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ON FINITE GROUPS WITH INDEPENDENT CYCLIC SYLOW SUBGROUPS

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The purpose of this paper is to classify all perfect groups with cyclic Sylow p-subgroups which satisfy the condition

(TI) two different Sylow p-subgroups of G contain only the unit element in common

and such that

$$o(G) < o(P)^3$$

where P is a Sylow p-subgroup of G.

The main result of this paper is the following

THEOREM 1. Let G be a perfect finite group with a cyclic Sylow p-subgroup P of order p^a and assume that the Sylow p-subgroups of G satisfy the (TI) condition. Assume, furthermore, that

$$o(G) < p^{3a} .$$

Then one of the following statements holds.

(I) $a = 1$, $G \cong PSL(2, p)$, where $p > 3$ is a prime.

(II) $a = 1$, $G \cong PSL(2, p - 1)$, where $p = 2^m + 1 > 5$ is a Fermat prime.

(III) $a = 1$, $G \cong SL(2, p)$, where $p > 3$ is a prime.

(IV) $a = 2$, $p = 3$, $G \cong PSL(2, 8)$.

Ten years ago E. Artin raised the following problem: what are the simple finite groups G of order g which are divisible by a prime $p > g^{1/3}$? This question was answered by R. Brauer and W. F. Reynolds in [1]. They found that the only groups satisfying the above conditions are $PSL(2, p)$, where $p > 3$ is a prime, and $PSL(2, p - 1)$ where $p > 3$ is a Fermat prime, $p = 2^m + 1$. In particular, the Sylow p-subgroups of these groups are of order p and therefore they are cyclic and satisfy the (TI) condition. Theorem 1 thus generalizes these results.

As a matter of fact we will prove a more general statement than Theorem 1.

THEOREM 1*. *Let G be a finite group with a cyclic Sylow p-subgroup P of order p^a and assume that the Sylow p-subgroups of G satisfy the (TI) condition. Assume, furthermore, that*

$$o(G) < p^{3a}$$

and no homomorphic image of G is isomorphic to $N_G(P)/W$, where

W is the normal complement of P in $C_G(P)$. Then one of the following statements holds.

(I)* $a = 1$, $G \cong PSL(2, p)$, *where* $p > 3$ *is a prime.*

(II)* $a = 1$, $G \cong PSL(2, p-1)$, *where* $p = 2^m + 1 > 5$ *is a Fermat prime.*

(III)* $a = 1$, $G \cong SL(2, p)$, *where* $p > 3$ *is a prime.*

(IV)* $a = 2$, $p = 3$, $G \cong PSL(2, 8)$.

(V)* $a = 1$, $G \cong PGL(2, p)$, *where* $p > 3$ *is a prime.*

(VI)* $a = 1$, $G \cong PSL(2, p) \times M$, *where* $p > 3$ *is a prime* and $o(M) = 2$.

Since $G = G'$ implies the last condition of Theorem 1*, Theorem 1 follows immediately from Theorem 1*. In this paper the group $N_G(P)/W$ will be referred to as the p-metacyclic group of order qp^a.

Theorem 1* follows from the following more general result:

THEOREM 2. *Let G be a finite group with a cyclic Sylow p-subgroup P of order* $p^a > 1$ *and assume that the Sylow p-subgroups of G satisfy the (TI) condition. Suppose that no homomorphic image of G is isomorphic to the p-metacyclic group of order* $p^a q$. *Then*

$$o(G) = qwp^a(1 + np^a)$$

where $wp^a = o(C_G(P))$, $q = [N_G(P) : C_G(P)] > 1$ *and* n *is a positive integer.*

Furthermore, let G_0 *be the minimal normal subgroup of G for which* G/G_0 *is solvable, and let M be the maximal normal subgroup of* G_0 *of order prime to p. Denote* G_0/M *by* G^*. *Then one of the following statements holds.*

(A) $n = (hvp^a + h + v^2 + v)/(v + 1)$

where h and v are positive integers and $v + 1 \mid h(p^a - 1)$.

(B) $a = 1$, $n = 1$, $G^* \cong PSL(2, p)$ *where* $p > 3$ *is a prime.*

(C) $a = 1$, $n = (p-3)/2$, $G^* \cong PSL(2, p-1)$ *where* $p = 2^m + 1 > 5$ *is a Fermat prime.*

(D) $a = 2$, $p = 3$, $n = (p^2 - 3)/2$, $G^* \cong PSL(2, 8)$.

Theorem 2 immediately yields

COROLLARY. *Let G satisfy the assumptions of Theorem 2 and suppose that* $n < (p^a + 3)/2$. *Then* G^* *is of type* (B), (C) *or* (D).

In § 2 some basic properties of groups with a Sylow subgroup satisfying the *TI*-property are derived. Section 3 contains the proof of Theorem 2, from which Theorem 1* is deduced in § 4.

We use the standard notation $C_G(T)$, $N_G(T)$, $o(T)$, $T^\#$, and $\langle T \rangle$, where T is a subset of the group G, to denote respectively: the centralizer, normalizer, number of elements, the nonunit elements and the group generated by T. We will say that $N_G(T)/C_G(T)$ acts frobeniusly on T if $\theta^\eta = \theta$ for $\theta \in T^\#$ and $\eta \in N_G(T)$ implies that $\eta \in C_G(T)$. An element of G is called a p'-element, where p is a prime number, if p does not divide its order. The principal character and the commutator subgroup of G will be denoted by 1_G and G' respectively. Finally, if a and b are integers, then (a, b) denotes their greatest common divisor and $a \mid b$ means: a divides b.

2. *TIP*-groups. A finite group will be called a *TIP*-group if its Sylow p-subgroups are nontrivial and satisfy the *TI*-property. This section deals with properties of *TIP*-groups in general, followed by a study of *TIP*-groups with a cyclic Sylow p-subgroup.

PROPOSITION 2.1. *Let G be a TIP-group with a Sylow p-subgroup P of order p^a. Then the following statements hold.*

(a) $C_G(P) = W \times P$
where $o(W) = w$ *and* $(w, p) = 1$.

(b) $o(G) = qwp^a(1 + np^a)$
where $q = [N_G(P) : C_G(P)]$ *and n is a nonnegative integer.*

(c) *Any normal subgroup L of G of order divisible exactly by $p^b > 1$ is a TIP-group of order $q_L w_L p^b(1 + np^a)$.*

(d) *If H is a normal subgroups of G of order prime to p, then G/H is a TIP-group.*

Proof. Let $C = C_G(P)$, $N = N_G(P)$.

(a) Since P is a normal Hall-subgroup of C, it has a complement W and $(w, p) = 1$. Since elements of W commute with elements of P, $C = W \times P$.

(b) Consider the conjugates $\{P_i\}$ of P, other than P. If $\sigma \in P$ and $P_i^\sigma = P_i$, where $P_i = P^\tau$, $\tau \in G$, then $P^{\tau\sigma\tau^{-1}} = P$, $\tau\sigma\tau^{-1} \in N_G(P)$ and $\sigma \in N_G(P^\tau)$, $\sigma \in P \cap P^\tau = \{1\}$. Thus P acts by conjugation fixed point free on $\{P_i\}$ and therefore $o\{P_i\} = np^a$ for some nonnegative integer n. Hence $[G : N] = 1 + np^a$ and $o(G) = qwp^a(1 + np^a)$.

(c)—(d) The proof of Lemma 1 in [6] obviously holds also for general *TIP*-groups, with $p \neq 2$. Thus any subgroup of G of order divisible by p is a *TIP*-group and (d) holds. Let $o(L) = q_L x_L p^b(1 + n_L p^b)$. Since L and G have the same number of Sylow p-subgroups $1 + n_L p^b = 1 + np^a$ proving (c).

PROPOSITION 2.2. *Let G be a TIP-group with a cyclic Sylow p-subgroup P of order p^a. Then in addition to properties* (a)—(d) *of Proposition* 2.1, *and using the same notation, the following state-*

ments hold.

(e) $C_G(P) = C_G(\sigma)$ *and* $N_G(P) = N_G(\langle\sigma\rangle)$ *for all* $\sigma \in P^{\#}$.

(f) q *divides* $p - 1$.

(g) $o(G/H) = q\bar{w}p^a(1 + \bar{n}p^a)$ *and there exists a nonnegative integer* z *such that*

$$n = z + \bar{n} + z\bar{n}p^a .$$

If $z = 0$ *then* $H \subset W$.

(h) *If* K *is a normal subgroup of* G *and* K *does not contain* P *then*

$$N_K(P) = C_K(P) .$$

(i) *If also* $o(K \cap P) > 1$, *then* G *can be mapped homomorphically on the p-metacyclic group of order* $p^a q$.

Proof. (e) Let $\sigma \in P^{\#}$; then by Lemma 2.1.b in [3] $C_G(\sigma) \cap N_G(P) = C_G(P)$. It follows from the *TI*-property that $C_G(\sigma) \subset N_G(P)$ and $N_G(\langle\sigma\rangle) \subset N_G(P)$. Thus $C_G(\sigma) = C_G(P)$ and since P is cyclic $N_G(\langle\sigma\rangle) = N_G(P)$.

(f) By Lemma 2.1.d of [3] N/C acts frobeniusly on P and P is cyclic. Therefore $q = [N : C]$ divides $p - 1$.

(g) The proof of Proposition 2 in [1] holds, with the obvious changes, also in the present case. It is clear from the proof in [1] that if $z = 0$ then $H \subset C_G(P)$.

(h) Suppose that $K \cap N \not\subset C$ and let $\sigma \in K \cap N - C$. Since N/C acts frobeniusly on P, it follows that the elements $\sigma\rho^{-1}\sigma^{-1}\rho$, $\rho \in P$, are distinct and belong to $P \cap K$. Thus P is contained in K, a contradiction. Consequently $K \cap N \subset C$ and $K \cap N = K \cap C$, as required.

(i) Let p^b be the exact power of p dividing $o(K)$. Then $1 < p^b < p^a$ and by Proposition 2.1.c and (h) $o(K) = w_K p^b(1 + np^a)$, where $w_K p^b = o(C_K(P \cap K)) = o(N_K(P \cap K))$. By the Burnside Theorem K has a characteristic subgroup T of order $w_K(1 + np^a)$. T is normal in G and $G = NT$. Consequently WT is a normal subgroup of G and G/WT is isomorphic to the p-metacyclic group of order $p^a q$.

3. **Proof of Theorem 2.** If either $p = 2$ or $q = 1$, then $C_G(P) = N_G(P)$ and by the Burnside Theorem P has a normal complement in G, in contradiction to our assumption. Thus $p > 2$ and $q > 1$.

If P is normal in G and $C_G(P) = W \times P$, then W is normal in G, again a contradiction. Thus P is not normal in G and the first statement of Theorem 2 follows from Proposition 2.1.b.

It follows from Proposition 3 in [1] and Proposition 2.2.i that

$P \subset G_0$. Indeed, if $P \not\subset G_0$ then either $a = 1$ or $a > 1$ and G contains a normal subgroup U such that $1 < o(U \cap P) < p^a$. In both cases the above mentioned propositions yield a contradiction to our assumptions.

The definition of G_0 forces it to be its own commutator subgroup and the same is true for G^*. Moreover, G^* does not have nontrivial normal subgroups of order prime to p.

From now on we will assume that (A) is not satisfied and will show that then one of the statements (B), (C), or (D) holds.

Let $o(G_0) = q_0 w_0 p^a(1 + np^a)$, $o(G^*) = q_0 w^* p^a(1 + n^* p^a)$. Since $G^* = (G^*)'$, $n^* \neq 0$. By Proposition 2.2.g there exists a nonnegative integer z such that

$$n = z + n^* + zn^*p^a .$$

If $z \neq 0$, let $h = (z + 1)n^*$, $v = z$. Then:

$$n = v + h/(v + 1) + vhp^a/(v + 1)$$

in contradiction to our assumptions. Thus $z = 0$ and $n^* = n$.

Consequently, it suffices to show that if G satisfies the assumptions of Theorem 2 and in addition, $G = G'$, G has no nontrivial normal subgroup of order prime to p and n does not satisfy (A), then G is isomorphic to one of the simple groups described in (B), (C), and (D).

We will use the following notation: $N = N_G(P)$, $C = C_G(P) = W \times P$ where $o(W) = w$ and $(w, p) = 1$.

Let B be the principal p-block of G. Then by Proposition 2.1 of [3] B contains $t = (p^a - 1)/q$ exceptional characters X_λ of degree x_0, $\lambda = 1, \cdots, t$ and q nonexceptional characters X_i of degree x_i, $i = 1, \cdots, q$. If $\sigma \in P^\#$ and π is a p'-element of $C_G(\sigma) = C$ then:

$$(1) \qquad \begin{aligned} X_\lambda(\sigma\pi) &= -\varepsilon_0 \sum_{\rho \in R} \zeta_\lambda^\rho(\sigma) && \text{for } \lambda = 1, \cdots, t \\ X_j(\sigma\pi) &= \varepsilon_j && \text{for } j = 1, \cdots, q \end{aligned}$$

where R is a set of coset representatives of C in N, $\{\zeta_\lambda \mid \lambda = 1, \cdots, t\}$ is a set of representatives of the t transitivity classes of characters of P under conjugation by N (see [3], Lemma 2.2), and $\varepsilon_j = \pm 1$ for $j = 0, 1, \cdots, q$. It follows also by Corollary 2.1 of [3] that the following relations hold:

$$(2) \qquad \begin{aligned} x_i &\equiv \varepsilon_i \pmod{p^a} && \text{for } i = 1, \cdots, q \\ tx_0 &\equiv \varepsilon_0 \pmod{p^a} \end{aligned}$$

and

$$(3) \qquad \sum_{i=0}^{q} \varepsilon_i x_i = 0 .$$

We are now ready to state

LEMMA 3.1.

$$tx_j \mid (p^a - 1)(1 + np^a) \qquad \textit{for } j = 0, \cdots, q \,. \tag{4}$$

Proof. If $\sigma \in P^{\#}$, then $C = C_G(\sigma)$ and it is well-known that the expression

$$\frac{o(G)\cdot X_j(\sigma)}{o(C)\cdot x_j}$$

is an algebraic integer for all j. It follows, from Proposition 2.1 and (1), that for $j = 1, \cdots, q$

$$qwp^a(1 + np^a)/wp^a x_j$$

is an algebraic integer and consequently

$$tx_j \mid tq(1 + np^a) = (p^a - 1)(1 + np^a) \,.$$

For $j = 0$, it follows from (1), Proposition 2.1 and Lemma 2.2 of [3] that

$$\sum_{\lambda=1}^{t} \frac{o(G)X_\lambda(\sigma)}{o(C)x_0} = \frac{qwp^a(1 + np^a)\varepsilon_0}{wp^a x_0}$$

is an algebraic integer and therefore $tx_0 \mid (p^a - 1)(1 + np^a)$.

Since the block B contains 1_G as a nonexceptional character, we may assume that $X_1 = 1_G$. We have then

LEMMA 3.2. *For $j = 0, 2, 3, \cdots, q$*

$$\bar{x}_j = \begin{cases} 1 + np^a & \text{if } \varepsilon_j = 1 \\ p^a - 1 & \text{if } \varepsilon_j = -1 \end{cases}$$

where $\bar{x}_j = x_j$ for $j = 2, \cdots, q$ and $\bar{x}_0 = tx_0$.

Proof. We will show first that if

$$up^a + \varepsilon \mid (p^a - 1)(1 + np^a) \,, \qquad \varepsilon = \pm 1$$

then either n satisfies statement (A) or one of the following relations holds:

$$\begin{array}{lll} up^a + \varepsilon = 1 & \text{or} \quad np^a + 1 & \text{if} \quad \varepsilon = 1 \\ up^a + \varepsilon = p^a - 1 & \text{or} \quad (p^a - 1)(np^a + 1) & \text{if} \quad \varepsilon = -1 \,. \end{array}$$

To do so, it suffices to show that if n does not satisfy (A) then the only solutions of

$$(5) \qquad (vp^a + 1)(wp^a - 1) = (p^a - 1)(1 + np^a)$$

in nonnegative integers v and w are: $v = 0$, $wp^a - 1 = (p^a - 1)(1 + np^a)$ and $v = n$, $w = 1$.

Suppose that $v \neq 0$ and $w > 1$. Then $vp^a + 1 < 1 + np^a$, $v < n$. By multiplying out equation (5), adding 1 to both sides and dividing by p^a we get

$$(6) \qquad wvp^a + w - v = np^a - n + 1 .$$

Now by (6):

$$\begin{aligned}(vp^a + 1)(n - wv) &= vp^a n - v(wvp^a) + n - wv \\ &= vp^a n + vw - v^2 - vnp^a + vn - v + n - wv \\ &= (n - v)(v + 1) .\end{aligned}$$

Since $n > v$, the left hand side of the equation is positive and so we may put $h = n - wv$, where h is a positive integer. Solving for n we get a contradiction to the assumption that n does not satisfy (A). Thus either $v = 0$ or $w = 1$ and the above assertion follows.

Now we have seen that for $j = 0, 2, 3, \cdots, q$

$$\bar{x}_j \equiv \varepsilon_j \pmod{p^a} \text{ and } \bar{x}_j \mid (p^a - 1)(1 + np^a) .$$

Since X_1 is the only character of G of degree 1, it follows that for $j = 0, 2, 3, \cdots, q$

$$\bar{x}_j = \begin{cases} 1 + np^a & \text{if } \varepsilon_j = 1 \\ p^a - 1 \quad \text{or} \quad (p^a - 1)(1 + np^a) & \text{if } \varepsilon_j = -1 . \end{cases}$$

Thus it suffices to show that for $j = 0, 2, 3, \cdots, q$

$$\bar{x}_j \neq (p^a - 1)(1 + np^a) .$$

Indeed, if the equality holds, then by (3):

$$0 = \sum_{i=0}^{q} \varepsilon_i x_i \leqq 1 + (q - 1)(1 + np^a) - (p^a - 1)(1 + np^a)/t = - np^a$$

a contradiction. The proof of Lemma 3.2 is complete.

We will proceed with the proof of Theorem 2. It follows from (3) that at least one of the ε_j's, $j = 0, 1, \cdots, q$, is negative. If $\varepsilon_0 = -1$, let $X = \sum_{\lambda=1}^{t} X_\lambda$ and if $\varepsilon_i = -1$ for some $i \geqq 2$ let $X = X_i$. In either case, by Lemma 3.2 X is a character of G of degree $p^a - 1$ and by (1) and Lemma 2.2 of [3]

$$X(\sigma\pi) = -1$$

for $\sigma \in P^\#$, $\pi \in W$, where $C = P \times W$. Denote the restriction of X to

C also by X; then X is a character of $P \times W$ and therefore for $\rho \in P$ and $\pi \in W$ we have:

$$X(\rho\pi) = \sum_{i=1}^{r} \psi_i(\pi)\varphi_i(\rho) \tag{7}$$

where ψ_i, $i = 1, \cdots, r$ are distinct irreducible characters of W and φ_i, $i = 1, \cdots, r$ are characters of P. Let $\sigma \in P^{\#}$, $\pi \in W$; as $X(\sigma\pi) = -1$, it follows from (7) and from the linear independence of the irreducible characters of W, that the principal character appears among the ψ_i, say $\psi_1 = 1_W$, and

$$\varphi_1(\sigma) = -1, \ \varphi_2(\sigma) = \cdots = \varphi_r(\sigma) = 0 .$$

Suppose that $r > 1$. Then φ_2 vanishes on $P^{\#}$ and therefore p^a divides $\varphi_2(1)$, in contradiction to (7) and the fact that $X(1) = p^a - 1$. Thus $r = 1$ and

$$X(\rho\pi) = \varphi_1(\rho) \qquad \text{for all } \rho \in P, \ \pi \in W .$$

In particular $X(\pi) = \varphi_1(1) = X(1)$ for all $\pi \in W$. Let V denote the kernel of X; then V is a normal subgroup G and $W \subset V$. Suppose that $W \neq \{1\}$. Then it follows from the assumption that G has no nontrivial subgroups of order prime to p and from Proposition 2.2.i that $P \subset V$, in contradiction to the fact that $X(\sigma) = -1$ for $\sigma \in P^{\#}$. Consequently $W = \{1\}$, P contains the centralizer of each of its nonunit elements and by Theorem 2 of [2] G is either of type (B), or of type C, or $G \cong PSL(2, p^a - 1)$, where $a > 1$ and $p^a - 1 = 2^b$. In view of Lemma 3.1 of [3], the only solution of the above equation with $a > 1$ is: $p = 3$, $a = 2$ and $b = 3$. Thus if $a > 1$, $G \cong PSL(2, 8)$. Since the groups of types (B), (C) and (D) satisfy the conditions of Theorem 2, the proof is complete.

4. Proof of Theorem 1*. It follows from Theorem 2 that one of the statements (B), (C) and (D) holds. Statement (A) could not occur, since then

$$n \geqq (p^a + 3)/2 , \qquad o(G) \geqq 2p^a(p^a + 3)p^a/2 > p^{3a}$$

a contradiction.

In cases (C) and (D) $o(G^*) > p^{3a}/2$ and therefore $G \cong G^*$, yielding statements (II)* or (IV)*. In case (B), $o(G^*) = (p^{3a} - p^a)/2$ and therefore either $[G : G_0] = 2$, $G_0 \cong G^*$, or $G = G_0$, $o(M) = 1$ or 2. If $[G : G_0] = 2$, then $o(M) = 1$, G is isomorphic to a subgroup of the automorphism group of $PSL(2, p)$ and by [5, Lemma 2] $G \cong PGL(2, p)$, yielding $(V)^*$. If $G = G_0$ and $o(M) = 1$, then $G \cong PSL(2, p)$, $p > 3$, and $(I)^*$ holds. Finally, if $G = G_0$ and $o(M) = 2$, then it follows from a theorem of

Schur [4, p. 120] that G is either isomorphic to $SL(2, p)$ and (III)* holds, or it is isomorphic to $PSL(2, p) \times M$ and (VI)* holds. Since the groups mentioned in Theorem 1* satisfy the conditions of that theorem, the proof is complete.

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