PACIFIC JOURNAL OF MATHEMATICS
Vol. 29, No. 2, 1969

EXTENSIONS OF PSEUDO-VALUATIONS

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Let w be a pseudo-valuation defined on a commutative ring R and let S be an overring of R. This paper investigates conditions needed to imply that w can be extended to S. These conditions are given in terms of a particular sequence of ideals $\{A_i\}_{i=0}^{\infty}$ in R which is called the best filtration for w. The main theorem states that if w is a pseudo-valuation on R with best filtration $\{A_i\}$ and each A_i is a contracted ideal with respect to S, then w can be extended to S. The converse of this result is then proved.

By using our main theorem and some recent results by Gilmer [1], we show in several important cases that if S is an overring of R and w is any pseudo-valuation on R possessing a best filtration, then w can be extended to S. In particular, if R is a Prüfer domain with quotient field K and if S is an overring of R such that $S \cap K = R$, then w can be extended from R to S.

We begin in §1 by defining and developing properties of a best filtration and determining classes of pseudo-valuations which have best filtrations. The main results and applications are then proved in §2.

1. Filtrations. All rings are commutative, associative, and have identity. If S is an overring of R, we assume that S and R have the same identity. A *pseudo-valuation* on the ring R is a mapping w from R into the extended real number system such that:

(i) $w(0) = \infty$, $w(1) = 0$,

(ii) $w(x - y) \geqq \min\{w(x), w(y)\}$,

(iii) $w(xy) \geqq w(x) + w(y)$, for each $x, y \in R$.

w is called a *homogeneous pseudo-valuation* in case:

(iv) $w(x^2) = 2w(x)$ for each $x \in R$.

w is called a *valuation* in case:

(v) $w(xy) = w(x) + w(y)$ for each $x, y \in R$.

Pseudo-valuations were first introduced by Rees [3]. Rees proved in [3] that (iv) is equivalent to the condition that $w(x^n) = nw(x)$ for each positive integer n and for each $x \in R$. These functions arise quite naturally in ring theory. If A is a proper ideal of R, define $v_A(x) = n$ if $x \in A^n, x \notin A^{n+1}$ and $v_A(x) = \infty$ if $x \in A^n$ for all n. Then v_A is a pseudo-valuation. We say that v_A is *associated with the ideal* A. A sequence of ideals $\{A_i\}_{i=0}^{\infty}$ of R such that $A_0 = R$, $A_{i+1} \subset A_i$, and $A_iA_j \subset A_{i+j}$ for all i and j is called a *filtration* on R. Notice that the nonnegative integral powers of an ideal A of R forms a filtration, where A° is defined to be R. Also note that any filtration $\{A_i\}$ determines a pseudo-valuation in exactly the same manner that the powers of an ideal A determines v_A. For an arbitrary pseudo-valuation w on R and a subset T of R, define $w(T) = \inf\{w(t): t \in T\}$.

DEFINITION 1. If w is a pseudo-valuation on R, define

$$\begin{aligned} A_0 &= R \\ A_i &= \{x \in R: w(x) > w(A_{i-1})\}\ , \\ &\quad \text{if } w(A_{i-1}) < \infty\ . \\ A_i &= A_{i-1}, \text{ if } w(A_{i-1}) = \infty\ . \end{aligned} \tag{1.1}$$

Each member of the sequence $\{A_i\}$ is an ideal of R. The sequence defined by (1.1) has the property that $A_0 \supset A_1 \supset A_2 \supset \cdots$. A_{i+1} is not necessarily a proper subset of A_i, as will be shown in Examples 1 and 2. Also note that for a given pseudo-valuation w, $\{x \in R: w(x) = \infty\} \subset \cap A_i$. The following example shows that there exists pseudo-valuations such that the sequence defined by (1.1) is not a filtration.

EXAMPLE 1. Let $\mathfrak{A}$ be an ideal of a ring R in which $\mathfrak{A}^i \supsetneqq \mathfrak{A}^{i+1}$ for all i. Define a sequence of ideals $\{B_i\}$ as follows: $B_0 = R$, $B_1 = \mathfrak{A}^2$, $B_2 = \mathfrak{A}^3$, $B_3 = B_4 = \mathfrak{A}^5$, and $B_i = \mathfrak{A}^7 (i \geqq 5)$. Then $\{B_i\}$ is a filtration in R and determines some pseudo-valuation w, where $w(x) = n$ if $x \in \varepsilon B_n$, $x \notin B_{n+1}$ and $w(x) = \infty$ if $x \in \cap B_n$. Now use Definition 1 to define A_i with respect to w. We obtain $A_i = B_i (i = 0, 1, 2, 3)$ and $A_i = B_{i+1} (i = 4, 5, \cdots)$. But $\{A_i\}$ is not a filtration, since $(A_2)^2 \not\subset A_4$.

DEFINITION 2. Let w be a pseudo-valuation on R and let $\{A_i\}$ be defined by (1.1). If $\{A_i\}$ is a filtration in R such that $x \in \cap A_i$ if and only if $w(x) = \infty$, then $\{A_i\}$ is called a *best filtration* for w. Let $B(R)$ denote the class of all pseudo-valuations on R which have a best filtration.

Example 1 then implies that there are pseudo-valuations which do not have best filtrations. It is clear from the definition that if w has a best filtration, then it is unique. From now on we will talk about *the* best filtration for w.

EXAMPLE 2. Let w be a pseudo-valuation on R and let $\{A_i\}$ be the sequence defined by (1.1). It is possible for $\{A_i\}$ to be a filtration in R, yet not be the best filtration for w. Let v be a real valued nondiscrete valuation on a field K and consider v as a pseudo-valuation on its valuation ring R_v. Since the value group of v has no smallest positive element, $v(A_1) = 0$. Then $A_2 = \{x \in R: v(x) > v(A_1) = 0\} = A_1$.

By induction, we see that $A_i = A_1$ for each $i \geqq 1$. Hence the sequence $\{A_i\}$ is such that $A_0 \supsetneqq A_1 = A_2 = \cdots$. Therefore $v \notin B(R)$. However, it is clear that $\{A_i\}$ is a filtration.

We will be interested only when the sequence defined by (1.1) is a filtration. This always happens in one important case.

REMARK 1. If v is a valuation on a ring R and if $\{A_i\}$ is the sequence of ideals defined by (1.1), then $\{A_i\}$ is a filtration in R.

Proof. It is clear that $A_i \supset A_{i+1}$ for each i. Hence, to complete the proof we need to show that $A_iA_j \subset A_{i+j}$ for all nonnegative integers i and j. We fix j and use induction on i. Clearly $A_0A_j \subset A_{0+j}$. Assume that $A_{i-1}A_j \subset A_{i+j-1}$ for $i \geqq 1$. Let $x \in A_iA_j$, then $x = \sum_{k=1}^{n} a_k b_k$ where $a_k \in A_i$ and $b_k \in A_j$. We may assume without loss of generality that $v(a_1) + v(b_1) = \min_{k=1}^{n}(v(a_k) + v(b_k))$. Then $v(x) \geqq v(a_1) + v(b_1)$. Case 1: If $v(A_{i-1}) < \infty$, then $v(a_1) > v(A_{i-1})$, and thus $v(x) > v(A_{i-1}) + v(A_j) = v(A_{i-1}A_j) \geqq v(A_{i+j-1})$. By Definition 1, $x \in A_{i+j}$. Case 2: If $v(A_{i-1}) = \infty$, then $v(x) = \infty$, which implies that $x \in A_{i+j}$. Therefore $A_iA_j \subset A_{i+j}$.

LEMMA 1. *Let $w \in B(R)$ and let $\{A_i\}$ be the best filtration for w. Then:*

(1) *$A_i = A_{i+1}$ if and only if $w(A_i) = \infty$.*

(2) *Let $x \in A_i$ and $x \notin A_{i+1}$. Then $y \in A_i$ and $y \notin A_{i+1}$ if and only if $w(x) = w(y)$. In fact, $w(x) = w(A_i)$.*

(3) *If $y \in A_i$ and $z \notin A_i$, then $w(y) > w(z)$.*

(4) *If $w(x) < \infty$, then there exists an integer i such that $x \in A_i$ and $x \notin A_{i+1}$.*

Proof. (1) Suppose $A_i = A_{i+1}$. By induction we see that $A_i = A_{i+t}$ for each positive integer t. If $w(A_i) < \infty$, then there is an element $x \in A_i$ such that $w(x) < \infty$. But $x \in \cap A_i$ which implies that $w(x) = \infty$, a contradiction. Conversely, if $w(A_i) = \infty$, then $A_i = A_{i+1}$ by definition of the best filtration.

(2) First note that $x \in A_i$, $x \notin A_{i+1}$ implies that $w(x) = w(A_i)$. If $y \in A_i$, $y \notin A_{i+1}$, then clearly $w(x) = w(y)$. Conversely, assume $w(x) = w(y)$. If $i = 0$, then $w(y) \leqq w(A_1)$ and hence y is in A_0, but not in A_1. If $i > 0$, then $w(A_{i-1}) \leqq w(A_i)$. If equality holds, then $A_i = A_{i+1}$, which implies that $x \in A_{i+1}$. Therefore $w(A_{i-1}) < w(A_i)$, which implies that $y \in A_i$. Also $y \notin A_{i+1}$, for if so, then $w(y) > w(A_i)$.

(3) and (4) are clear.

The converse of the above result is also true.

LEMMA 2. *Let w be a pseudo-valuation on R and let $\{B_i\}$ be a filtration in R satisfying properties* (1)–(4). *Then $\{B_i\}$ is the best filtration for w.*

Proof. Clearly $x \in \cap B_i$ if and only if $w(x) = \infty$. Suppose that $w(B_{i-1}) < \infty$. By properties (2) and (3) $B_i = \{x \in R: w(x) \geqq w(B_i)\}$. Thus $B_i \subset \{x \in R: w(x) > w(B_{i-1})\}$. On the other hand, suppose that $w(x) > w(B_{i-1})$. If $w(x) = \infty$, then $x \in \cap B_j$ and hence $x \in B_i$. If $w(x) < \infty$, choose k such that $x \in B_k$ and $x \notin B_{k+1}$. Suppose that $k \leqq i-1$, then $B_k \supset B_{i-1}$, so $w(x) = w(B_k) \leqq w(B_{i-1})$, a contradiction. So we must have $k > i - 1$ and hence $x \in B_i$. Therefore $B_i = \{x \in R: w(x) > w(B_{i-1})\}$.

By (1), if $w(B_{i-1}) = \infty$, then $B_{i-1} = B_i$.

We assume from now on that all pseudo-valuations w which are considered have the property that there exists at least one x such that $0 < w(x) < \infty$.

LEMMA 3. (a) *If w is a homogeneous pseudo-valuation on R and if $\{A_i\}$ is the sequence of ideals defined by* (1.1), *then $w(A_i) < \infty$ for each i.*

(b) *If w is a pseudo-valuation on a ring R and if $\{A_i\}$ is the sequence of ideals defined by* (1.1) *such that each A_i is finitely generated, then $w(A_{i-1}) < \infty$ implies that $w(A_i) > w(A_{i-1})$.*

Proof. (a) Suppose, to the contrary, that i is the smallest positive integer such that $w(A_i) = \infty$. Since w is nontrivial, $i \geqq 2$. Choose $x \in A_{i-1}$, $x \notin A_i$. Then $0 < w(x) < \infty$, and $w(x^2) > w(x) \geqq w(A_{i-1})$, so $x^2 \in A_i$. But, $w(A_i) \leqq w(x^2) = 2w(x) < \infty$, contradicting the assumption that $w(A_i) = \infty$.

(b) Let $a_1, \cdots, a_n$ be a basis for A_i. Choose a_k such that $w(a_k) = \min\{w(a_1), \cdots, w(a_n)\}$. Then $w(A_i) = w(a_k)$. Since $a_k \in A_i$, $w(a_k) > w(A_{i-1})$ and therefore $w(A_i) > w(A_{i-1})$.

The following theorem shows that there are many pseudo-valuations with best filtrations.

THEOREM 1. (1) *Any pseudo-valuation associated with an ideal is in $B(R)$. More generally, any pseudo-valuation determined by a filtration $\{B_i\}$, where $B_i = B_{i+1}$ implies that $B_i = B_{i+k}$ for each positive integer k, is in $B(R)$.*

(2) *If the sequence $\{A_i\}$ of ideals defined by* (1.1) *is a filtration and if $\lim_{i\to\infty} w(A_i) = \infty$, then $w \in B(R)$. Both of these conditions are satisfied if w is a valuation on R and R is noetherian.*

(3) *A pseudo-valuation w on R such that the range of w is equal to the set of all multiples of some positive real number $t > 0$ is in $B(R)$. This includes all integrally valued homogeneous pseudo-valuations w such that there is an $x \in R$ for which $w(x) = 1$.*

(4) *All integrally valued pseudo-valuations and pseudo-valuations on a noetherian ring such that* (1.1) *forms a filtrations are in $B(R)$.*

Proof. (1) Clear.

(2) Let w and $\{A_i\}$ satisfy the hypothesis of the first statement of (2). Clearly $w(x) = \infty$ implies that $x \in \cap A_i$. Let $x \in \cap A_i$, then $w(x) \geqq w(A_{i-1})$ for each i. Since $\lim_{i\to\infty} w(A_i) = \infty$, $w(x) = \infty$.

We will now prove the second statement of (2). Let v be a valuation on a noetherian ring R. By Remark 1, the sequence of ideals $\{A_i\}$ defined by (1.1) is a filtration in R. We need to show that $\lim_{i\to\infty} v(A_i) = \infty$. Consider a basis $\{y_1, \cdots, y_r\}$ for the ideal A_1. Let $v(y_1) = \min\{v(y_1), \cdots, v(y_r)\}$. Then y_1 is an element of R with the property that $v(y_1) = \varepsilon$ is a minimal positive element in the range of v. Assume that $\lim_{i\to\infty} v(A_i) = t < \infty$. By Lemma 3 (b), $v(A_i) > v(A_{i-1})$ for each i. Thus we can choose a sequence $\{x_j\} \in R$ so that $v(x_j) = a_j$ where $(t-\varepsilon) < a_1 < a_2 < \cdots$, and each $a_j < t$. Let B be the ideal generated by $\{x_j\}$. Since R is noetherian there exists a positive integer n so that $\{x_1, \cdots, x_n\}$ is a basis of B. Let $p > n$, then $x_p \in B$ and so $x_p = \sum_{i=1}^{n} \alpha_i x_i$, $\alpha_i \in R$. Then $v(x_p) \geqq \min\{v(\alpha_1 x_1), \cdots, v(\alpha_n x_n)\}$. Let $v(\alpha_j x_j)$ be this minimum. Case 1: If $v(\alpha_j) \neq 0$, then $v(x_p) \geqq v(\alpha_j) + a_j \geqq \varepsilon + a_j \geqq t$, which is a contradiction. Case 2: If $v(\alpha_j) = 0$, then $v(x_p) = a_p > a_j = v(x_j) = v(\alpha_j x_j)$. By properties of a valuation, $v(\alpha_j x_j) = v(\alpha_k x_k)$ for some $k \leqq n$, $k \neq j$. Since $v(x_j) \neq v(x_k)$, $v(\alpha_k) \neq 0$. Hence, $v(x_p) \geqq v(\alpha_k) + a_k \geqq t$, a contradiction. This proves that $\lim_{i\to\infty} v(A_i) = \infty$.

(3) Define $B_0 = R$ and inductively, $B_i = \{x \in R : w(x) \geqq i \cdot t\}$. The sequence $\{B_i\}$ satisfies the hypothesis of Lemma 2 and is a best filtration for w.

(4) The first part is clear. For the second part use the same technique as in (2).

2. The main results.

The following notation will be used in this section. Let S be an overring of R. If A is an ideal of R, then the extension of A to S, $A \cdot S$, will be denoted by A^e. If B is an ideal of S, then the contraction of B to R, $B \cap R$, will be denoted by B^c.

THEOREM 2. *Suppose that S is an overring of R, $w_0 \in B(R)$, and $\{A_i\}$ is the best filtration for w_0. If each A_i is a contracted ideal with respect to S, then w_0 can be extended to S.*

Proof. Define $B_i = A_i^e$ for each i. Then $\{B_i\}$ is a filtration on S.

Define a mapping w on S as follows: $w(x) = w_0(A_i)$ if $x \in B_i$, $x \notin B_{i+1}$ and $w(x) = \infty$ if $x \in \bigcap B_i$. We will show that w is a pseudo-valuation on S which extends w_0 to S. Property (i) of the definition of pseudo-valuation is obviously satisfied. Suppose that $x \in B_i$, $x \notin B_{i+1}$ and $y \in B_j$, $y \notin B_{j+1}$. Without loss of generality, assume that $i \leqq j$. Then $x - y \in B_i$ and hence, $w(x-y) \geqq w_0(A_i) = \min\{w_0(A_i), w_0(A_j)\} = \min\{w(x), w(y)\}$. Similarly if either $x \in B_i$ for all i or $y \in B_j$ for all j, then $w(x-y) \geqq \min\{w(x), w(y)\}$. This proves property (ii).

Finally, we wish to show $w(xy) \geqq w(x) + w(y)$. Again let $x \in B_i$, $x \notin B_{i+1}$ and $y \in B_j$, $y \notin B_{j+1}$. Then $xy \in B_iB_j \subset B_{i+j}$, so that

$$w(xy) \geqq w_0(A_{i+j}) .$$

If $w_0(A_i) + w_0(A_j) \leqq w_0(A_{i+j})$, then $w(xy) \geqq w(x) + w(y)$. On the other hand, if $w_0(A_i) + w_0(A_j) > w_0(A_{i+j})$, there are two cases to consider. Case 1: Suppose there is a positive integer t such that

$$w_0(A_i) + w_0(A_j) \leqq w_0(A_{i+j+t}) ,$$

but $w_0(A_i) + w_0(A_j) > w_0(A_{i+j+t-1})$. Then $A_iA_j \subset A_{i+j+t}$, and hence $B_iB_j \subset B_{i+j+t}$. Since $xy \in B_iB_j \subset B_{i+j+t}$, $w(xy) \geqq w(x) + w(y)$. Case 2: Suppose that $w_0(A_i) + w_0(A_j) > w_0(A_{i+j+t})$ for all t. Then $w_0(A_iA_j) > w_0(A_k)$ for all k, which implies that $A_iA_j \subset A_k$ for all k. Hence,

$$xy \in A_i^eA_j^e \subset \left(\bigcap\nolimits_{k=1}^{\infty} A_k\right)^e \subset \bigcap\nolimits_{k=1}^{\infty} (A_k^e) = \bigcap\nolimits_{k=1}^{\infty} B_k .$$

Therefore $w(xy) = \infty$ and $w(xy) \geqq w(x) + w(y)$. When either $x \in \bigcap B_k$ or $y \in \bigcap B_k$, clearly $w(xy) = w(x) + w(y) = \infty$. This proves property (iii), showing that w is a pseudo-valuation on S.

It is easy to see that w extends w_0. Take $z \in R$. If $z \in A_i$, $z \notin A_{i+1}$ then by Lemma 1, $w_0(z) = w_0(A_i)$. Clearly $z \in B_i$. Suppose $z \in B_{i+1}$, since z is also in R, $z \in A_{i+1}{}^{ec} = A_{i+1}$ a contradiction. Therefore $z \notin B_{i+1}$, and hence $w(z) = w_0(A_i)$. If $z \in \bigcap A_i$, then $z \in \bigcap B_i$ which implies that $w_0(z) = w(z) = \infty$.

A subring R of a ring S is said to have *property* C with respect to S in case each ideal of R is a contraction of an ideal in S. In [1], Gilmer shows that in several cases, if S is an overring of R which is integrally dependent on R, then R has property C with respect to S. Using Gilmer's theory we obtain several applications of Theorem 2, which are listed in the corollaries below. A *Prüfer domain* is a domain R with identity in which each finitely generated ideal is invertible, or equivalently, in which R_P is a valuation ring for each prime ideal P in R. An ideal A of a ring R is called a *valuation ideal* in case there exists a valuation ring R_v containing R and an ideal B of R_v such that $B \cap R = A$.

COROLLARY 1. *Suppose that R is a Prüfer domain with quotient field K and that R is a subdomain of R_1. If $R_1 \cap K = R$, then every $w \in B(R)$ can be extended to R_1.*

Proof. By [1; p. 563, Corollary 2], R has property C with respect to R_1. Then each ideal in a best filtration for w is a contracted ideal with respect to R_1. By Theorem 2, w can be extended to R_1.

COROLLARY 2. *Let R be a domain, let $w \in B(R)$, suppose that R_1 is integral over R, and let $\{A_i\}$ be the best filtration for w. If each A_i is an intersection of valuation ideals of R, then w can be extended to R_1.*

Proof. Apply [1; p. 564, Th. 2] and Theorem 2.

It is known that if R is an integrally closed domain, A is a complete ideal in R if and only if A is the intersection of valuation ideals. Now let R be an integrally closed domain with quotient field K, L a finite algebraic extension of K, and R' the integral closure of R in L. By [1; p. 569, Th. 6] and Theorem 2, we have:

COROLLARY 3. *If R' has an integral basis over R and if $w \in B(R)$, then w can be extended to R'.*

THEOREM 3. *Suppose that R is a subring of the ring S and suppose that w_0 is a pseudo-valuation on R which has an extension to a pseudo-valuation w on S. If α belongs to the set of extended reals, then the ideals $A_\alpha = \{x \in R: w_0(x) > \alpha\}$ and $B_\alpha = \{x \in R: w_0(x) \geqq \alpha\}$ are contractions of ideals of S.*

Proof. A_α is the contraction of $A'_\alpha = \{x \in S: w(x) > \alpha\}$ and B_α is the contraction of $B'_\alpha = \{x \in S: w(x) \geqq \alpha\}$.

The converse of Theorem 2 is also true.

THEOREM 4. *Let S be an overring of R, let $w_0 \in B(R)$, and let $\{A_i\}$ be the best filtration for w_0. If w_0 can be extended to S, then each A_i is a contracted ideal with respect to S.*

Proof. Apply Theorem 3.

COROLLARY 4. *Suppose $w_0 \in B(R)$ can be extended to some w on R_v, where R_v is a valuation ring. Then each ideal in the best filtration of w_0 is a valuation ideal.*

REMARK 2. Let R be a domain with quotient field K and w_0 a nonnegative pseudo-valuation on R. (Nonnegative pseudo-valuations were the most important types of pseudo-valuations studied in [2] and [3]). w_0 can always be extended to a nonnegative pseudo-valuation w on a subring R', where $R \subset R' \subset K$, in the following way. Let M be the set of $y \in R$ such that $w_0(y) < \infty$ and $w_0(xy) = w_0(x) + w_0(y)$ for all $x \in R$. Then M is a multiplicative subset of R not containing zero. Hence we can form the quotient ring of R with respect to M, R_M. A function w' can be defined on R_M by $w'(x/y) = w_0(x) - w_0(y)$. w' is not necessarily nonnegative. However, if $R' = \{z \in R_M : w'(z) \geqq 0\}$ and w is the restriction of w' to R', then R' is a ring and w is an extension of w_0 to R'. R' is called the *natural domain* of w_0. This type of extension was discussed and used in [2].

I would like to express my gratitude to the referee for his many suggestions on this manuscript. In particular, I would like to thank him for his suggested statement and proof of Theorem 3.

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Received June 10, 1968. This work was partially supported by a summer research grant from University of Missouri.

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