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CONDITIONS FOR A MAPPING TO HAVE THE SLICING STRUCTURE PROPERTY

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Let $p: E \rightarrow B$ be a fibering in the sense of Serre. As is well known the fibering need not be a fibering in any stronger sense. However it is expected that if certain conditions are placed on E, p or B then p might be a fibration in a stronger sense. This paper gives such conditions.

The main result of this paper is:

THEOREM 1. Let p be an n-regular perfect map from a complete metric space (E, d) onto a locally equiconnected space B. If $\dim E \times B \leqq n$ then p has the slicing structure property (in particular p is a Hurewicz fibration).

The following definitions will be needed.

DEFINITION 1. A space X is *locally equiconnected* if for each point x, there exists a neighborhood U_x of x and a map

$$N: U_x \times U_x \times I \rightarrow X$$

satisfying $N(a, b, 0) = a$, $N(a, b, 1) = b$, and $N(a, a, t) = a$.

DEFINITION 2. A map p from E to B is n-regular if it is open and satisfies the following property: given any x in E and any neighborhood U of x there exists a neighborhood V of x such that if $f: S^m \rightarrow V \cap p^{-1}(y)$ for some $y \in B$ $(m \leqq n)$ then there exists

$$F: B^{m+1} \rightarrow U \cap p^{-1}(y)$$

which is an extension of f.

DEFINITION 3. A family $\mathscr{S}$ of sets of Y is equi-LC^n if for every $y \in S \in \mathscr{S}$ and every neighborhood U of y in Y there exists a neighborhood V of y such that for every $S \in \mathscr{S}$, every continuous image of an m-sphere $(m \leqq n)$ in $S \cap V$ is contractible in $S \cap U$.

Note 1. If $p: E \rightarrow B$ is n-regular then the collection $\{p^{-1}(b) \mid b \in B\}$ is equi-LC^n.

DEFINITION 4. A family $\mathscr{S}$ of sets of a metric space (Y, d) is *uniformly* equi-LC^n with respect to d if given $\varepsilon > 0$ there exists $\delta > 0$ such that if $f: S^m \rightarrow S \cap N(x, \delta)$ $(m \leqq n$ and $S \in \mathscr{S})$ then there exists $F: B^{m+1} \rightarrow S \cap N(x, \varepsilon)$ which is an extension of f.

DEFINITION 5. A map $p: E \rightarrow B$ has the *covering homotopy pro-*

perty for a class of spaces if given any space X in the class and maps $F: X \times I \to B$ and $g: X \to E$ such that $F(x, 0) = pg(x)$ then there exists a map: $G: X \times I \to E$ such that $pG = F$ and $G(x, 0) = g(x)$.

DEFINITION 6. A map $p: E \to B$ is a *Serre fibration* if p has the covering homotopy property for the class of polyhedra. It is a *Hurewicz fibration* if it has the covering homotopy property for all spaces.

DEFINITION 7. A map $p: E \to B$ has the *slicing structure property* (SSP) if for each point $b \in B$ there exists a neighborhood U_b of b and a map $\psi_b: p^{-1}(U_b) \times U_b \to p^{-1}(U_b)$ such that (1) $\psi_b(e, p(e)) = e$ and (2) $p\psi_b = \pi_2$ (the projection onto U_b).

DEFINITION 8. A function $\varphi: X \to 2^Y$ (Y metric) is continuous if given $\varepsilon > 0$; every $x_0 \in x$ has a neighborhood U such that for every $x \in U$, $\varphi(x_0) \subset N_\varepsilon(\varphi(x))$ and $\varphi(x) \subset N_\varepsilon(\varphi(x_0))$.

DEFINITION 9. A selection for a function $\varphi: X \to 2^Y$ is a map $g: X \to Y$ such that $g(x) \in \varphi(x)$.

A mapping is a continuous function. All spaces will be Hausdorff. The n-dimensional sphere will be denoted by S^n and the ball which it bounds B^{n+1}. If f is a mapping $\mathrm{Gr}(f)$ will denote the graph of f.

The following theorem of Michael will be needed:

THEOREM M. *Let Z be paracompact, let $X = Z \times I$ and let Y be a complete metric space with metric ρ. Let $\mathscr{S} \subset 2^Y$ be uniformly* equi-LC^n *with respect to ρ and let $\varphi: X \to \mathscr{S}$ be continuous with respect to ρ. Let* dim *$Z \leqq n$ and let $A = (Z \times 0) \cup (C \times I)$ where C is closed in Z. Then every selection for $\varphi \mid A$ can be extended to a* selection for φ.

2. Proof of Theorem 1 and its consequences.

Proof. Let $b_0 \in B$. Since B is locally equiconnected at b_0 there exists a neighborhood U of b_0 and a map $N_U: U \times U \times I \to B$ such that $N_U(x, y, 0) = x$, $N_U(x, y, 1) = y$, and $N_U(x, x, t) = x$. Let $P_U = p \mid p^{-1}(U)$ and define $g: \mathrm{Gr}(p_U) \to p^{-1}(U)$ by $g(e, p(e)) = e$. Also define $F: p^{-1}(U) \times B \to B$ by $F(e, b) = b$ and

$$H: p^{-1}(U) \times U \times I \to p^{-1}(U) \times B$$

by $H(e, b, t) = (e, N_U(p(e), b, t))$. Note $H(e, b, 0) = (e, N_U(p(e), b, 0)) = (e, p(e))$ and $H(e, b, 1) = (e, N_U(p(e), b, 1)) = (e, b)$. Further define

$$g'\colon (p^{-1}(U) \times U \times 0) \cup (\mathrm{Gr}(P_U) \times I) \to E$$

by $g'(e, b, t) = e$ and $K\colon p^{-1}(U) \times U \times I \to B$ by

$$K(e, b, t) = F(H(e, b, t))$$

and note that $pg' = K \mid (p^{-1}(U) \times U \times 0) \cup \mathrm{Gr}(P_U) \times I$.

Therefore we have the following commutative diagram.

$$\begin{tikzcd}
& & E \arrow[d, "p"] \\
(p^{-1}(U) \times U \times 0) \cup \mathrm{Gr}(p_U) \times I \arrow[urr, bend left, "g'"] & \subset p^{-1}(U) \times U \times I \arrow[ur, dotted, "G"] \arrow[r, "K"] & B
\end{tikzcd}$$

Now Theorem M will be applied. Let $Z = p^{-1}(U) \times U$, $Y = E$, and $\varphi\colon Z \times I \to \mathscr{S} \subset 2^Y$ be defined by $\varphi(z, t) = p^{-1}K(z, t)$ and let $C = \mathrm{Gr}(p_U)$. Note Z is paracompact and φ is continuous since p is perfect. Since p is n-regular $\{p^{-1}(b)\}$ in equi-LC^n and by Proposition 2.1 [3] there exists a metric σ on E agreeing with the topology such that $\sigma \geqq d$ and $\{p^{-1}(b)\}$ is uniformly equi-LC^n. Since $\sigma \geqq d$, (E, σ) is a complete metric space. It should also be noted that $\dim Z \leqq n$ and that g' is a selection for $\varphi \mid (Z \times 0) \cup (C \times I)$. Hence by Theorem M, g' could be extended to a selection G for φ (i.e.,

$$G\colon p^{-1}(U) \times U \times I \to E$$

in such a way that the above diagram will still be commutative with the addition of G).

Define $\varphi_U\colon p^{-1}(U) \times U \to p^{-1}(U)$ by $\varphi_U(e, b) = G(e, b, 1)$. Note if $(e, b) \in p^{-1}(U) \times U$ then

$$\begin{aligned} G(e, b, 1) \in p^{-1}K(e, b, 1) &= p^{-1}FH(e, b, 1) = p^{-1}F(e, N_U(p(e), b, 1)) \\ &= p^{-1}F(e, b) = p^{-1}(b) \in p^{-1}(U)\ . \end{aligned}$$

Hence the range of φ_U is as stated. It is now easy to see that φ_U satisfies the conditions to be a slicing function. This completes the proof.

Note 2. The hypothesis that p be perfect was used only to show that $\{p^{-1}(b) \mid b \in B\}$ is a continuous collection and that B is paracompact. Hence if this could be shown some other way a stronger theorem will be obtained.

COROLLARY 1. *If* $p: E \rightarrow B$ *is a Serre fibration and* E *and* B *are finite dimensional compact* ANR's *then* p *has the SSP.*

Proof. It is well known that ANR's are locally equiconnected.
It also follows from [2] that p is n-regular for all n. Hence the proof follows from Theorem 1.

Theorem 1 and Corollary 1 allow us to get the following generalizations of Raymond's results in [5].

COROLLARY 2. *Let* $p: E \rightarrow B$ *be a Serre fibration of a connected compact metric finite dimensional* ANR *onto a compact metric finite dimensional* ANR. *Suppose that* E *is an* n*-gm over* L (*a field or the integers*). *Then*:

(*a*) *each fiber* F_b *is a* k*-gm over* L

(*b*) B is an $(n\text{-}k) - gm$ *over* L.

COROLLARY 3. *Let* $p: E \rightarrow B$ *is a Serre fibration of a connected compact metric finite dimensional* ANR *onto a compact metric finite dimensional* ANR *base* B. *Suppose that* E *is a* (*generalized*) *manifold* (*over a principal ideal domain*) *and some fiber has a component of dimension* $\leqq 2$. *Then* p *is locally trivial.*

Another theorem which follows from Michael's Theorem 1.2 [3] is the following:

THEOREM 2. *Let* $p: E \rightarrow B$ *be an* n*-regular map from a complete metric space* E *onto a paracompact space* B. *Assume that*

$$\dim E \times B \leqq n + 1$$

and $p^{-1}(b)$ *is* C^n *for every* $b \in B$. *Then* p *has the* SSP *and the slicing structure could be chosen with only one slicing function.*

Proof. Define $g: \mathrm{Gr}(p) \rightarrow E$ by $g(e, p(e)) = e$ and $F: E \times B \rightarrow B$ by $F(e, b) = e$. The $\varphi(e, b) = p^{-1}F(e, b)$ is a carrier and g is a selection for $\varphi \mid \mathrm{Gr}(p)$. Hence by Theorem 1.2 [3] g could be extended to a selection G for φ. It is easily seen that G is the desired slicing function.

Note 3. Theorem 2 has corollaries similar to those of Theorem 1 and the author leaves them to the reader to develop.

REFERENCES

1. S. T. Hu, *Homotopy theory,* Academic Press, New York, 1959.

2. L. McAuley and P. Tulley, *Fiber spaces and N-regularity,* Topology Seminar Wisconsin 1965, Edited by R. Bing and R. Bean, Ann. of Math Studies, No. 60, Princeton University Press, Princeton, N. J., 1966.

3. E. Michael, *Continuous selections II,* Ann. of Math. **64** (1956), 567-580.

4. ———, *Continuous selections III,* Ann. of Math. **65** (1957), 375-390.

5. F. Raymond *Local triviality for Hurewicz fiberings of manifolds,* Topology **3** (1956), 43-57.

6. G. S. Ungar, *Light fiber maps,* Fund. Math. **62** (1968), 31-45.

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