PACIFIC JOURNAL OF MATHEMATICS
Vol. 29, No. 3, 1969

MATRICES WITH PRESCRIBED CHARACTERISTIC POLYNOMIAL AND A PRESCRIBED SUBMATRIX

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Let A be an arbitrary (complex) $n \times n$ matrix and let $f(\lambda)$ be a polynomial (with complex coefficients) of degree $n+1$ with leading coefficient $(-1)^{n+1}$. In this paper we solve the problem: under what conditions does there exist an $(n+1) \times (n+1)$ (complex) matrix B of which A is the submatrix standing in the top left-hand corner and such that $f(\lambda)$ is its characteristic polynomial?

In [1] Farahat and Ledermann proved that if A is a nonderogatory matrix over a field Φ and $f(\lambda)$ is a monic polynomial over Φ, then there exists an $(n+1) \times (n+1)$ matrix B over Φ with A standing in its top left-hand corner and such that $f(\lambda) = \det(\lambda E_{n+1} - B)$. Now, our main results are:

THEOREM 1. *Let A be an $n \times n$ complex matrix whose distinct characteristic roots are w_α $(\alpha = 1, \cdots, t)$. Let us suppose that in the Jordan normal form of A, w_α appears in r_α distinct diagonal blocks of orders $v_1^{(\alpha)}, \cdots, v_{r_\alpha}^{(\alpha)}$ respectively. We assume that*

$$v_1^{(\alpha)} \leqq \cdots \leqq v_{r_\alpha}^{(\alpha)} .$$

Let $\theta_\alpha = \sum_{j=1}^{r_\alpha - 1} v_j^{(\alpha)}$. There exists an $(n+1) \times (n+1)$ complex matrix B having A in the top left-hand corner and with $f(\lambda)$ as characteristic polynomial (i.e., $f(\lambda) = \det(B - \lambda E_{n+1})$) *if and only if $f(\lambda)$ is divisible by $\prod_{\alpha=1}^{t} (w_\alpha - \lambda)^{\theta_\alpha}$.*

THEOREM 2. *Let A be a real $n \times n$ symmetric matrix whose distinct characteristic roots are w_α $(\alpha = 1, \cdots, t)$. Let r_α be the multiplicity of w_α. There exists a real $(n+1) \times (n+1)$ symmetric matrix B having A in the top left-hand corner and with $f(\lambda)$ (now with real coefficients) as characteristic polynomial if and only if*

(a) $\qquad f(\lambda)$ *is divisible by* $\prod_{\alpha=1}^{t} (w_\alpha - \lambda)^{r_\alpha - 1}$

and

(b) $$\left[\frac{f(\lambda)}{(w_\beta - \lambda)^{r_\beta - 1}}\right]_{\lambda = \lambda_\beta} \cdot \prod_{\substack{\alpha=1 \\ \alpha \neq \beta}}^{t} (w_\alpha - w_\beta)^{r_\alpha} \quad (\beta = 1, \cdots, t)$$

is real and nonpositive.

REMARK. There is no difficulty in seeing that the conditions (a) and (b) imposed on $f(\lambda)$ are equivalent to the following: $f(\lambda)$ has only real roots wich are interlaced by the n characteristic roots of A.

2. We start with the following

LEMMA. *Let A be any $n \times n$ complex matrix with normal Jordan form J. In order that the matrix B referred in Theorem 1 exists, it is necessary and sufficient that there should exist a column X_1 (with n elements), a row Y_1 (with n elements) and a number q_1 such that*

$$\begin{bmatrix} J & X_1 \\ Y_1 & q_1 \end{bmatrix}$$

has $f(\lambda)$ as characteristic polynomial.

Proof. Let T be an $n \times n$ nonsingular matrix such that $TAT^{-1} = J$. Suppose B exists and is given by

$$B = \begin{bmatrix} A & X \\ Y & q \end{bmatrix}.$$

Let

$$S = \begin{bmatrix} T & 0 \\ 0 & 1 \end{bmatrix}.$$

We have

$$SBS^{-1} = \begin{bmatrix} J & TX \\ YT^{-1} & q \end{bmatrix}.$$

and so we can take $Y_1 = YT^{-1}$, $X_1 = TX$ and $q_1 = q$.

The converse is easily proved in a similar way.

Our next step is to deduce the characteristic polynomial of the matrix:

$$C_i = \begin{bmatrix} J_i & 0 & \cdots & 0 & X_i \\ 0 & J_{i+1} & \cdots & 0 & X_{i+1} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & J_m & X_m \\ Y_i & Y_{i+1} & \cdots & Y_m & q \end{bmatrix} \tag{2.1}$$

where, with obvious notation,

$$J_j = \begin{bmatrix} \lambda_j & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_j \end{bmatrix} \quad \text{(of type } s_j \times s_j\text{)}, \tag{2.2}$$

$$X_j = \begin{bmatrix} x_1^j \\ \vdots \\ x_{s_j}^j \end{bmatrix}, \quad Y_j = [y_1^j, \cdots, y_{s_j}^j] \qquad (j = i, \cdots, m)$$

and q is a complex number.

We expand $\det(C_i - \lambda E_i)$ (where E_i is the identity matrix of the same order as C_i) by Laplace Theorem in terms of its first s_i rows. In order to do this let us find all the nonzero minors contained in these rows. They are: $J_i \ \lambda E^{(i)}$ ($E^{(i)}$ denotes the identity matrix of the same order as J_i) and the s_i minors formed with $s_i - 1$ columns of $J_i - \lambda E^{(i)}$ and the column X_i. These s_i minors are given by

$$H_\rho = (-1)^{s_i - \rho} \begin{vmatrix} \overbrace{\lambda_i - \lambda \quad 1 \quad \cdots}^{\rho - 1 \text{ columns}} & x_1^i & 0 & \cdots & 0 \\ 0 \quad \lambda_i - \lambda \quad \cdots & x_2^i & 0 & \cdots & 0 \\ \cdots\cdots & \cdots & \cdots & \cdots & \cdots \\ 0 \quad 0 \quad \cdots & x_\rho^i & 1 & \cdots & 0 \\ 0 \quad 0 \quad \cdots & x_{\rho+1}^i & \lambda_i - \lambda & \cdots & 0 \\ \cdots\cdots & \cdots & \cdots & \cdots & \cdots \\ 0 \quad 0 \quad \cdots & x_{s_i}^i & 0 & \cdots & \lambda_i - \lambda \end{vmatrix}$$

$$(\rho = 1, \cdots, s_i) .$$

We have

$$H_\rho = (-1)^{s_i - \rho} (\lambda_i - \lambda)^{\rho - 1} P_\rho$$

with

$$P_\rho = \begin{vmatrix} x_\rho^i & 1 & 0 & \cdots & 0 \\ x_{\rho+1}^i & \lambda_i - \lambda & 1 & \cdots & 0 \\ x_{\rho+2}^i & 0 & \lambda_i - \lambda & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{s_i}^i & 0 & 0 & \cdots & \lambda_i - \lambda \end{vmatrix} .$$

Expanding P_ρ in terms of the first row we get

$$P_\rho = x_\rho^i(\lambda_i - \lambda)^{s_i-\rho} - P_{\rho+1}$$

and by induction it can be easily seen that

$$P_\rho = \sum_{\tau=0}^{s_i-\rho}(-1)^\tau x_{\rho+\tau}^i(\lambda_i - \lambda)^{s_i-\rho-\tau};$$

so we can write

$$H_\rho = \sum_{\tau=0}^{s_i-\rho}(-1)^{\tau+s_i-\rho}x_{\rho+\tau}^i(\lambda_i - \lambda)^{s_i-\tau-1}.$$

Let us now calculate the complementary minor $\widetilde{H}_\rho$ of H_ρ in $C_i - \lambda E_i$. There is no difficulty in seeing that

$$\widetilde{H}_\rho = \begin{array}{r|cccccc|}
 & \overbrace{}^{\text{1 column}} & & & & & \\
 & 0 & J_{i+1}-\lambda E^{(i+1)} & 0 & \cdots & 0 & 0 \\
 & 0 & 0 & J_{i+2}-\lambda E^{(i+2)} & \cdots & 0 & 0 \\
 & 0 & 0 & 0 & \cdots & 0 & 0 \\
 & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
 & 0 & 0 & 0 & \cdots & 0 & J_m-\lambda E^{(m)} \\
\text{1 row}\{ & y_\rho^i & Y_{i+1} & Y_{i+2} & \cdots & Y_{m-1} & Y_m
\end{array}$$

We have

$$\widetilde{H}_\rho = (-1)^\sigma y_\rho^i \prod_{j=i+1}^{m}(\lambda_j - \lambda)^{s_j},$$

with

$$\sigma = \sum_{k=i+1}^{m} s_k .$$

Bearing in mind that H_ρ was formed from the rows $1, \cdots, s_i$ and columns $1, \cdots, \rho - 1, \rho + 1, \cdots, s_i, \sum_{k=i}^m s_k + 1$, we have

$$\det(C_i - \lambda E_i) = \sum_{\rho=1}^{s_i}\sum_{\tau=0}^{s_i-\rho}(-1)^{\tau+1}y_\rho^i x_{\rho+\tau}^i(\lambda_i - \lambda)^{s_i-\tau-1}\prod_{j=i+1}^{m}(\lambda_j - \lambda)^{s_j}$$
$$+ \det(J_i - \lambda E^{(i)})\det[\operatorname{comp}(J_i - \lambda E^{(i)})],$$

where the symbol $\operatorname{comp}(J_i - \lambda E^{(i)})$ means the complementary minor of $J_i - \lambda E^{(i)}$ in the matrix $C_i - \lambda E_i$. Interchanging the order of the first two sums, noting that $\det(J_i - \lambda E^{(i)}) = (\lambda_i - \lambda)^{s_i}$ and that comp $(J_i - \lambda E^{(i)}) = \det(C_{i+1} - \lambda E_{i+1})$ we get

$$\det(C_i - \lambda E_i) = \sum_{\tau=0}^{s_i-1}\sum_{\rho=1}^{s_i-\tau}(-1)^{\tau+1}y_\rho^i x_{\rho+\tau}^i(\lambda_i - \lambda)^{s_i-\tau-1}\prod_{j=i+1}^{m}(\lambda_j - \lambda)^{s_j}$$
$$+ (\lambda_i - \lambda)^{s_i}\det(C_{i+1} - \lambda E_{i+1}).$$

Putting here successively $i = 1, 2, \cdots, m$ and writing for the sake of simplicity

$$b_{k\mu} = \sum_{\rho=1}^{\mu+1} (-1)^{s_k-\mu} y_\rho^k x_{\rho+s_k-1-\mu}^k \qquad (\mu = 0, \cdots, s_k - 1) , \tag{2.3}$$

we get after some manipulation

$$\begin{aligned} \det(C_1 - \lambda E_1) = \sum_{k=1}^{m} \left\{ \left[\sum_{\mu=0}^{s_k-1} b_{k\mu}(\lambda_k - \lambda)^\mu \right] \left[\prod_{\substack{j=1 \\ j\neq k}}^{m} (\lambda_j - \lambda)^{s_j} \right] \right\} \\ + (q - \lambda) \prod_{j=1}^{m} (\lambda_j - \lambda)^{s_j} . \end{aligned} \tag{2.4}$$

We are now ready for the proof of Theorem 1. Because of the lemma it is sufficient to prove the theorem assuming that A is in the Jordan normal form $J = \operatorname{diag}(J_1, \cdots, J_m)$ with J_j $(j = 1, \cdots, m)$ given by (2.2). So what we have to do is to find out under what conditions it is possible to find columns $X_1, \cdots, X_m$, rows $Y_1, \cdots, Y_m$ and a number q such that the characteristic polynomial (2.4) of the matrix C_1 be $f(\lambda)$.

As in the Jordan normal form the order in which the diagonal blocks occur is arbitrary, we can suppose without loss of generality that

$$\begin{gathered} \lambda_1 = \cdots = \lambda_{u_1} \quad (= w_1) \\ \cdots\cdots\cdots\cdots \\ \lambda_{u_{\alpha-1}+1} = \cdots = \lambda_{u_\alpha} \quad (= w_\alpha) \\ \cdots\cdots\cdots\cdots \\ \lambda_{u_{t-1}+1} = \cdots = \lambda_{u_t} \quad (= w_t) \end{gathered}$$

$$(u_\alpha = \textstyle\sum_{\beta=1}^{\alpha} r_\beta;\ r_\beta \text{ defined in Theorem 1})$$

with $w_\alpha \neq w_\beta$ if $\alpha \neq \beta$. With this notation, in J the characteristic root w_α appears in the diagonal blocks $J_{u_{\alpha-1}+1} \cdots, J_{u_\alpha}$ which are of orders $s_{u_{\alpha-1}+1} \cdots, s_{u_\alpha}$ respectively. We will assume that

$$s_{u_{\alpha-1}+1} \leqq \cdots \leqq s_u$$

for every α.

Let

$$\theta_\alpha = \sum_{\mu=u_{\alpha-1}+1}^{u_\alpha-1} s_\mu .$$

From (2.4) we have

$$\det(C_1 - \lambda E_1) = (w_\alpha - \lambda)^{\theta_\alpha} \varphi_\alpha(\lambda)$$

where $\varphi_\alpha(\lambda)$ is a polynomial in λ which is not necessarily divisible by $w_\alpha - \lambda$. As $\alpha \neq \beta$ implies $w_\alpha \neq w_\beta$ we will have

$$\det(C_1 - \lambda E_1) = \prod_{\alpha=1}^{t} (w_\alpha - \lambda)^{\theta_\alpha} \psi(\lambda) \tag{2.5}$$

where $\psi(\lambda)$ is a polynomial in λ not necessarily divisible by any factor of $h(\lambda) = \prod_{\alpha=1}^{t} (w_\alpha - \lambda)^{\theta_\alpha}$. Therefore, if $f(\lambda)$ is not divisible by $h(\lambda)$ it is impossible to find X_i, Y_i $(i = 1, \cdots, m)$ and q such that $f(\lambda) = \det(C_1 - \lambda E_1)$. Let us now suppose that $f(\lambda) = h(\lambda) f_1(\lambda)$. All we have to prove is that it is possible to find X_i, Y_i $(i = 1, \cdots, m)$ and q such that $\psi(\lambda) = f_1(\lambda)$.

Setting

$$S_k(\lambda) = \sum_{\mu=0}^{s_k - 1} b_{k\mu} (\lambda_k - \lambda)^\mu \tag{2.6}$$

and

$$\xi_\alpha = \sum_{\mu = u_{\alpha-1}+1}^{u_\alpha} s_\mu ,$$

(2.4) gives

$$\det(C_1 - \lambda E_1) = \sum_{\beta=0}^{t-1} \sum_{k=u_\beta+1}^{u_{\beta+1}} S_k(\lambda) \frac{\prod_{\alpha=1}^{t} (w_\alpha - \lambda)^{\xi_\alpha}}{(w_{\beta+1} - \lambda)^{s_k}} + (q - \lambda) \prod_{\alpha=1}^{t} (w_\alpha - \lambda)^{\xi_\alpha} \qquad (u_0 = 0) . \tag{2.7}$$

Let us choose $b_{k\mu} = 0$ for every $k \neq u_{\beta+1}$ $(\beta = 0, \cdots, t-1;\ \mu = 0, \cdots, s_k - 1)$. With this choice (2.7) gives

$$\det(C_1 - \lambda E_1) = \prod_{\gamma=1}^{t} (w_\gamma - \lambda)^{\theta_\gamma} \Big[\sum_{\beta=0}^{t-1} S_{u_{\beta+1}}(\lambda) \prod_{\substack{\alpha=1 \\ \alpha \neq \beta+1}}^{t} (w_\alpha - \lambda)^{s_{u_\alpha}} + (q - \lambda) \prod_{\alpha=1}^{t} (w_\alpha - \lambda)^{\xi_\alpha - \theta_\alpha} \Big]$$

and so by (2.5)

$$\psi(\lambda) = \sum_{\beta=0}^{t-1} S_{u_{\beta+1}}(\lambda) \prod_{\substack{\alpha=1 \\ \alpha \neq \beta+1}}^{t} (w_\alpha - \lambda)^{s_{u_\alpha}} + (q - \lambda) \prod_{\alpha=1}^{t} (w_\alpha - \lambda)^{s_{u_\alpha}} .$$

By (2.6) $S_{u_{\beta+1}}(\lambda)$ is a polynomial in $(w_{\beta+1} - \lambda)$ of degree $s_{u_{\beta+1}} - 1$. For the sake of simplicity we now change the notation (in an obvious way) writing

$$\psi(\lambda) = \sum_{\beta=0}^{t-1} R_\beta(\lambda) \prod_{\substack{\alpha=1 \\ \alpha \neq \beta+1}}^{t} (w_\alpha - \lambda)^{t_\alpha} + (q - \lambda) \prod_{\alpha=1}^{t} (w_\alpha - \lambda)^{t_\alpha} .$$

Let

$$R_\beta(\lambda) = \sum_{\mu=0}^{t_{\beta+1} - 1} \delta_{\beta\mu} (w_{\beta+1} - \lambda)^\mu .$$

We can write

$$\frac{\psi(\lambda)}{\prod_{\alpha=1}^{t}(w_\alpha - \lambda)^{t_\alpha}} = \sum_{\beta=0}^{t-1}\sum_{\mu=0}^{t_{\beta+1}-1} \frac{\delta_{\beta\mu}}{(w_{\beta+1} - \lambda)^{t_{\beta+1}-\mu}} + q - \lambda\,. \tag{2.8}$$

Let us resolve $f_1(\lambda)/\prod_{\alpha=1}^{t}(w_\alpha - \lambda)^{t_\alpha}$ into partial fractions. We will get

$$\frac{f_1(\lambda)}{\prod_{\alpha=1}^{t}(w_\alpha - \lambda)^{t_\alpha}} = \sum_{\beta=0}^{t-1}\sum_{\mu=0}^{t_{\beta+1}-1} \frac{A_{\beta\mu}}{(w_{\beta+1} - \lambda)^{t_{\beta+1}-\mu}} + Q - \lambda\,.$$

If now in (2.8) we take $\delta_{\beta\mu} = A_{\beta\mu}$ and $q = Q$ we will have $\psi(\lambda) = f_1(\lambda)$ as required. So we have given a process to choose all the $b_{k\mu}$ appearing in (2.6). To conclude the proof we show that it is always possible to find valuse x^i_σ, y^i_σ satisfying (2.3), no matter what values we have given to the $b_{k\mu}$. In fact, let us give to the x^i_σ arbitrary nonzero values ($x^i_\sigma = 1$, for example). Then, for each k, (2.3) becomes a system of linear equations in the y^k_ρ with a triangular matrix whose principal elements are different from zero. This means that the system is compatible. The proof of Theorem 1 is now complete.

COROLLARY. *If A is a complex nonderogatory matrix, then the matrix B of Theorem* 1 *always exists.*

Proof. If A is nonderogatory in its Jordan normal form there are no two diagonal blocks corresponding to the same characteristic root. So in Theorem 1 we have $r_\alpha = 1$ and so $\theta_\alpha = 0$. This means that B exists.

Proof of Theorem 2. If A is real and symmetric, the matrix T such that $TAT^{-1} = J$ can be chosen orthogonal and J will be a diagonal matrix. So using Theorem 1 we have $v_1^{(\alpha)} = \cdots = v_{r_\alpha}^{(\alpha)} = 1$ and $\theta_\alpha = r_\alpha - 1$. It follows that (a) is necessary and sufficient for the existence of a matrix B (not necessarily real and symmetric) of type $(n+1)\times(n+1)$ having A in the top left-hand corner and with $f(\lambda)$ as characteristic polynomial. Let us now find out the conditions for B to be real and symmetric. Choosing T orthogonal for B to fulfill this condition it is necessary and sufficient that there exist real X_j, Y_j, q $(j = 1, \cdots, m)$ with $X_j = Y_j$. Let us write $x^i_\rho = y^i_\rho$. We have now $\xi_\alpha = r_\alpha$, $\theta_\alpha = \xi_\alpha - 1$ and $S_k(\lambda) = b_{k0}$. Let

$$c_{\beta 0} = \sum_{k=u_\beta+1}^{u_{\beta+1}} b_{k0}\,. \tag{2.9}$$

The formula (2.7) gives

$$\det(C_1 - \lambda E_1) = \prod_{\gamma=1}^{t}(w_\gamma - \lambda)^{r_\gamma - 1}\left[\sum_{\beta=0}^{t-1} c_{\beta 0} \prod_{\substack{\alpha=1\\ \alpha\neq\beta+1}}^{t}(w_\alpha - \lambda) + (q-\lambda)\prod_{\alpha=1}^{t}(w_\alpha - \lambda)\right]$$

and so

$$\psi(\lambda) = \sum_{\beta=0}^{t-1} c_{\beta 0} \prod_{\substack{\alpha=1\\ \alpha\neq\beta+1}}^{t}(w_\alpha - \lambda) + (q-\lambda)\prod_{\alpha=1}^{t}(w_\alpha - \lambda) \,. \tag{2.10}$$

We are assuming that $f(\lambda)$ is divisible by

$$h(\lambda) = \prod_{\alpha=1}^{t}(w_\alpha - \lambda)^{r_\alpha - 1} \,.$$

Let $f(\lambda)/h(\lambda) = f_1(\lambda)$. Resolving $f_1(\lambda)/\prod_{\alpha=1}^{t}(w_\alpha - \lambda)$ into partial fractions we get

$$\frac{f_1(\lambda)}{\prod_{\alpha=1}^{t}(w_\alpha - \lambda)} = \sum_{\beta=0}^{t-1}\frac{B_\beta}{w_{\beta+1} - \lambda} + Q_1 - \lambda$$

with

$$B_\beta = \frac{f_1(w_{\beta+1})}{\prod_{\substack{\alpha=1\\ \alpha\neq\beta+1}}^{t}(w_\alpha - w_{\beta+1})} \,.$$

From (2.10) we have

$$\frac{\psi(\lambda)}{\prod_{\alpha=1}^{t}(w_\alpha - \lambda)} = \sum_{\beta=0}^{t-1}\frac{c_{\beta 0}}{w_{\beta+1} - \lambda} + q - \lambda \,.$$

So we must take

$$c_{\beta 0} = \frac{f_1(w_{\beta+1})}{\prod_{\substack{\alpha=1\\ \alpha\neq\beta+1}}^{t}(w_\alpha - w_{\beta+1})} \,, \quad q = Q_1 \,.$$

The equations (2.3) now take the form

$$b_{k0} = -[x_1^k]^2$$

or, by (2.9)

$$c_{\beta 0} = -\sum_{k=u_\beta+1}^{u_{\beta+1}}[x_1^k]^2 \,.$$

So B can be real and symmetric if and only if $c_{\beta 0} \leqq 0$ and Q_1 is real. The condition $c_{\beta 0} \leqq 0$ is equivalent to (b). Bearing in mind that $\sum_{\alpha=1}^{t} w_\alpha$ is real we can see easily that Q_1 is always real. With this the proof is complete.

In a similar way we could prove a theorem analogous to Theorem 2 but with 'real symmetric' substituted by 'hermitian'.

Note. After I had written this paper I noticed that Theorem 2 is not new. It is essentially equivalent to Theorem 1 in Fan and Pall, *Imbedding Conditions for Hermitian and Normal Matrices*, Canad. J. Math. 9 (1957), 298–304. However, the proof I have given here is a bit different from the proof of Fan and Pall. For further details see my forthcoming paper *Matrices with prescribed characteristic polynomial and a prescribed submatrix*–II (submitted to Pacific J. Math.).

I wish to thank the referee for his comments.

Reference

1. Farahat and Ledermann, *Matrices with prescribed characteristic polynomial*, Proc. Edinburgh Math. Soc. (2) **11** (1959), 143–146.

Received March 5, 1968.

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