ and $\rho^2 = \|x\|^2 + (t - t_0)^2$. Consider now the $(n + 1)$-dimensional sphere $B = \{(x, t): \rho^2 < r_0^2\}$, which is exterior to

[1] Γ has the inner (exterior) sphere property at x^0 if there exists an open ball B contained in Ω (the complement of $\bar{\Omega}$) such that $\bar{B} \cap \Gamma = \{x^0\}$.

D at the point (x^0, t_0), and the concentric sphere $B' = \{(x, t): \rho^2 < (r_0 + \varepsilon)^2\}$ where $\varepsilon > 0$ is chosen to be suitably small.

Define Ω' to be the open subset of D enclosed by B' for $0 < t < t_0$, Γ' to be the closure of $B' \cap \{\Gamma \times (0, t_0)\}$, and S' to be the subset of the surface of B' contained in D for $0 < t \leqq t_0$.

In the closure of Ω' consider the function $h(x, t) = e^{-\delta r_0^2} - e^{-\delta\rho^2}$, where δ is a constant to be chosen subsequently. Note that $h > 0$ in the closure of Ω' with the exception of the point (x^0, t_0) where we have $h(x^0, t_0) = 0$.

In Ω' we have

$$\begin{aligned} Lh - k\frac{\partial h}{\partial t} &= ch + 2\delta e^{-\delta\rho^2}\{1 - 2\delta \sum_{i,j=1}^{n} a_{ij}x_i x_j + \sum_{i=1}^{n} b_i x_i - k(t - t_0)\} \\ &\leqq 2\delta e^{-\delta\rho^2}\{A - 2\delta\alpha \| x^0 \|^2\} \\ &< 0 \text{ if } \delta \text{ is chosen to be sufficiently large.} \end{aligned}$$

It is an easy consequence of the weak form of the maximum principle mentioned earlier that $u(x, t) \leqq \max\{0, M\}$ in D and hence there exists a positive constant C such that $u \leqq Ch$ for $(x, t) \in S'$.

If we now define $w = u - Ch$ we have $Lw - k(\partial w/\partial t) > 0$ in Ω' while $w \leqq 0$ in $S' \cup \Gamma'$. It follows from the weak maximum principle that $w \leqq 0$ in the closure of Ω'. In particular we have $u(x, t_0) \leqq C\{e^{-\delta r_0^2} - e^{\delta r^2}\}$ for $x \in \bar{\Omega}$, $r \leqq r_0 + \varepsilon$, which immediately implies the result of the lemma. Note that the constant C is independent of t_0 in this case.

For the more general case when $c(x, t) \leqq \gamma, \gamma > 0$, we apply the above result to the function $v(x, t) = e^{-\gamma t/k_0}u(x, t)$ which satisfies the inequality $Lv - k(\partial v/\partial t) \geqq 0$ in D for an operator of type (2.1) with $c(x, t) \leqq 0$. We obtain $v(x, t_0) \leqq C\{e^{-\delta r_0^2} - e^{-\delta r^2}\}$ and thus $u(x, t_0) \leqq Ce^{\gamma t_0/k_0}\{e^{-\delta r^2 0} - e^{-\delta r^2}\}$.

3. Extremal operators and the class $\mathscr{L}_\alpha$. Let us denote by $\mathscr{L}_\alpha$ the class of uniformly elliptic operators of the form

$$L' = \sum_{i,j=1}^{n} a_{ij}(x, t)\frac{\partial^2}{\partial x_i \partial x_j}$$

with real valued coefficients defined in D and satisfying there the hypotheses $I(1)$ and $I(2)$. In recent papers [15, 16] Pucci has introduced the *maximizing* and *minimizing* operators, relative to the class $\mathscr{L}_\alpha$, defined respectively by

$$\begin{aligned} M_\alpha[u](x, t) &= \sup_{L' \in \mathscr{L}_\alpha}\{L'u(x, t)\} \\ m_\alpha[u](x, t) &= \inf_{L' \in \mathscr{L}_\alpha}\{L'u(x, t)\} \end{aligned} \tag{3.1}$$

for any given function $u(x, t)$ twice differentiable with respect to x in D.

He has shown that these operators have the nonlinear representations

$$M_\alpha[u] = \alpha\Delta u + (1 - n\alpha)C_n(u) \tag{3.2}$$
$$m_\alpha[u] = \alpha\Delta u + (1 - n\alpha)C_1(u)$$

where Δ denotes the n-dimensional Laplacian and $C_1(u) \leqq C_2(u) \leqq \cdots \leqq C_n(u)$ denote the ordered eigenvalues of the $n \times n$ Hessian matrix $(u_{ij}(x, t))$.

For functions $u(x, t) = R(r, t)$ which depend only upon t and the distance $r \equiv \| x \|$ from a fixed point (taken to be the origin) the representations (3.2) reduce to the simpler forms

$$M_\alpha[u] = h_1 \frac{\partial^2 R}{\partial r^2} + \frac{(1 - h_1)}{r} \frac{\partial R}{\partial r} \tag{3.3}$$
$$m_\alpha[u] = h_2 \frac{\partial^2 R}{\partial r^2} + \frac{(1 - h_2)}{r} \frac{\partial R}{\partial r}$$

where $h_1 = \alpha$ and $h_2 = 1 - (n - 1)\alpha$ if $(\partial^2 R/\partial r^2) < (1/r)(\partial R/\partial r)$ while $h_1 = 1 - (n - 1)\alpha$ and $h_2 = \alpha$ if $(\partial^2 R/\partial r^2) \geqq (1/r)(\partial R/\partial r)$.

From the definitions it is clear that for any function $u(x, t)$, twice differentiable with respect to x in D, and any $L' \in \mathscr{L}_\alpha$ we have the inequality $m_\alpha[u](x, t) \leqq L'u(x, t) \leqq M_\alpha[u](x, t)$ for each $(x, t) \in D$. However, it may also be shown that there exist operators $L_1', L_2' \in L_\alpha$ (with coefficients determined by u) such that $L_1'u(x, t) = m_\alpha[u](x, t)$ and $L_2'u(x, t) = M_\alpha[u](x, t)$ for all $(x, t) \in D$, thereby justifying the terminology.

We refer to the papers [8, 11–13, 16] for a more detailed development of the theory of these extremal operators and their applications. Their significance for our purposes in this paper rests on the fact that they provide a systematic method for determining comparison functions for our parabolic operator (2.1). The construction of these functions is based upon the following lemma. Here J_μ denotes the Bessel function of the first kind of order μ in the notation of [1], and $r = \| x \| = (\sum_{i=1}^n x_i^2)^{1/2}$.

LEMMA 2. *Let $\alpha, \beta, \gamma, \kappa$ be constants with $0 < \alpha \leqq 1/n$, and $\beta > -1$. If we set*

$$\mu = \frac{\beta + 1}{2[1 - (n - 1)\alpha]} - 1 \quad \textit{and} \quad \lambda = \frac{\xi_0^2[1 - (n - 1)\alpha] - \gamma r_0^2}{\kappa r_0^2},$$

where ξ_0 is the first positive zero of $J_\mu(\xi)$, then the function $r^{-\mu} J_\mu\{(\xi_0 r/r_0)\} e^{-\lambda t}$ is a solution of the equation

$$M_\alpha[u] + \frac{\beta}{r}\frac{\partial u}{\partial r} + \gamma u - \kappa \frac{\partial u}{\partial t} = 0$$

in the $(n+1)$-dimensional cylinder $Q=\{(x, t): r < r_0, t > 0\}$ which is positive in Q, decreasing with respect to r, and zero on the cylindrical surface $r = r_0$. If, instead, we set

$$\mu = \frac{(\beta+1)}{2\alpha} - 1 \quad \textit{and} \quad \gamma = \frac{\alpha\xi_0^2 - \gamma r_0^2}{\kappa r_0^2}$$

then this function is a solution of the equation

$$m_\alpha[u] + \frac{\beta}{r}\frac{\partial u}{\partial r} + \gamma u - \kappa\frac{\partial u}{\partial t} = 0$$

in Q which is positive in Q, decreasing with respect to r, and zero on the surface $r = r_0$.

4. **The decay estimates.** With the aid of the maximum principle and the previous lemmas we can now establish our main results on the decay rate of solutions associated with our parabolic operator (2.1).

To obtain an upper bound on the decay rate we suppose that the set Ω has diameter $2r_0$ and is contained in the sphere of radius r_0 centered at the point $x^0 \in E^n$. The set D is then enclosed in the $(n+1)$-dimensional half cylinder $Q = \{(x, t): \|x - x^0\| < r_0, t > 0\}$.

THEOREM 1. *Suppose that hypotheses I hold and that*

$$\inf_D \sum_{i=1}^n b_i(x_i - x_i^0) \equiv \beta > -1 \quad \textit{and} \quad \sup_D c(x, t) = \gamma\,.$$

Define

$$\mu = \frac{(\beta+1)}{2[1-(n-1)\alpha]} - 1 \quad \text{and} \quad \lambda = \frac{\xi_0^2[1-(n-1)\alpha] - \gamma r_0^2}{\kappa r_0^2}$$

where ξ_0 is the first positive zero of $J_\mu(\xi)$ and $\kappa = k_1$ if $\xi_0^2[1-(n-1)\alpha] \geqq \gamma r_0^2$, while $\kappa = k_0$ if $\xi_0^2[1-(n-1)\alpha] < \gamma r_0^2$.

Suppose that $u(x, t)$ is a real valued function, differentiable with respect to t and twice differentiable with respect to x in D, continuous on $\bar{\Omega} \times (0, \infty)$, and satisfying $Lu - k(\partial u/\partial t) \geqq 0$ in D. If $u(x, t) \leqq 0$ on $\Gamma \times (0, \infty)$ and $\overline{\lim}_{(x,t)\to\bar{\Omega}\times\{0\}} u(x, t) \leqq M$ then there exists a positive constant C such that $u(x, t) \leqq C\rho e^{-\lambda t}$ in $\bar{\Omega} \times (0, \infty)$ where ρ denotes the distance from x to the lateral surface of Q.

Proof. We need only consider the case $M > 0$ for otherwise the result follows trivially from the weak maximum principle. Moreover without loss of generality we may suppose that the axes have been translated so that the axis of the cylinder Q is the t-axis.

Using the definition of the maximizing operator, the assumptions of the theorem, and Lemma 2 we note that the function $V(r, t) = r^{-\mu}J_{\mu}(\xi_0 r/r_0)e^{-\lambda t}$ satisfies the differential inequality

$$LV - k\frac{\partial V}{\partial t} \leqq M_\alpha[V] + \frac{(\sum_{i=1}^{n} b_i x_i)}{r}\frac{\partial V}{\partial r} + cV - k\frac{\partial V}{\partial t}$$

$$\leqq M_\alpha[V] + \frac{\beta}{r}\frac{\partial V}{\partial r} + \gamma V - \kappa\frac{\partial V}{\partial t} = 0$$

in D, where κ is prescribed in the hypothesis of the theorem.

Let $0 < \varepsilon < 1$ be an arbitrary constant and denote by D_ε the half cylinder $D_\varepsilon = \Omega \times (\varepsilon, \infty)$. Note that Γ has the exterior sphere property at any point $x^0 \in \Gamma$ such that (x^0, ε) is a common boundary point of both D_ε and Q. The result of Lemma 1 then implies that $\overline{\lim}_{x\to\Gamma}\{u(x, \varepsilon)/V(r, \varepsilon)\}$ is bounded above by a constant which may be chosen independent of ε. Moreover from the weak maximum principle we have $u(x, \varepsilon)$ bounded above, by M if $\gamma \leqq 0$ and by $Me^{\gamma\varepsilon/k_0}$ if $\gamma > 0$, for all $x \in \Omega$. It follows that there exists a positive constant a, independent of ε, such that $u(x, \varepsilon) - aV(r, \varepsilon) \leqq 0$ for all $x \in \Omega$.

Combining these results we conclude that the function $w(x, t) = u(x, t) - aV(r, t)$ satisfies the differential inequality $Lw - k(\partial w/\partial t) \geqq 0$ in D_ε and the inequality $w \leqq 0$ on $\Gamma \times [\varepsilon, \infty]$ and on $\Omega \times \{\varepsilon\}$. By the weak maximum principle we have $w \leqq 0$ in $\bar{\Omega} \times [\varepsilon, \infty)$, i.e., there exists a positive constant, a, such that

$$u(x, t) \leqq ar^{-\mu}J_\mu\left(\frac{\xi_0 r}{r_0}\right)e^{-\lambda t} \quad \text{in} \quad \bar{\Omega} \times [\varepsilon, \infty)$$

and since a is independent of ε the same result holds in $\bar{\Omega} \times (0, \infty)$. This clearly implies the result of the theorem.

To obtain a lower bound for the decay rate we define r_1 to be the maximum of the radii of all open spheres contained in Ω and suppose one such largest sphere is B, centred at the point $x^1 \in \Omega$. The set D then contains the $(n+1)$-dimensional half cylinder $Q_1 = B \times (0, \infty) = \{(x, t): \|x - x^1\| < r_1, t > 0\}$.

THEOREM 2. *Suppose that hypotheses 1 hold and that* $\sup_{Q_1}\sum_{i=1}^{n} b_i(x_i - x_i^1) = \beta$, $\inf_{Q_1} c(x, t) = \gamma$. *Define* $\mu = \{(\beta + 1)/2\alpha\} - 1$ and $\lambda = \dfrac{\alpha\xi_0^2 - \gamma r_1^2}{Kr_1^2}$, *where* ξ_0 *is the first positive zero of* $J_\mu(\xi)$ *and* $K = k_0$ *if* $\alpha\xi_0^2 \geqq \gamma r_1^2$ *while* $K = k_1$ *if* $\alpha\xi_0^2 < \gamma r_1^2$.

Suppose that $u(x, t)$ *is a real valued function, differentiable with respect to t and twice differentiable with respect to x in D, continuous on* $\bar{\Omega} \times (0, \infty)$, *and satisfying* $Lu - k(\partial u/\partial t) \leqq 0$ *in D. If* $u \geqq 0$ *on* $\Gamma \times (0, \infty)$, $\underline{\lim}_{(x,t)\to\Gamma\times\{0\}} u(x, t) \geqq 0$, *and* $\underline{\lim}_{(x,t)\to\Omega\times\{0\}} u(x, t) > 0$ *then there exists a* *positive constant m such that $u(x, t) \geqq m\rho e^{-\lambda t}$ in $B \times [\varepsilon, \infty)$ where ρ denotes the distance from x to the boundary of B and $\varepsilon > 0$ is arbitrarily small.*

Proof. Again we may assume that the axes are translated so that the sphere B is centred at the origin. Using the definition of the minimizing operator, the assumptions of the theorem, and Lemma 2 we note that the function $v(r, t) = r^{-\mu}J_{\mu}\{(\xi_0 r/r_1)\}e^{-\lambda t}$ satisfies the differential inequality

$$Lv - k\frac{\partial v}{\partial t} \geqq m_{\alpha}[v] + \frac{(\sum_{i=1}^{n} b_i x_i)}{r}\frac{\partial v}{\partial r} + cv - k\frac{\partial v}{\partial t}$$
$$\geqq m_{\alpha}[v] + \frac{\beta}{r}\frac{\partial v}{\partial r} + \gamma v - K\frac{\partial v}{\partial t} = 0$$

in Q_1, where K is prescribed in the hypotheses of the theorem.

Let $0 < \varepsilon < 1$ be an arbitrary constant. Note that $u > 0$ on $B \times \{\varepsilon\}$ by the strong minimum principle and the condition $\underline{\lim}_{(x,t)\to\Omega\times\{0\}} u(x,t) > 0$. Moreover the boundary point principle implies that $\underline{\lim}_{x\to\partial B} \{u(x, \varepsilon)/v(r, \varepsilon)\}$ is bounded below by a positive constant, since Γ has the inner sphere property at any point common to Γ and ∂B. It follows that there exists a positive constant δ such that $u(x, \varepsilon) - \delta v(r, \varepsilon) \geqq 0$ for all $x \in B$.

Combining these results we conclude that the function $w(x, t) = u(x, t) - \delta v(r, t)$ satisfies the differential inequality $Lw - k(\partial w/\partial t) \leqq 0$ in $B \times (\varepsilon, \infty)$ and the inequality $w \geqq 0$ on $\partial B \times [\varepsilon, \infty)$ and on $B \times [\varepsilon]$. By the weak minimum principle we have $w \geqq 0$ in $B \times [\varepsilon, \infty)$, i.e., there exists a positive constant δ such that $u(x, t) \geqq \delta r^{-\mu}J_{\mu}\{\xi_0 r/r_1)\}e^{-\lambda t}$ in $B \times [\varepsilon, \infty)$. The result of the theorem now follows directly.

5. Concluding remarks. We emphasize that for the class of operators of the form (2.1) with coefficients $b_i \equiv 0, i \equiv 1, \cdots, n$ and the case that Ω is a sphere our Theorems 1 and 2 cannot be improved. In fact from the discussion of the properties of the extremal operators it follows that there exists an operator in this class for which our comparison function $V(r, t)\,(v(r, t))$ is a solution in $\Omega \times (0, \infty)$.

Finally let us note that Theorems 1 and 2 may be used to obtain comparison theorems for more general nonhomogeneous parabolic equations by the device of subtracting off a suitable approximate solution of the equations.

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