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CONJUGATE SURFACES FOR MULTIPLE INTEGRAL PROBLEMS IN THE CALCULUS OF VARIATIONS

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The Jacobi equation of the second variation for a multiple integral problem in the calculus of variations is a linear second order elliptic type partial differential equation provided certain hypotheses hold in the multiple integral problem. By means of the theory of quadratic forms in Hilbert space already present in the literature pertinent properties of solutions of such partial differential equations can be established. Here the pertinent property discussed is the vanishing of a solution on the boundary of a region, i.e. the existence of a conjugate surface of the differential equation. After developing the notion of focal point and stating the index theorems of the associated quadratic form, the existence of one parameter families of conjugate surfaces is shown, and illustrations of the theory are given.

1. **Introduction.** Fundamental theorems for quadratic forms in Hilbert space which are pertinent to problems in the calculus of variations are established in [7], [9] by Hestenes. Included in [7] is a theory of indices for an important class of quadratic forms arising in variational theory, and a general theory of focal points applicable to simple or multiple integral problems. Illustrations of the applications of focal point theory to one independent variable variational problems are given, as well as to boundary value problems for ordinary differential equations. The theory is, however, also applicable to multiple integral problems, and to boundary value problems for elliptic partial differential equations (indeed for integro-differential equations), and the author had this in mind in the formulation of the theory. In [9] there are general theorems on properties of quadratic forms applicable to variational problems involving functionals defined on classes of vector valued functions of m independent variables and with higher order derivatives. These theorems have as consequences further theorems on properties of systems of partial differential equations, and existence and differentiability theorems are established.

The purpose here is to set down an extension of the highly developed theory of conjugate points for simple integral problems in the calculus of variations to multiple integral problems. The extension is afforded by the theory established in [7]. Here the multiple integral problems in mind are those where the integrand involves at most first order partial derivatives of a real valued function of m real variables. The Jacobi equation of the second variation is then a linear second order partial differential equation, and it is relative to this equation that the notion of conjugate surface is considered.

For consistency all definition, terminology, and notation conventions used (apart from certain minor differences explicitly written down) are carried over from [7] and [9]. For brevity this material is not repeated, except where necessary for readability.

2. **Hilbert space and subspaces.** In the sequel m is a fixed positive integer, and Ω denotes m dimensional real euclidean space. Points of Ω are written $t = (t_1, \cdots, t_m)$, $s = (s_1, \cdots, s_m)$, $\cdots$ and $|t|$ is the usual length in Ω. If S is a subset of Ω, then $I(S)$, S^*, $\bar{S}$ and $c(S)$, mean, respectively, the interior of S, the boundary of S, the closure of S, and the complement of S. An interval $a_k \leqq t_k \leqq b_k$, $k = 1, \cdots, m$, is abbreviated $[a, b]$, and similarly for (a, b). A region is a bounded open connected subset of Ω. Letters such as x, y, z, u, v are used for real valued functions defined on subsets of Ω. The summation convention for repeated indices in a product is adopted, and all summations are from 1 to m. The subscripts i, j, k always have the range $1, \cdots, m$, while the subscripts p, q have the range $1, 2,, \cdots$. Subscripts k, p, q are never used as summation indices. A partial derivative $\partial x/\partial t_k$ is often written $\dot{x}_k$.

Let T be a fixed region of class B^1 (see [9], [10], [11]). Simple examples are: the interior of a sphere or interval in Ω, the interior of the union of a finite number of closed contiguous nonoverlapping intervals. Also, the image of one of these regions under a continuous one-to-one transformation, which is such that the transformation and its inverse satisfy a uniform Lipschitz condition on every compact subset of their respective domains, is also a region of class B^1 (henceforth the superscript 1 is omitted).

The basic Hilbert space $\mathscr{H}$ is the class of functions x of class $D^{(1)}$ on T ([9], [10], [11]), which together with their first partial derivatives are square integrable on T. This is the space $\mathscr{H}_1$ of [9], except that the functions are real scalar valued. A function $x \in \mathscr{H}$ need not be continuous on T, and is characterized by the following properties:

(i) x is essentially absolutely continuous on T in the sense of Calkin and Morrey ([3], [10], [11]);

(ii) x and the derivatives $\dot{x}_k$, $k = 1, \cdots, m$, are square integrable on T;

x is normalized by taking $x(t) = \lim_{h \to 0} x^h(t)$ for each $t \in T$ for which the limit exists, and $x(t) = 0$ elsewhere. Here x^h denotes the h-average of x ([9], p. 314). This normalization of members of $\mathscr{H}$ is convenient in the sequel. The space $\mathscr{H}$ is also called a Sobolev space ([11], p. 19). The inner product on $\mathscr{H}$ is

$$(x, y) = \int_T \dot{x}_j(t)\dot{y}_j(t)dt + \int_T x(t)y(t)dt \tag{2.1}$$

and the norm is

$$\| x \| = (x, x)^{1/2} . \tag{2.2}$$

An important subspace of $\mathscr{H}$ is the class of $x \in \mathscr{H}$ which "vanish" on the boundary T^*. More exactly, let C_0^∞ denote the subclass consisting of all functions x having continuous partial derivatives of all orders on T and whose support set (closure of the set of points t such that $x(t) \neq 0$) is contained in T. Then the subspace of interest is $\mathscr{A} = \overline{C_0^\infty}$, the closure under the norm in Eq. (2-2). This subspace is denoted by $\mathscr{H}_{10}$ in [9]. It can also be shown that $\mathscr{A} = \overline{\mathscr{K}_0}$, where $\mathscr{K}_0$ denotes the class of all Lipschitzian functions having support set contained in T. If S is any subset of Ω for which the function classes are defined, then it will be convenient at times to write $\mathscr{H}(S)$, $\mathscr{A}(S)$, $C_0^\infty(S)$, etc. For example, $C_0^p(S)$ denotes the class of all functions whose derivatives of order $\leqq p$ are continuous on T and whose support set is contained in S.

In the sequel the following alternate characterization of $\mathscr{A}$ is useful. Consider the space $\mathscr{H} = \mathscr{H}(T)$. Extend each $x \in \mathscr{H}$ to Ω by setting $x(t) = 0$ for $t \in c(T)$. A function x so extended need not belong to the Hilbert space $\mathscr{H}(\Omega)$. However the class of functions which do belong to $\mathscr{H}(\Omega)$ when so extended constitute the subspace $\mathscr{A}$.

3. Divergence theorem.

Use is made in the sequel of the following extension of the divergence theorem established by Hestenes. See also Morrey [12], and Carson ([4]).

THEOREM 3.1. *Let S be a nonempty open set in Ω, and let $M, N_1, \cdots, N_m$ be given integrable functions on S. Consider the linear functional*

$$L(x) = \int_S [M(t)x(t) + N_j(t)\dot{x}_j(t)]dt$$

on various linear manifolds. The following statements are equivalent.

(a) *$L(x) = 0$ on $\mathscr{K}_0(S)$.*

(b) *$L(x) = 0$ on $C_0^p(S)$, p a given positive integer.*

(c) *$L(x) = 0$ on $C_0^\infty(S)$.*

(d) *Let $\varepsilon > 0$ be given. Then $L(x) = 0$ on $\mathscr{K}_\varepsilon(S)$, the class of all $x \in \mathscr{K}_0(S)$ whose support set has diameter less than ε.*

(e) *If k is a given integer in the range $1, \cdots, m$, and S_k denotes the projection of S onto the t_k axis, then there exists a set Z_k*

of linear measure zero in S_k *such that for each interval* $[a, b]$ *in* S *having neither* a_k *nor* b_k *in* Z_k *the relation*

$$\sum_{j=1}^{m} \int_{a'_j}^{b'_j} [N(b_j, t') - N(a_j, t'_j)] dt'_j = \int_a^b M(t) dt \tag{3.1}$$

holds (*where primes denote the remaining* $m - 1$ *coordinates, e.g.,*

$$(b, t'_j) = (t_1, \cdots, t_{j-1}, b, t_{j+1}, \cdots, t_m) ,$$

and

$$dt'_j = dt_1 \cdots dt_{j-1} dt_{j+1} \cdots dt_m) .$$

In fact, for almost all intervals or spheres R *such that* $\bar{R}$ *is contained in* S,

$$\int_{R^*} N_i(\sigma) l_i d\sigma = \int_R M(t) dt \tag{3.2}$$

holds, where l_i, $i = 1, \cdots, m$, *are the direction cosines of the outer normal to* R^* *and* $d\sigma$ *donotes the surface element on* R^*.

If M *is continuous and each* N_k *has continuous first partial derivatives on* S, *and one of the statements* (a)—(e) *holds, then* Eq. (3.2) *holds for each sphere* R *such that* $\bar{R}$ *is contained in* S, *and* Eqs. (3.1) *and* (3.2) *are each equivalent to*

$$\sum_{j=1}^{m} \frac{\partial N_j}{\partial t_j} = M(t) \tag{3.3}$$

holding in S.

4. **The quadratic form.** Let $P, Q_1, \cdots, Q_m, R_{ij}, i, j = 1, \cdots, m$ be given integrable functions on T. It is assumed that $R_{ji}(t) = R_{ij}(t)$, $t \in T$, for all $i, j = 1, \cdots, m$. Then

$$J(x) = \int_T \{P(t)x^2(t) + [2Q_i(t)\dot{x}_i(t)]x(t) + R_{ij}(t)\dot{x}_i(t)\dot{x}_j(t)\} dt \tag{4.1}$$

defines a quadratic form J on $\mathscr{H}$. The associated bilinear form is

$$J(x, y) = \int_T \{Pxy + Q_i(x\dot{y}_i + \dot{x}_i y) + R_{ij}\dot{x}_i\dot{y}_j\} dt . \tag{4.2}$$

Let

$$\omega(t, x, \dot{x}) = \frac{1}{2}(Px^2 + 2Q_i\dot{x}_i x + R_{ij}\dot{x}_i\dot{x}_j) . \tag{4.3}$$

Then

$$J(x, y) = \int_T (\omega_x y + \omega_{\dot{x}_i} \dot{y}_i) dt$$
$$= \int_T (\omega_y x + \omega_{\dot{y}_i} \dot{x}_i) dt = J(y, x)$$

for each $x \in \mathscr{H}$, $y \in \mathscr{H}$. The quadratic form J can be written

$$J(x) = K(x) + R(x) \tag{4.4}$$

where

$$K(x) = \int_T \{P(t)x^2(t) + [2Q_i(t)\dot{x}_i(t)]x(t)\} dt \tag{4.5}$$

$$R(x) = \int_T R_{ij}(t)\dot{x}_i(t)\dot{x}_j(t) dt \tag{4.6}$$

are quadratic forms on $\mathscr{H}$.

It is further assumed that $P, Q_1, \cdots, Q_m$ are bounded on T. Then (Theorem 5.1, [7]), K is w-continuous on $\mathscr{H}$. Further, thefunctions R_{ij} are assumed continuous on $\bar{T}$, with the strong Legendre condition

$$R_{ij}(t)\xi_i\xi_j > 0 \qquad t \in \bar{T} \tag{4.7}$$

holding for each m-tuple $\xi \neq (0, \cdots, 0)$. Accordingly R is positive definite on $\mathscr{A}$ and hence (Theorem 8.1, [9]), the quadratic form J is a Legendre form on $\mathscr{A}$.

Let $\mathscr{B}$ be a linear manifold in $\mathscr{H}$. A function $x \in \mathscr{H}$ is said to be *J-orthogonal to* $\mathscr{B}$ if, for every $y \in \mathscr{B}$, $J(x, y) = 0$. The set of all such x is called the *J-orthogonal complement* of $\mathscr{B}$, denoted by $\mathscr{B}^J$. There may exist one or more $x \in \mathscr{B}$ which are J-orthogonal to $\mathscr{B}$, i.e., $x \in \mathscr{B} \cap \mathscr{B}^J$. A function x having this property is called a *J-null vector* of $\mathscr{B}$. The set of *J-null vectors* of $\mathscr{B}$ is denoted by $\mathscr{B}_0$. Observe $J(x) = 0$ on $\mathscr{B}_0$. The *nullity* of J on $\mathscr{B}$ is the dimension of the submanifold $\mathscr{B}_0$ of $\mathscr{B}$. The *index* of J on $\mathscr{B}$ is the dimension of the maximal linear submanifold $\mathscr{B}$ on which $J(x) < 0$. The following basic theorem is proven in [7].

THEOREM 4.1. *J is of finite index and nullity on $\mathscr{A}$. If $\mathscr{B}$ is a subspace of $\mathscr{A}$, $L(x)$ a linear form (functional) on $\mathscr{B}$ such that $L(x) = 0$ on the submanifold of J-null vectors of $\mathscr{B}$, then there exists a function $y \in \mathscr{B}$ such that $L(x) = J(x, y)$ on $\mathscr{B}$. The fuction y can be chosen orthogonal to the submanifold of J-null vectors of $\mathscr{B}$, and if so chosen is unique.*

5. **Extremals of J.** Let x be a function in $\mathscr{H}$ such that

$$J(x, y) = \int_T (\omega_x y + \omega_{\dot{x}_i} \dot{y}_i) dt = 0$$

holds for every $y \in \mathscr{A}$, i.e., x is J-orthogonal to the subspace $\mathscr{A}$. From the Divergence Theorem and the fact that $C_0^\infty(T)$ is dense in $\mathscr{A}$ it follows that

$$\int_{R^*} \omega_{\dot{x}_i} l_i d\sigma = \int_R \omega_x dt \tag{5.1}$$

holds for almost all intervals and spheres R whose closure lies in T. Eq. (5.1) is the *Euler equation* for the functional J on $\mathscr{H}$. A function $x \in \mathscr{H}$ such that Eq. (5.1) holds for almost all intervals and spheres with the stated property is called an *extremal of J*.

Accordingly the linear manifold of extremals of J is just $\mathscr{A}^J$, the J-orthogonal complement of $\mathscr{A}$. The submanifold $\mathscr{A}_0$ of J-null vectors of $\mathscr{A}$ is the class of extremals of J which vanish on T^*. In view of Theorem 4.1 this submanifold is finite dimensional.

With additional hypotheses on the coefficients the usual differential equation characterization of extremals is obtained. First, if the functions R_{ij} and Q_i are of class $C^1(T)$ and $P \in C(T)$, and if it is known that the extremal x is of class $C^2(T)$, then the Euler equation (5.1) is equivalent to

$$E(x) = \frac{\partial}{\partial t_i}\Big(R_{ij}\frac{\partial x}{\partial t_j}\Big) - x\Big(P - \sum_{i=1}^{m}\frac{\partial Q_i}{\partial t_i}\Big) = 0 \tag{5.2}$$

holding in T. Eq. (5.2) is the Euler equation associated with J as usually written. The differential operator E appearing in Eq. (5.2) may be referred to as the Euler operator. Under the preceding hypotheses it is an elliptic operator.

In order to insure that the extremal $x \in \mathscr{H}$ is of class $C^2(T)$ still further hypotheses are placed on the coefficients. The following is a special case of Sobolev's theorem as given by Friedrichs ([6]). Let p be an integer, $p > m/2 + 2$. In addition to the previous requirements assume the functions R_{ij} and Q_i are of class $C^p(T)$, and let $P \in C^{p-1}(T)$. Then if $x \in \mathscr{H}$ is an extremal, $x \in C^2(T)$. Henceforth it is assumed that the coefficients satisfy these additional requirements. Thus x is an extremal of J on T if, and only if, $x \in \mathscr{H} \cap C^2(T)$ and Eq. (5.2) holds on T. From Theorem 4.1 it follows that there are at most a finite number of linearly independent solutions of the Euler equation (5.2) which vanish on the boundary. The dimension of the submanifold of such solutions of (5.2) is the nullity of the form J on $\mathscr{A}$.

6. **Focal points of J. Index theorems.** In order to apply the theory of indices given in [7] there is considered in subsequent sections a one parameter family $\{\mathscr{A}(\lambda)\}$ of subspaces of $\mathscr{A}$, where the real parameter λ is restricted to an interval $\lambda' \leqq \lambda \leqq \lambda''$. The family

$\{\mathscr{A}(\lambda)\}$ has the following properties:

(a) $\mathscr{A}(\lambda')$ has as its sole member the function which is zero everywhere in T, and $\mathscr{A}(\lambda'') = \mathscr{A}$;

(b) if λ_1, λ_2 are such that $\lambda' \leqq \lambda_1 < \lambda_2 \leqq \lambda''$, then $\mathscr{A}(\lambda_1) \subset \mathscr{A}(\lambda_2)$;

(6.1) (c) if λ_0 is a value such that $\lambda' \leqq \lambda_0 < \lambda''$, then

$$\mathscr{A}(\lambda_0) = \Pi\mathscr{A}(\lambda) \qquad \lambda_0 < \lambda \leqq \lambda'' ;$$

(d) if λ_0 is a value such that $\lambda' < \lambda_0 \leqq \lambda''$, then

$$\mathscr{A}(\lambda_0) = \overline{\Sigma\mathscr{A}(\lambda)} \qquad \lambda' \leqq \lambda < \lambda_0 .$$

For each $\lambda, \lambda' \leqq \lambda \leqq \lambda''$, the symbols $\iota(\lambda), \nu(\lambda)$ denote, respectively, the index and the nullity of J on $\mathscr{A}(\lambda)$. Observe that the index ι is an integer valued function, monotone nondecreasing with increasing λ on the interval. Moreover $\iota(\lambda') = 0, \iota(\lambda'') = \iota_a$, the index of J on $\mathscr{A}$. In general $\iota_a \neq 0$, so that there exist one or more values λ_0 (though but a finite number of such values) in the interval at which ι is discontinuous, with jump

$$c(\lambda_0) = \iota(\lambda_0+) - \iota(\lambda_0-) > 0 .$$

Such a value λ_0 is called a *focal point of* J relative to the family $\{\mathscr{A}(\lambda)\}$. The value $c(\lambda_0)$ is termed the *order of* λ_0 as a focal point. In virtue of property (d) in (6.1) the left hand limit $\iota(\lambda_0-) = \iota(\lambda_0)$, $\lambda' < \lambda_0 \leqq \lambda''$ (see [7], § 16). The value $\iota(\lambda''+)$ is defined to be ι_a. The following theorem, a restatement of results established in [7], is applied in the sequel.

THEOREM 6.1. *Let $\{\mathscr{A}(\lambda)\}$ be a family of subspaces of $\mathscr{A}$ having the properties* (6.1). *Then, for $\lambda' \leqq \lambda < \lambda''$, the order $c(\lambda)$ of a focal point of J relative to $\{\mathscr{A}(\lambda)\}$ is the dimension of the maximal submanifold $\mathscr{C}(\lambda)$ of $\mathscr{A}_0(\lambda)$ having the property that no nontrivial function in $\mathscr{C}(\lambda)$ is J-orthogonal to a subspace $\mathscr{A}(\lambda_1), \lambda_1 > \lambda$. In the event the family $\{\mathscr{A}(\lambda)\}$ has the additional property that, whenever λ_1, λ_2 are values such that $\lambda' \leqq \lambda_1 < \lambda_2 \leqq \lambda''$, there exists no nontivial function in $\mathscr{A}$ which is J-orthogonal to both $\mathscr{A}(\lambda_1)$ and $\mathscr{A}(\lambda_2)$, then $c(\lambda') = c(\lambda'') = 0$, and for $\lambda' < \lambda < \lambda''$, the order $c(\lambda) = \nu(\lambda)$, the nullity of J on $\mathscr{A}(\lambda)$.*

7. **Conjugate surfaces in T.** Assume that there has been constructed a one parameter family $\{T(\lambda)\}$ of subsets of T, defined for $\lambda' \leqq \lambda \leqq \lambda''$, having the following properties:

(a) $T(\lambda')$ consists of a point of Ω, or else has $m-1$ dimensional measure zero, while $T(\lambda'') = T$;

(b) $T(\lambda)$ is a region of class B, $\lambda' < \lambda \leqq \lambda''$;

(c) if λ_1, λ_2 are such that $\lambda'' \leqq \lambda_1 < \lambda_2 < \lambda''$, then $\overline{T(\lambda_1)} \subset \overline{T(\lambda_2)}$; and $T^*(\lambda_1) \cap T(\lambda_2)$ is not empty:

(7.1) (d) if λ_0 is value such that $\lambda' \leqq \lambda_0 < \lambda''$, then

$$\overline{T(\lambda_0)} = \Pi\, \overline{T(\lambda)} \qquad \lambda_0 < \lambda \leqq \lambda'' ;$$

(e) if λ_0 is a value such that $\lambda' < \lambda_0 \leqq \lambda''$, then

$$T(\lambda_0) = \Sigma\, T(\lambda) \qquad \lambda' \leqq \lambda < \lambda_0 .$$

Examples of families of sets which have these properties are given subsequently.

THEOREM 7.1. *Let* $\{T(\lambda)\}$ *be a family of subsets of* T *having properties* (7.1). *Define the family* $\{\mathscr{A}(\lambda)\}$ *of subsets of* $\mathscr{A}$ *as follows*:

(i) $\mathscr{A}(\lambda')$ *is the set whose sole member is the function which is identically zero on* T, *and* $\mathscr{A}(\lambda'') = \mathscr{A}$;

(ii) *If* λ *is such that* $\lambda' < \lambda < \lambda''$, *the* $\mathscr{A}(\lambda)$ *is the set of all* $x \in \mathscr{A}$ *having support set* $\bar{S}_x$ *contained in* $\overline{T(\lambda)}$.

Then the family $\{\mathscr{A}(\lambda)\}$ *is a family of subspaces of* $\mathscr{A}$ *for which the properties* (6.1) *hold.*

Proof. Let λ_0 be a fixed but otherwise arbitrary value, $\lambda' < \lambda_0 < \lambda''$. It is readily verifiable that $\mathscr{A}(\lambda_0)$ is a closed linear manifold in $\mathscr{A}$. Let $\mathscr{H}(T(\lambda_0))$, $\mathscr{A}(T(\lambda_0))$ refer to, respectively, the Hilbert space, and the subspace of $\mathscr{H}(T(\lambda_0))$ composed of functions vanishing on $T^*(\lambda_0)$. Recall $\mathscr{A}(T(\lambda_0))$ is characterized as being the subset of all $x \in \mathscr{H}(T(\lambda_0))$ which when extended to vanish identically on $c(T(\lambda_0))$, belong to $\mathscr{H}(\Omega)$. It follows that $\mathscr{A}(\lambda_0)$ is just the subspace $\mathscr{A}(T(\lambda_0))$. For, if $x \in \mathscr{A}(\lambda_0)$, then x restricted to $T(\lambda_0)$ belongs to $\mathscr{H}(T(\lambda_0))$, while if x is extended to Ω by setting $x(t) = 0$ on $c(T(\lambda_0))$, then $x \in \mathscr{H}(\Omega)$ since $x \in \mathscr{A}$. Thus $x \in \mathscr{A}(T(\lambda_0))$. On the other hand

$$\mathscr{A}(T(\lambda_0)) = \overline{C_0^\infty(T(\lambda_0))} \subset \mathscr{A}(\lambda_0) .$$

To show that property (c) of (6.1) holds, let λ_0 be fixed, $\lambda' \leqq \lambda_0 < \lambda''$. If $x \in \mathscr{A}(\lambda_0)$, then the support set

$$\bar{S}_x \subset \overline{T(\lambda_0)} \subset \overline{T(\lambda)}$$

whenever $\lambda_0 < \lambda \leqq \lambda''$. Hence

$$x \in \Pi \mathscr{A}(\lambda) \qquad \lambda_0 < \lambda \leqq \lambda'' .$$

On the other hand, suppose x belongs to the intersection of the subspaces $\mathscr{A}(\lambda)$, for $\lambda_0 < \lambda \leqq \lambda''$. Then whenever $\lambda_0 < \lambda \leqq \lambda''$, $\bar{S}_x \subset \overline{T(\lambda)}$, so

$$\bar{S}_x \subset \Pi\, \overline{T(\lambda)} = \overline{T(\lambda_0)}$$

and hence $x \in \mathscr{A}(\lambda_0)$.

To establish property (d) of (6.1), let λ_0 be a value such that $\lambda' < \lambda_0 \leqq \lambda''$. Then it is clear from definitions that

$$\Sigma \mathscr{A}(\lambda) \subset \mathscr{A}(\lambda_0)$$

where the union is over all subspaces $\mathscr{A}(\lambda)$ for which $\lambda' \leqq \lambda < \lambda_0$. On the other hand, as noted above,

$$\mathscr{A}(\lambda_0) = \overline{C_0^\infty(T(\lambda_0))} \ .$$

Let $x \in C_0^\infty(T(\lambda_0))$, so the support set $\bar{S}_x \subset T(\lambda_0)$. Now there must exist λ_x, $\lambda' < \lambda_x < \lambda_0$, such that

$$\bar{S}_x \subset T(\lambda_x) \subset T(\lambda_0) \ .$$

For, if not, a sequence $\{\lambda_p\}$ exists such that $\lambda' < \lambda_p < \lambda_{p+1} < \lambda_0$, $p = 1, 2, \cdots$, and $\lambda_p \rightarrow \lambda_0$, and there exists a sequence $\{t_p\}$ of points such that $t_p \in \bar{S}_x \cap c(T(\lambda_p))$, $p = 1, 2, \cdots$, and $t_p \rightarrow t_0 \in \bar{S}_x$. The sequence $\{t_p\}$ has the property that given a value $\lambda, \lambda' < \lambda < \lambda_0$, there exists an integer q_λ such that $t_p \in c(T(\lambda))$ whenever $p > q_\lambda$. Accordingly $t_0 \in c(T(\lambda))$ for $\lambda' \leqq \lambda < \lambda_0$, and hence

$$t_0 \in \Pi c(T(\lambda) = c(\Sigma T(\lambda)) = c(T(\lambda_0))$$

where the intersection and union are taken for λ such that $\lambda' \leqq \lambda < \lambda_0$. This is a contradiction in view of $t_0 \in \bar{S}_x$, and $\bar{S}_x \subset T(\lambda_0)$. Thus a value λ_x having the stated property must exist, and so $x \in \mathscr{A}(\lambda_x)$. Accordingly

$$C_0^\infty(T(\lambda_0)) \subset \Sigma \mathscr{A}(\lambda)$$

for $\lambda' \leqq \lambda < \lambda_0$, and so

$$\mathscr{A}(\lambda_0) \subset \overline{\Sigma \mathscr{A}(\lambda)} \subset \mathscr{A}(\lambda_0) \ .$$

The following are examples of families of sets $\{T(\lambda)\}$ which have the properties (7.1).

EXAMPLE 1. Let $t_0 \in \Omega$ be fixed, and let $T(\lambda)$ denote the interior of the sphere $|t - t_0| = \lambda$, for $0 < \lambda \leqq r$, r a fixed positive number. Let $T(0) = \{t_0\}$, $T = \{t: |t - t_0| < r\}$.

EXAMPLE 2. Let T be a given interval (a, b) having positive measure, and let λ denote length measured along the diagonal joining the points a and b, where $\lambda'' = |b - a|$. Let c_k denote the k-th direction cosine of the line joining a to b. Let

$$T(\lambda) = (a, a + \lambda c) \qquad 0 < \lambda \leqq \lambda''$$

while $T(0) = \{a\}$.

EXAMPLE 3. Let $T = (a, b)$, and let t_0 denote the center of T. Define the family $\{T(\lambda)\}$ for $0 < \lambda \leqq 1$ by

$$T(\lambda) = \left(t_0 - \frac{\lambda}{2}(b - a), t_0 + \frac{\lambda}{2}(b - a)\right)$$

and let $T(0) = \{t_0\}$.

EXAMPLE 4. Let S denote an interval (a, b) of positive measure, and let t_0 be a point on the boundary S^*. Let V denote a hypercube $(t_0 - h/2, t_0 + h/2)$, where $h > 0$ is fixed. Let T be the union of S with V. Let $\{S(\lambda)\}$ be the family of expanding subsets constructed for the interval S in the same manner as for the interval in Example 2, for $0 \leqq \lambda \leqq \lambda''$, where $\lambda'' = |b - a|$. Let $\{V(\lambda)\}$ be the family of cubes

$$V(\lambda) = \left(t_0 - \frac{\lambda h}{2}, t_0 + \frac{\lambda h}{2}\right) \qquad 0 < \lambda \leqq 1$$

centered about the point t_0, and let $V(0) = \{t_0\}$. Define the family $\{T(\lambda)\}$ of subsets of T by

$$\begin{aligned} T(\lambda) &= S(\lambda) & 0 \leqq \lambda \leqq \lambda'' \\ &= S \cup V(\lambda - \lambda'') & \lambda'' \leqq \lambda \leqq \lambda'' + 1 \,. \end{aligned}$$

Instead of expanding to fill S and then T, one can have the family of subsets $\{T(\lambda)\}$ expand to fill V first, then T. Alternatively one can have the family $\{T(\lambda)\}$ expand into both sets simultaneously if

$$T(0) = \{t_0\} \quad T(\lambda) = S(\lambda) \cup V(\lambda/\lambda'') \quad 0 < \lambda \leqq \lambda'' \,.$$

In any case the desired properties (7.1) hold.

The following theorem shows that there exists a wide class of families $\{T(\lambda)\}$, each of which have the properties (7.1). The proof follows from the fact that closure and inclusion properties of sets is preserved under the transformations considered.

THEOREM 7.2. *Let T be a region of class B, and let $(T(\lambda))$ be a family of subsets of T having the properties* (7.1). *Let S be the* *image of* T, $S(\lambda)$ *the image of* $T(\lambda)$, $\lambda' \leqq \lambda \leqq \lambda''$, *under a continuous one-to-one transformation which is such that the transformation and its inverse satisfy a uniform Lipschitz condition on every compact subset of their respective domains. Then the family* $\{S(\lambda)\}$ *has the properties in* (7.1) *relative to the set* S, *which is a region of class* B.

With the foregoing in mind one can relate the index theory given in [7] to the notion of conjugate surface for the Euler equation as follows. Assume that Eq. (5.2) has the following weak unique continuation property: if T_1 is a region of class B, $\bar{T}_1 \subset T$, and if x is a solution of the differential equation in T which vanishes identically on $T - T_1$, then x vanishes identically on T. Now let u be a nontrivial solution of Eq. (5.2) on such a region T_1 and suppose u vanishes on the boundary T_1^*. Let y be the extension of u to T such that $y(t) = 0$ on $T - T_1$. Then $y \in \mathscr{A}$, but $y \notin \mathscr{A}_0$. For if y is a J-null vector of $\mathscr{A}$, then y is a solution of Eq. (5.2) on T. But then y vanishes identically on T. Hence u is identically zero on T_1, contrary to the supposition. Thus y cannot be a J-null vector of $\mathscr{A}$. However, since y restricted to T_1 is an extremal on T_1, it must be J-orthogonal to the subspace $\mathscr{A}(T_1)$. Observe y is a J-null vector of $\mathscr{A}(T_1)$, but not of any subspace $\mathscr{A}(S)$ where $S \supset \bar{T}_1$.

The index ι_a of J on $\mathscr{A}$ is given by the dimension of a maximal submanifold $\mathscr{C}$ of $\mathscr{A}$ on which $J(x) \leqq 0$ and which contains no nontrivial J-null vector of $\mathscr{A}$. In view of the preceding paragraph and the fact that $J(y) = 0$ it is seen that y belongs to such a submanifold $\mathscr{C}$. Thus $\iota_a \geqq 1$. *A conjugate surface* (of the Euler equation) is the boundary of a region of class B on which there vanishes a nontrivial solution of the Euler equation. Accordingly T_1^* is a conjugate surface. The existence of another conjugate surface T_2^* distinct from T_1^*, where $T_2 \supset T_1$, leads to the conclusion that the index $\iota_a \geqq 2$. These properties are analogous to those given for one dimensional problems in the calculus of variations.

The results given by Cordes ([5]) together with the smoothness assumptions stated in § 5 for the coefficients and the strong Legendre condition imply that Eq. (5.2) has the weak unique continuation property. Another result of this type is given by Aronszajn ([1]) and Calderon ([2]).

THEOREM 7.3. *Assume the coefficients in Eq.* (5.2) *have the properties stated heretofore. Let* $\{T(\lambda)\}$ *be a family of subsets of* T *having properties* (7.1), *and let* $\{\mathscr{A}(\lambda)\}$ *be the corresponding family of subspaces of* $\mathscr{A}$ *given by Theorem* 7.1. *Then*

(a) *a value* λ, $\lambda' < \lambda < \lambda''$, *is a focal point of* J *relative to the* *family* $\{\mathscr{A}(\lambda)\}$ *if, and only if,* $T^*(\lambda)$ *is a conjugate surface; moreover* λ' *and* λ'' *are not focal points;*

(b) *there are at most a finite number of conjugate surfaces in the family* $\{T^*(\lambda)\}$;

(c) *if* λ_j *is a focal point, then the order* $c(\lambda_j)$ *is* $\nu(\lambda_j)$, *the nullity of* J *on the subspace* $\mathscr{A}(\lambda_j)$, *and this is just the number of linearly independent solutions of Eq.* (5.2) *which vanish on* $T^*(\lambda_j)$, *in the maximal set of such nontrivial solutions;*

(d) *there exists a least focal point* $\lambda_1>0$ *in the interval* $\lambda'<\lambda<\lambda''$, *so that for* $\lambda' \leqq \lambda < \lambda_1$ *the index* $\iota(\lambda) = 0$ *and* $\nu(\lambda) = 0$;

(e) *let* $\lambda_1, \cdots, \lambda_N$ *be the focal points arranged in order of increasing magnitude, with respective orders* $\nu(\lambda_j), i = 1, \cdots, N$, *then the index of* J *on* $\mathscr{A}$ *is*

$$\iota_a = \sum_{j=1}^{N} \nu(\lambda_j) \ . \tag{7.2}$$

Proof. Since Eq. (5.2) has the requisite properties an argument exactly like that used above shows that whenever λ_1, λ_2 are values such that $\lambda' < \lambda_1 < \lambda_2 \leqq \lambda''$, then there exists no nontrivial functions in $\mathscr{A}$ which are J-orthogonal to both $\mathscr{A}(\lambda_1)$ and $\mathscr{A}(\lambda_2)$. Hence by Theorem 6.1 the order of a focal point λ_j is exactly the nullity $\nu(\lambda_j)$ of J on $\mathscr{A}(\lambda_j)$. Clearly λ' is not a focal point. One sets $\iota(\lambda''+) = \iota(\lambda'')$, so λ'' is not a focal point. There are but a finite number of focal points in the interval $\lambda' < \lambda < \lambda''$. If λ_1 is the least, then $\nu(\lambda) = 0$ for $\lambda' \leqq \lambda < \lambda_1$, so $\iota(\lambda) = 0$ on that subinterval.

It is noted that the index ι_a given by Eq. (7.2) is the same for every choice of a family $\{T(\lambda)\}$ of expanding subsets of T having properties (7.1).

8. Oscillation and comparison theorem.

The following theorem is a corollary of Theorem 7.1, [9]. It is observed that the proof does not depend on the weak unique continuation property assumed above for Eq. (5.2).

THEOREM 8.1. *There exists an* $\varepsilon > 0$ *such that if* S *is a region of class* $B, S \subset T$, *with the diameter of* S *at most* ε, *then* $J(x) > 0$ *holds for all nontrivial* $x \in \mathscr{A}(S)$. *Accordingly there are no conjugate surfaces contained in* S.

Proof. If S is a region of class $B, S \subset T$, then each $x \in \mathscr{A}(S)$ is extended to vanish identically on $c(S)$, and $\mathscr{A}(S)$ is a subspace of $\mathscr{A}$. Thus, in virtue of the theorem cited, there exists an $\varepsilon > 0$ and an $h>0$ such that if $S \subset T$ and the diameter of S is at most ε, then

$J(x) \geqq h \| x \|^2$ holds on $\mathscr{A}(S)$. Suppose S_1^* is a conjugate surface contained in S. Let u be the corresponding nontrivial solution of the Euler equation on S_1 which vanishes on S_1^*. Let y be the extension of u which vanishes on $c(S_1)$. Then $y \in \mathscr{A}(S)$, and moreover

$$J(y) = \int_S (\omega_y y + \omega_{\dot{y}_i} \dot{y}_i) dt = \int_{S_1} (\omega_u u + \omega_{\dot{u}_i} \dot{u}_i) dt = 0 \, .$$

Hence y is the trivial function on S, and so u is the trivial solution on S_1, contrary to the assumption. Thus there are no conjugate surfaces within S.

The following theorem is a consequence of Eq. (7.2) and the fact that if $J(x) \geqq 0$ holds on $\mathscr{A}$, then the index $\iota_a = 0$.

THEOREM 8.2. *If* $J(x) \geqq 0$ *holds on* $\mathscr{A}$, *then there are no conjugate surfaces properly contained in* T.

COROLLARY 8.3. *In addition to the assumptions of the strong Legendre condition and smoothness conditions made heretofore let* $P(t) > 0$ *on* T. *Then no solution on* T *of the differential equation*

$$\frac{\partial}{\partial t_i}\left(R_{ij} \frac{\partial x}{\partial t_j}\right) - P(t)x = 0$$

oscillates in T *in the sense that there exists no conjugate surface properly contained in* T.

Theorem 8.4 is a consequence of Theorem 16.3 [7], and the discussion in § 7.

THEOREM 8.4. *Let*

$$J^*(x) = \int_T \{P^*(t)x^2 + 2Q_i^*(t)x\dot{x}_i + R_{ij}^*(t)\dot{x}_i\dot{x}_j\} dt$$

$(i, j = 1, 2, \cdots, m)$ *be a quadratic form on* $\mathscr{H}$ *having suitable coefficients* $P^*(t), Q_i^*(t), R_{ij}^*(t)$ *such that the properties of* J *hold also for* J^*. *Moreover, suppose that*

$$J^*(x) \geqq J(x)$$

holds for all vectors $x \in \mathscr{A}$. *Let*

$$E^*(x) = \frac{\partial}{\partial t_i}\left(R_{ij}^* \frac{\partial x}{\partial t_j}\right) - x\left(P^* - \sum_{i=1}^{n} \frac{\partial Q_i^*}{\partial t_j}\right) = 0 \tag{8.1}$$

be the Euler equation corresponding to J^*. *Let* $\{T(\lambda)\}$ *be a family* *of subsets of T having the properties* (7.1). *Then the theorems on focal points and conjugate surfaces hold for Eq.* (8.1). *Let T_1^*, T_2^*, $\cdots$, T_N^*, be the distinct conjugate surfaces of Eq.* (5.2) *ordered according to the increasing and distinct focal points of J in the interval, and let $T_1^{*\prime}$, $T_2^{*\prime}$, $\cdots$, $T_{N^*}^{*\prime}$ be the distinct conjugate surfaces of Eq.* (8.1) *ordered according to the increasing and distinct focal points of J^* in the same interval. Let T_r, $r = 1, 2, \cdots, N$, be the member of the family $\{T(\lambda)\}$ having as its boundary T_r^* and let T_r', $r = 1, 2, \cdots, N^*$, be the member of the family $\{T(\lambda)\}$ having as its boundary $T_r^{*\prime}$. Then $T_r \subset T_r'$, $r = 1, 2, \cdots, N^*$. If $J^*(x) > J(x)$ holds for all nontrivial functions $x \in \mathscr{A}$, then $\bar{T}_r \subset T_r'$, $r = 1, 2, \cdots, N^*$.*

Order relations between the conjugate surfaces stated in the conclusion of Theorem 8.4 hold for the conjugate surfaces of the differential equations

$$E(x) = \frac{\partial}{\partial t_i}\left(R_{ij}\frac{\partial x}{\partial t_j}\right) - P(t)x = 0 \tag{8.2}$$

$$E^*(x) = \frac{\partial}{\partial t_i}\left(R_{ij}^*\frac{\partial x}{\partial t_j}\right) - P^*(t)x = 0 \tag{8.3}$$

provided the operator E is strongly elliptic and

$$R_{ij}^*(t)\xi_i\xi_j \geqq R_{ij}(t)\xi_i\xi_j \qquad t \in T \tag{8.4}$$

holds for each $\xi \in \Omega$, and $P^*(t) \geqq P(t)$ and for each $t \in T$. If strict inequality holds for some $t \in T$ in at least one of these inequalities, then the proper inclusion of the conjugate surfaces T_r^* in T_r' hold, for $r = 1, \cdots, N^*$.

If, in Eq. (8.2), $P = P(t, \mu)$, μ a real parameter, and is monotone strictly increasing with increasing μ, for each $t \in T$ (for example if $P(t; \mu) = P_1(t) + \mu$), then the proper inclusion of conjugate surfaces holds for the equations

$$\frac{\partial}{\partial t_i}\left(R_{ij}\frac{\partial x}{\partial t_j}\right) - P(t; \mu)x = 0$$

$$\frac{\partial}{\partial t_i}\left(R_{ij}\frac{\partial x}{\partial t_j}\right) - P(t; \mu^*)x = 0$$

where $\mu^* > \mu$.

9. Examples. In order to illustrate some of the results of the preceding sections the special case

$$J(x) = \int_T \{\dot{x}_i\dot{x}_i - \mu x^2\}dt \tag{9.1}$$

is considered, where μ denotes a constant. The corresponding Euler equation is

$$\Delta x + \mu x = 0 . \tag{9.2}$$

As a first case let T be the open interval $(0, b)$ in Ω. The class of extremals is the class of solutions of Eq. (9.2). If x is an extremal, then the J-orthogonality

$$J(x, y) = \int_0^b \{\dot{x}_i \dot{y}_i - \mu xy\} dt = 0$$

holds for every $y \in \mathscr{A}$; moreover $x(t)$ is analytic on $(0, b)$.

The class $\mathscr{A}_0$ of J-null vectors of $\mathscr{A}$ consists of all solutions $x(t)$ of the problem

$$\begin{aligned} \Delta x + \mu x &= 0 \quad \text{in} \quad (0, b) \\ x &= 0 , \qquad t \in T^* . \end{aligned} \tag{9.3}$$

There are at most a finite number of linearly independent solutions of this problem. Separable solutions of Problem (9.3) are of the form

$$x = \prod_{k=1}^{m} \sin \frac{n_k \pi t_k}{b_k} , \tag{9.4}$$

where the set $(n_1, \cdots, n_m)$ of positive integers satisfy the equation

$$\sum_{j=1}^{m} \left(\frac{n_j}{b_j} \right)^2 = \frac{\mu}{\pi^2} . \tag{9.5}$$

The set of functions of the form (9.4) spans the class $\mathscr{A}_0$ of J-null vectors of $\mathscr{A}$. There is but a finite number ν of linearly independent functions of this type, and the number ν is the nullity of J on $\mathscr{A}$. For if x is a function of the form (9.4) with positive integers satisfying (9.4), then $x \in \mathscr{A}_0$. Since the nullity of J on $\mathscr{A}$ must be finite, there are at most a finite number of linearly independent functions of this type. Suppose now that $x \in \mathscr{A}_0$, and let the Fourier series for $x(t)$ in T be

$$x(t) = \sum_{p_1, \cdots, p_m = 1}^{\infty} a_{p_1 \cdots p_m} \prod_{k=1}^{m} \sin \frac{p_k \pi t_k}{b_k} .$$

Since x must satisfy Eq. (9.2),

$$\sum_{p_1, \cdots, p_m = 1}^{\infty} a_{p_1 \cdots p_m} \left(-\sum_{j=1}^{m} \pi^2 \left(\frac{p_j}{b_j} \right) + \mu \right) \prod_{k=1}^{m} \sin \frac{p_k \pi t_k}{b_k} = 0 ,$$

holds on every closed set in T. Hence whenever $a_{p_1 \cdots p_m} \neq 0$, then the set $\{p_j\}$ of positive integers must satisfy

$$\sum_{j=1}^{m} \left(\frac{p_j}{b_j} \right)^2 = \frac{\mu}{\pi^2} \; .$$

There are but a finite number of distinct sets $\{p_j\}$ of positive integers which satisfy this last relation. Thus $x(t)$ must be a finite linear combination of functions of the type (9.4). The number ν of linearly independent functions of this type in a maximal set is the nullity of J on $\mathscr{A}$. In fact the nullity of J on $\mathscr{A}$ is given by M, where M denotes the sum of the counts of all distinct sets $(p_1, \cdots, p_m)$ of positive integers which satisfy (9.5). A set $(p_1, \cdots, p_m)$ is counted $m!/r!$ times whenever it has r of its elements alike.

Let l denote the length of the diagonal from 0 to the point b. Define the family $\{T(\lambda)\}$ of subintervals by $T(\lambda) = (0, \lambda b/l), 0 < \lambda < l$. Let $\{\mathscr{A}(\lambda)\}$ be the corresponding family of subspaces of $\mathscr{A}$. A function x is a J-null vector of $\mathscr{A}(\lambda)$ if and only if x is a linear combination of functions of the form

$$y(t) = \prod_{k=1}^{m} \sin \frac{p_k \pi t_k}{c_k} \qquad c_k = \frac{\lambda b_k}{l}, \; k = 1, \cdots, m$$

with $(p_1, \cdots, p_m)$ a set of positive integers satisfying

$$\sum_{i=1}^{m} \left(\frac{p_i}{b_i} \right)^2 = \frac{\lambda^2 \mu}{\pi^2 l^2} \; . \tag{9.6}$$

There is a set $\lambda_1, \cdots, \lambda_N$ of values λ in the interval $0 < \lambda < l$, such that for each λ_j there exists at least one set $(p_1, \cdots, p_m)$ of positive integers satisfying (9.6). These values λ_j of length along the diagonal are the distinct focal points of J relative to the family $\{\mathscr{A}(\lambda)\}$. The corresponding intervals $(0, \lambda_j b/l)$ have boundaries which are the distinct conjugate surfaces $T^*(\lambda_j)$ of J in T. Let $M(\lambda_j)$ denote the sum of the counts of sets $(p_1, \cdots, p_m)$ of positive integers satisfying (9.6) with λ replaced by λ_j, the count being made as indicated previously, for $j = 1, \cdots, N$. Then the index of J on $\mathscr{A}$ is

$$\iota_a = \sum_{j=1}^{N} M(\lambda_j) \; .$$

Consider now the equation

$$\Delta x + P(t)x = 0 \tag{9.7}$$

where $P \in C^{p-1}(T)$, $p > m/2 + 2$, and also bounded and integrable on $(0, b)$. Let $\lambda_1, \cdots, \lambda_N$ be the distinct focal points of J relative to the family $\{T(\lambda)\}$ of subintervals of T. Suppose $P(t) \geqq \mu$, $t \in T$. Then there are values $\lambda'_1, \lambda'_2, \cdots, \lambda'_{N'}$ of lengths along the diagonal in $0 < \lambda < l$, such that for each λ'_j there is at least one solution x of Eq. (9.7) in the subinterval $(0, \lambda_j' b/l)$ which vanishes on the boundary of the subinterval; moreover

$$[0, \lambda_j' b/l] \subset [0, \lambda_j b/l] \ .$$

If $P(t) > \mu$,

$$[0, \lambda_j' b/l] \subset (0, \lambda_j b/l)$$

$j = 1, 2, \cdots, N$. Let $\nu(\lambda_j')$ denote the number of linearly independent solutions of Eq. (9.7) which vanish on the boundary of $(0, \lambda_j' b/l)$, $j = 1, 2, \cdots, N'$. Then

$$\iota_a' = \sum_{j=1}^{N'} \nu(\lambda_j') > \sum_{j=1}^{N} M(\lambda_m) = \iota_a$$

where the numbers $M(\lambda_j)$ are those described in the preceding paragraph.

For a different example, let T be the interior of the circle of radius R about the origin, and let $m = 2$. Separable solutions of Eq. (9.2) in polar coordinates which are single valued in T are of the form

$$x = J_p(\mu r)[c_1 \cos p\theta + c_2 \sin p\theta] \ ,$$

where c_1, c_2 are constants, $p = 0, 1, 2, \cdots$, and J_p is the Bessel function of the first kind of order p. The class $\mathscr{A}_0$ of J-null vectors contains no nontrivial functions unless $\mu R > t_{01}$, where t_{01} is the first zero of $J_0(t)$, and in any case the nullity will be either zero or one. Let $T(\lambda)$ be the interior of the circle of radius λ about the origin, for $0 \leqq \lambda \leqq R$. Then $T^*(\lambda)$ is a conjugate curve if and only if

$$J_p(\mu\lambda) = 0$$

for some $p = 0, 1, 2, \cdots$. Let $J_0, J_1, \cdots, J_p$ be the Bessel functions of integral order which have at least one zero in the interval $0 < \lambda < \mu R$, and let ν_q be the number of zeros of $J_q(t)$ in this interval, for $q = 0, 1, \cdots, p$. Then the index

$$\iota_a = \sum_{q=0}^{p} \nu_q \ .$$

This value will be the same for any mode of expansion in sets $\{T(\lambda)\}$ having properties (7.1).

REFERENCES

1. N. Aronszajn, *A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order*, J. Math. Pure Appl. (9) **36** (1957), 235–249.

2. A. Calderon, *Uniqueness in the Cauchy problem for partial differential equations*,

Amer, J. Math. **80** (1958), 16-36.
3. J. W. Calkin, *Functions of several variables and absolute continuity* I, Duke Math. J. **6** (1940), 170.
4. A. B. Carson, *An analogue of Green's theorem for multiple integral problems in the calculus of variations*, Contributions to the Calculus of Variations, 1938-1941, University of Chicago Press.
5. H. O. Cordes, *Über die eindeutige Bestimmtheit der Lösungen elliptischer differentialgleichungen durch Anfangsvorgaben*, Akademie der Wissenschaften Göttingen, Math.- physikalische Klasse IIa. Nachrichten, 1956, 239.
6. K. Friedrichs, *On the differentiability of the solutions of linear elliptic differential equations*, Comm. Pure Appl. Math. **6** (1953), 299-326.
7. M. R. Hestenes, *Applications of the theory of quadratic forms in Hilbert space to the calculus of variations*, Pacific J. Math. **1** (1951), 525.
8. ———, *Calculus of variations and optimal control theory*, J. Wiley and Sons, Inc., New York, 1966.
9. ———, *Quadratic variational theory and linear elliptic partial differential equations*, Trans. Amer. Math. Soc. **101** (1961), 306-350.
10. C. Morrey Jr., *Functions of several variables and absolute continuity* II, Duke Math. J. **6** (1940), 439.
11. ———, *Multiple integrals in the calculus of variations*, Springer-Verlag, New York, 1966.
12. ———, *Multiple integral problems in the calculus of variations and related topics*, Univ. of California Publ. in Math. **1** (1943).

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