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A NONIMBEDDING THEOREM OF ASSOCIATIVE ALGEBRAS

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Let A and B be associative algebras and define the Frattini subalgebra of $A, \phi(A)$, to be the intersection of all maximal subalgebras of A if maximal subalgebras of A exist and as A otherwise. Conditions on B will be found such that B cannot be an ideal of A contained in $\phi(A)$.

Hobby in [2] has shown that a nonabelian group G cannot be the Frattini subgroup of any p-group if the center of G is cyclic. Chao in [1] has shown that a nonabelian Lie algebra L can not be the Frattini subalgebra of any nilpotent Lie algebra if the center of L is one dimensional. In this note, we find a similiar result in the theory of associative algebras. However, in this case, it is not necessary to place any restrictions on the containing algebra.

Let A be an associative algebra over a field F and let B be an ideal of A. If $x \in A$, then x induces an endomorphism of the additive group of B by $L_x(b) = xb$ for all $b \in B$. Let $E(B, A)$ be the collection of all endomorphisms of this type. Then $E(B, A)$ is a subspace of the vector space of all linear transformations from B into B and is an associative algebra under the compositions $L_x + L_y = L_{x+y}$, $\alpha L_x = L_{\alpha x}$ and $L_x L_y = L_{xy}$ for all $x, y \in A$ and all $\alpha \in F$. Clearly $E(B, B)$ is an ideal of $E(B, A)$. If C is an ideal of A contained in B, then let $E(B, A, C) = \{E \in E(B, A); E(c) = 0 \text{ for all } c \in C\}$. Then $E(B, A, C)$ is an ideal of $E(B, A)$ and $E(B, A)/E(B, A, C)$ is isomorphic to $E(C, A)$. Note that the mapping from A onto $E(B, A)$ which assigns to each $a \in A$ the element L_a is an algebra homomorphism. We define the right annihilating series of B inductively. Let $r_1(B) = \{c \in B; bc = 0$ for all $b \in B\}$ and let $r_j(B)$ be the ideal of B such that $r_j(B)/r_{j-1}(B)$ $r_1(B/r_{j-1}(B))$ for $j > 1$. Since B is an ideal in A, $r_i(B)$ is an ideal in A for all i.

The following lemma is immediate.

LEMMA. *If A and A' are associative algebras and π is a homomorphism from A onto A', then $\pi(\phi(A)) \subseteq \phi(\pi(A))$. Furthermore, if the kernel of π is contained in $\phi(A)$, then $\pi(\phi(A)) = \phi(\pi(A))$.*

THEOREM. *Let B be an associative algebra such that* $\dim r_1(B) = 1$ *and* $\dim r_2(B) = k$ *where* $1 < k < \infty$. *Then B cannot be an ideal contained in the Frattini subalgebra of any associative algebra.*

Proof. Suppose that to the contrary B is an ideal contained in the Frattini subalgebra of the associative algebra A. Then

$$E(B, B) \subseteq \phi(E(B, A)) .$$

For if T is the mapping from A onto $E(B, A)$ defined by $T(a) = L_a$ for all $a \in A$, then, by the lemma,

$$E(B, B) = T(B) \subseteq T(\phi(A)) \subseteq \phi(T(A)) = \phi(E(B, A)) .$$

Let $z_1, \cdots, z_k$ be a basis for $r_2(B)$ such that z_k is a basis $r_1(B)$. For notational convenience, let $r_i = r_i(B)$ for all i. Let π be the natural homomorphism from $E(B, A)$ onto $E(r_2, A)$. Since

$$\begin{aligned} E(B, B) + E(B, A, r_2)/E(B, A, r_2) &\simeq E(B, B)/E(B, A, r_2) \cap E(B, B) \\ &= E(B, B)/E(B, B, r_2) \simeq E(r_2, B) \end{aligned}$$

it follows that

$$E(r_2, B) \simeq \pi(E(B, B)) \subseteq \pi(\phi(E(B, A))) \subseteq \phi(E(r_2, A)) .$$

We now show that $E(r_2, B) \not\subseteq \phi(E(r_2, A))$ by showing that $E(r_2, B)$ is complemented in $E(r_2, A)$. For $i = 1, \cdots, k - 1$, define linear transformations e_i from r_2 onto r_1 by

$$e_i(z_j) = \begin{cases} \delta_{ij} z_k & \text{for } j = 1, \cdots, k-1 \\ 0 & \text{for } j = k \end{cases}$$

where δ_{ij} is the Kronecker delta. Let $S = ((e_1, \cdots, e_{k-1}))$. We claim that $S = E(r_2, B)$. Since $r_1 = ((z_k))$ and $B \cdot r_2 \subseteq r_1, E(r_2, B) \subseteq S$. To show that $S = E(r_2, B)$, we shall show that $\dim E(r_2, B) = k - 1 = \dim S$. For each $x \in B$, L_x induces a linear transformation from r_2 into $r_1 \simeq F$, where F is the ground field. Therefore, we may consider each L_x, $x \in B$ as a linear functional on r_2. That is, $E(r_2, B) \subseteq (r_2)^*$ where $(r_2)^*$ is the dual space of r_2. Consequently, $\dim E(r_2, B) = \dim r_2 - \dim r_2^B$ where $r_2^B = \{z \in r_2; L_x(z) = 0 \text{ for all } x \in B\}$. Clearly $r_2^B = r_1$. Then, since $\dim r_2 = k$ and $\dim r_1 = 1$, $\dim E(r_2, B) = k - 1$ and $S = E(r_2, B)$.

We now show that S is complemented in $E(r_2, A)$. Let

$$M = \{E \in E(r_2, A); E(z_i) = \textstyle\sum_{j=1}^{k-1} \lambda_{ij} z_j, \lambda_{ij} \in F, i = 1, \cdots, k-1$$

and $E(z_k) = \lambda_k z_k, \lambda_k \in F\}$. M is clearly a subalgebra of $E(r_2, A)$ and $M \cap S = 0$. We claim that $M + S = E(r_2, A)$. Let $E \in E(r_2, A)$. Then $E(z_i) = \sum_{j=1}^{k-1} \lambda_{ij} z_j + \lambda_{ik} z_k$ for $i = 1, \cdots, k - 1$ and $E(z_k) = \lambda_k z_k$. However $E = E - \sum_{i=1}^{k-1} \lambda_{ik} e_i + \sum_{i=1}^{k-1} \lambda_{ik} e_i$ where $E - \sum_{i=1}^{k-1} \lambda_{ik} e_i \in M$ and $\sum_{i=1}^{k-1} \lambda_{ik} e_i \in S$. Therefore $M + S = E(r_2, A)$. We claim that $M \neq 0$. If $M = 0$, then $E(r_2, A) = E(r_2, B)$ which contradicts

$$E(r_2, B) \subseteq \phi(E(r_2, A)) \subset E(r_2 A) .$$

Consequently, S is complemented in $E(r_2, A)$, contradicting $S \subseteq \phi(E(r_2, A))$. This contradiction establishes the result.

COROLLARY. *Let B be a finite dimensional nontrivial nilpotent associative algebra with* $\dim r_1(B) = 1$. *Then B cannot be an ideal contained in the Frattini subalgebra of any associative algebra.*

REFERENCES

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