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LINEAR TRANSFORMATIONS OF TENSOR PRODUCTS PRESERVING A FIXED RANK

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In this paper T is a linear transformation from a tensor product $X \otimes Y$ into $U \otimes V$, where X, Y, U, V are vector spaces over an infinite field F. The main result gives a characterization of surjective transformations T for which there is a positive integer k ($k < \dim U$, $k < \dim V$) such that whenever $z \in X \otimes Y$ has rank k then also $Tz \in U \otimes V$ has rank k. It is shown that $T = A \otimes B$ or $T = S \circ (C \otimes D)$ where A, B, C, D are appropriate linear isomorphisms and S is the canonical isomorphism of $V \otimes U$ onto $U \otimes V$.

Let F be an infinite field and X, Y, U, V vector spaces over F. We denote by T a linear transformation of the tensor product $X \otimes Y$ into $U \otimes V$. The rank of a tensor $z \in X \otimes Y$ is denoted by $\rho(z)$. By definition $\rho(o) = 0$. The subspace of X spaned by the vectors $x_1, \cdots, x_n \in X$ will be denoted by $< x_1, \cdots, x_n >$.

LEMMA 1. *Let k be a positive integer such that $z \in X \otimes Y$ and $\rho(z) = k$ imply that $\rho(Tz) = k$. Then $\rho(z) \leqq k$ implies that $\rho(Tz) \leqq k$ for all z.*

Proof. If this is not true then for some $z \in X \otimes Y, z \neq 0$, we have $\rho(z) < k$ and $\rho(Tz) > k$. There exists $t \in X \otimes Y$ such that $\rho(t) + \rho(z) = k$ and moreover $\rho(z + \lambda t) = k$ for all $\lambda \neq 0, \lambda \in F$. Let

$$Tz = \sum_{i=1}^{m} u_i \otimes v_i \ , \qquad m = \rho(Tz) \ .$$

Since $u_i \in U$ are linearly independent and also $v_i \in V$ we can consider them as contained in a basis of U and V, respectively. The matrix of coordinates of Tz has the form

$$\left(\begin{array}{c|c} I_m & 0 \\ \hline 0 & 0 \end{array}\right)$$

where I_m is the identity $m \times m$ matrix. Let

$$\left(\begin{array}{c|c} A_m & B \\ \hline C & D \end{array}\right)$$

be the matrix of coordinates of Tt. Then the minor $| I_m + \lambda A_m |$ of the matrix of $T(z + \lambda t)$ has the form

$$1 + \alpha_1\lambda + \alpha_2\lambda^2 + \cdots .$$

Since F is infinite we can choose $\lambda \neq 0$ so that $|I_m + \lambda A_m| \neq 0$. For this value of λ we have

$$\rho(z + \lambda t) = k , \qquad \rho(T(z + \lambda t)) \geqq m > k$$

which contradicts our assumption. This proves the lemma.

LEMMA 2. *Let k be a positive integer such that $z \in X \otimes Y$ and $\rho(z) \leqq k$ imply $\rho(Tz) \leqq k$. If T is surjective and $k < \dim U$, $k < \dim V$ then $\rho(z) \geqq \rho(Tz)$ for all z.*

Proof. Assume that for some z we have $\rho(z) < \rho(Tz)$. Clearly, we can assume in addition that $\rho(z) = 1$. Therefore $k > 1$. By assumption $\rho(z) \leqq k$ implies that $\rho(Tz) \leqq k$. Let $s \leqq k$ be the maximal integer such that there exists $z \in X \otimes Y$ satisfying $\rho(z) < s$ and $\rho(Tz) = s$. Let

$$Tz = \sum_{i=1}^{s} u_i \otimes v_i .$$

We can choose $u_{s+1} \in U$, $v_{s+1} \in V$ such that $u_{s+1} \notin < u_1, \cdots, u_s >$ and $v_{s+1} \notin < v_1, \cdots, v_s >$. Since $u_i \in U$ are linearly independent and $v_i \in V$ also linearly independent we can assume that these vectors are contained in a basis of U and V, respectively. Since T is surjective there exists $t \in X \otimes Y$ such that $\rho(t) = 1$ and the $(s + 1, s + 1)$-coordinate $a_{s+1, s+1}$ of Tt is nonzero. The minor of order $s + 1$ in the upper left corner of the matrix of $T(z + \lambda t)$ has the form

$$a_{s+1,s+1}\lambda + \alpha_2\lambda^2 + \cdots .$$

Since $a_{s+1,s+1} \neq 0$ we can choose $\lambda \neq 0$ so that the minor is nonzero. For this value of λ we have

$$\rho(z + \lambda t) \leqq \rho(z) + 1 \leqq s \leqq k ,$$
$$\rho(T(z + \lambda t)) \geqq s + 1 .$$

If $s = k$ this contradicts our assumption. If $s < k$ this contradicts the maximality of s. Hence, Lemma 2 is proved.

LEMMA 3. *Let k be a positive integer such that $z \in X \otimes Y$ and $\rho(z) = k$ imply that $\rho(Tz) = k$. If T is surjective and $k < \dim U$, $k < \dim V$ then $\rho(z) = \rho(Tz)$ for each $z \in X \otimes Y$ satisfying $\rho(z) \leqq k$.*

Proof. The assertion is trivial if $\rho(z) = 0$ or k. Let $0 < \rho(z) < k$. Choose $t \in X \otimes Y$ such that

$$\rho(z + t) = \rho(z) + \rho(t) = k .$$

Using this and Lemmas 1 and 2 we deduce

$$\begin{aligned} &\rho(T(z + t)) = \rho(Tz + Tt) = k , \\ &\rho(Tz) + \rho(Tt) \geqq k , \\ &\rho(Tz) + \rho(t) \geqq k , \\ &\rho(Tz) \geqq \rho(z) . \end{aligned}$$

Since by Lemma 2, $\rho(Tz) \leqq \rho(z)$ we are ready.

The following Theorem is an immediate consequence of Lemma 3 and Theorem 3.4 of [3]:

THEOREM 1. *Let k be a positive integer such that $z \in X \otimes Y$ and $\rho(z) = k$ imply that $\rho(Tz) = k$. If T is surjective and $k <$ dim U, $k <$ dim V then*

$$T = A \otimes B , \tag{1}$$

or

$$T = S \circ (C \otimes D) , \tag{2}$$

where

$$\begin{aligned} A &: X \rightarrow U , \qquad & B &: Y \rightarrow V , \\ C &: X \rightarrow V , \qquad & D &: Y \rightarrow U , \end{aligned}$$

are bijective linear transformations and S is the canonical isomorphism of $V \otimes U$ onto $U \otimes V$.

This theorem gives a partial answer to a conjecture of Marcus and Moyls [2].

From Lemma 2 and Theorem 3.4 of [3] we get the following variant:

THEOREM 2. *Let k be a positive integer such that $z \in X \otimes Y$ and $\rho(z) \leqq k$ imply that $\rho(Tz) \leqq k$. If T is bijective and $k <$ dim U, $k <$ dim V then* (1) *or* (2) *holds.*

When $X = Y = U = V$, dim $X = n$, $k = n - 1$ we get a result of Dieudonné [1].

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