PACIFIC JOURNAL OF MATHEMATICS
Vol. 31, No. 1, 1969

SYMMETRIC POSITIVE DEFINITE MULTILINEAR FUNCTIONALS WITH A GIVEN AUTOMORPHISM

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Let V be an n-dimensional vector space over the real numbers R and let φ be a multilinear functional,

$$\varphi\colon \overset{m}{\underset{1}{\times}} V \longrightarrow R \tag{1}$$

i.e., $\varphi(x_1, \cdots, x_m)$ is linear in each x_j separately, $j = 1, \cdots, m$. Let H be a subgroup of the symmetric group S_m. Then φ is said to be *symmetric* with respect to H if

$$\varphi(x_{\sigma(1)}, \cdots, x_{\sigma(m)}) = \varphi(x_1, \cdots, x_m) \tag{2}$$

for all $\sigma \in H$ and all $x_j \in V$, $j = 1, \cdots, m$. (In general, the range of φ may be a subset of some vector space over R.) Let $T\colon V \to V$ be a linear transformation. Then T is an *automorphism* with respect to φ if

$$\varphi(Tx_1, \cdots, Tx_m) = \varphi(x_1, \cdots, x_m) \tag{3}$$

for all $x_i \in V$, $i = 1, \cdots, m$. It is easy to verify that the set $\mathfrak{A}(H, T)$ of all φ which are symmetric with respect to H and which satisfy (3) constitutes a subspace of the space of all multilinear functionals symmetric with respect to H. We denote this latter set by $M_m(V, H, R)$.

We shall say that φ is *positive definite* if

$$\varphi(x, \cdots, x) > 0 \tag{4}$$

for all nonzero x in V, and we shall denote the set of all positive definite φ in $\mathfrak{A}(H, T)$ by $P(H, T)$. It can be readily verified that $P(H, T)$ is a convex cone in $\mathfrak{A}(H, T)$.

Our main results follow.

THEOREM 1. *If $P(H, T)$ is nonempty then*

(a) *m is even*

and

(b) *every eigenvalue of T has modulus* 1.

If, in addition, T has only real eigenvalues then

(c) *every elementary divisor of T is linear.*

Conversely if (a), (b) *and* (c) *hold then $P(H, T)$ is nonempty. Moreover, if $P(H, T)$ is nonempty then $\mathfrak{A}(H, T)$ is the linear closure of $P(H, T)$.*

In Theorem 2 we shall investigate the dimension of $\mathfrak{A}(H, T)$ in the event $P(H, T)$ is not empty. To do this we must introduce some combinatorial notation. Let $\Gamma_{m,n}$ denote the set of all sequences

$\omega = (\omega_1, \cdots, \omega_m)$ of length $m, 1 \leqq \omega_i \leqq n, i = 1, \cdots, m$. Introduce an equivalence relation $\sim$ in $\Gamma_{m,n}$ as follows: $\alpha \sim \beta$ if there exists a $\sigma \in H$ such that

$$\alpha^\sigma = \beta$$

where $\alpha^\sigma = (\alpha_{\sigma(1)}, \cdots, \alpha_{\sigma(m)})$. Let Δ be a system of distinct representatives for $\sim$, i.e., Δ is a set of sequences, one from each equivalence class with respect to $\sim$. We specify Δ uniquely by choosing each element $\alpha \in \Delta$ to be lowest in lexicographic order in the equivalence class in which α occurs.

THEOREM 2. *If $P(H, T)$ is nonempty and T has real eigenvalues $\gamma_1, \cdots, \gamma_n$ then $\gamma_i = \pm 1$, $i = 1, \cdots, n$. Suppose*

$$\gamma_{i_1} = \cdots = \gamma_{i_p} = 1 , \qquad \gamma_j = -1 , \qquad j \neq i_1, \cdots, i_p .$$

Let μ_p be the number of sequences ω in Δ such that the total number of occurrences of $i_1, \cdots, i_p$ in ω is even. Then

$$\dim \mathfrak{A}(H, T) = \mu_p . \tag{5}$$

COROLLARY. *If $H = S_m$ in Theorem 2 and T has p eigenvalues 1 and $n - p$ eigenvalues -1 then*

$$\dim \mathfrak{A}(H, T) = \sum_{k=0}^{m/2} \binom{p - 1 + 2k}{p - 1} \binom{n - p - 1 + m - 2k}{n - p - 1} .$$

In case $m = 2$, $H = S_2$, $\mathfrak{A}(H, T)$ is the totality of symmetric bilinear functionals φ for which

$$\varphi(Tx_1, Tx_2) = \varphi(x_1, x_2) , \qquad x_1, x_2 \in V ,$$

and $P(H, T)$ is just the cone of positive definite φ in $\mathfrak{A}(H, T)$ i.e.,

$$\varphi(x, x) \geqq 0$$

with equality only if $x = 0$. In this case we need not assume that T has real eigenvalues in order to analyze $\mathfrak{A}(H, T)$. We can easily prove the following result by our methods, most parts of which are known (see e.g. [1], Chapter 7).

THEOREM 3. *Assume that $m = 2$ and $H = S_2$. Then $P(H, T)$ is nonempty if and only if*

(a) *T has linear elementary divisors over the complex field,*

(b) *every eigenvalue of T has modulus 1.*

Suppose that T has distinct complex eigenvalues

$$\gamma_k = a_k + ib_k \qquad (\textit{and } \bar{\gamma}_k = a_k - ib_k)$$

of multiplicity e_k, $k = 1, \cdots, p$ *and real eigenvalues*

$$\gamma_k = r_k \,, \qquad k = \sum_{j=1}^{p} 2e_j + 1, \cdots, n \,.$$

If $P(H, T)$ *is nonempty then the elementary divisors of* T *over the real field are*

$$\begin{array}{lll} \lambda^2 - 2\lambda a_k + 1 \,, & e_k \text{ times}, & k = 1, \cdots, p \,, \\ \lambda - 1 \,, & q \text{ times}, & \\ \lambda + 1 \,, & l \text{ times}, & \end{array}$$

where

$$\sum_{j=1}^{p} 2e_j + q + l = n \,.$$

Moreover, $\mathfrak{A}(H, T)$ *is the linear closure of* $P(H, T)$,

$$\dim \mathfrak{A}(H, T) = \frac{q(q+1)}{2} + \frac{l(l+1)}{2} + \sum_{j=1}^{p} e_j^2 \,,$$

and there exists a basis E *of* V *such that* $\mathfrak{A}(H, T)$ *consists of the set of all* φ *whose matrix representations with respect to* E, $[\varphi]_E^E$, *have the following form*:

$$[\varphi]_E^E = \sum_{k=1}^{p}{}^{\boldsymbol{\cdot}} (X_k \otimes I_2 + Y_k \otimes F) \dotplus H_q \dotplus H_l \,. \tag{6}$$

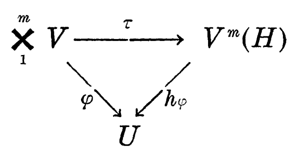
In (6), the dot indicates direct sum, $\otimes$ denotes the Kronecker product, $F = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, X_k is e_k-square symmetric, Y_k is e_k-square skew-symmetric, H_q and H_l are q-square and l-square symmetric respectively.

2. Proofs. Let $V^m(H)$ denote the symmetry class of tensors associated with H [2]. That is, there exists a fixed multilinear function $\tau\colon \times_1^m V \to V^m(H)$ symmetric with respect to H, for which

(i) the linear closure of $\tau(\times_1^m V)$ is $V^m(H)$

(ii) the pair $(V^m(H), \tau)$ is universal: given any space U and any multilinear function $\varphi\colon \times_1^m V \to U$ symmetric with respect to H, there exists a (unique) linear $h_\varphi\colon V^m(H) \to U$ satisfying

$$h_\varphi \tau = \varphi. \tag{7}$$



We shall denote $\tau(x_1, \cdots, x_m)$ by $x_1 * \cdots * x_m$, and $x_1 * \cdots * x_m$ is called a decomposable tensor or a symmetric product of $x_1, \cdots, x_m$. If we take $\varphi(x_1, \cdots, x_m)$ to be $Tx_1 * \cdots * Tx_m$ in (7) then h_φ is denoted by $K(T)$ and is called the *induced transformation* on $V^m(H)$.

Before we embark on the proof of Theorem 1 we can define $\mathfrak{A}(H, T)$ in terms of $V^m(H)$. First observe that the mapping θ from the space of multilinear functionals φ symmetric with respect to H to the dual space of $V^m(H)$,

$$\theta: M_m(V, H, R) \longrightarrow (V^m(H))^* ,$$

defined by

$$\theta(\varphi) = h_\varphi ,$$

is one-to-one linear, and onto. That is, the correspondence $\varphi \leftrightarrow h_\varphi$ is linear bijective. Now, the subspace $\mathfrak{U}(H, T)$ of $M_m(V, H, R)$ is defined by

$$\varphi(Tx_1, \cdots, Tx_m) = \varphi(x_1, \cdots, x_m)$$

or in view of (7) by

$$h_\varphi(Tx_1 * \cdots * Tx_m) = h_\varphi(x_1 * \cdots * x_m) ,$$

for all $x_i \in V$, $i = 1, \cdots, m$. In other words, since the decomposable tensors span $V^m(H)$ (see (i) above), $\varphi \in \mathfrak{A}(H, T)$ if and only if $\theta(\varphi) = h_\varphi$ satisfies

$$h_\varphi K(T) = h_\varphi ,$$

or

$$h_\varphi(K(T) - I) = 0 \tag{8}$$

where I is the identity mapping on $V^m(H)$. We have proved the following.

LEMMA 1. *$\mathfrak{A}(H, T)$ is nonempty if and only if $K(T) - I$ is singular. Also,*

$$\dim \mathfrak{A}(H, T) = \eta(K(T) - I) \tag{9}$$

where η is the nullity of the indicated transformation.

LEMMA 2. *If $P(H, T)$ is nonempty then m is even and every eigenvalue of T has modulus 1. Moreover, corresponding to real eigenvalues, T has only linear elementary divisors.*

Proof. If $\varphi \in P(H, T)$ and $x \neq 0$ then

$$\varphi(-x, \cdots, -x) = (-1)^m \varphi(x, \cdots, x)$$

and hence $(-1)^m > 0$ and m is even. Suppose that γ is a real eigenvalue of T with corresponding eigenvector x. Then

$$\begin{aligned} \varphi(Tx, \cdots, Tx) &= \varphi(\gamma x, \cdots, \gamma x) \\ &= \gamma^m \varphi(x, \cdots, x) . \end{aligned}$$

Since $\varphi \in P(H, T)$, $\varphi(Tx, \cdots, Tx) = \varphi(x, \cdots, x) > 0$ and hence $\gamma^m = 1$ and $\gamma = \pm 1$. If γ were involved in an elementary divisor of degree greater than 1 then there would exist linearly independent vectors u_1 and u_2 such that $Tu_1 = \gamma u_1$, $Tu_2 = \gamma u_2 + u_1$ and hence

$$\varphi(Tu_1, \cdots, Tu_1, Tu_2) = \varphi(\gamma u_1, \cdots, \gamma u_1, \gamma u_2 + u_1) .$$

Now

$$\begin{aligned} \varphi(u_1, \cdots, u_1, u_2) &= \gamma^m \varphi(u_1, \cdots, u_1, u_2) \\ &= \varphi(\gamma u_1, \cdots, \gamma u_1, \gamma u_2) \end{aligned}$$

so that

$$\begin{aligned} 0 &= \varphi(\gamma u_1, \cdots, \gamma u_1, \gamma u_2 + u_1) - \varphi(\gamma u_1, \cdots, \gamma u_1, \gamma u_2) \\ &= \varphi(\gamma u_1, \cdots, \gamma u_1, u_1) \\ &= \gamma^{m-1} \varphi(u_1, \cdots, u_1) , \end{aligned}$$

a contradiction.

We now show that any complex eigenvalue of T has modulus 1. Since $\gamma = a + ib$ is now assumed not to be real there exists a pair of linearly independent vectors v_1 and v_2 in V such that

$$\begin{aligned} Tv_1 &= av_1 - bv_2 \\ Tv_2 &= bv_1 + av_2 . \end{aligned} \tag{10}$$

Let $\bar{V}$ be the extension of V to an n-dimensional space over the complex field. Now $\varphi \in \mathfrak{A}(H, T)$ means that

$$\varphi(Tx_1, \cdots, Tx_m) - \varphi(x_1, \cdots, x_m) = 0 \tag{11}$$

is an identity in $x_1, \cdots, x_m$. If we express the vectors in $\bar{V}$ in terms of a basis in V (using in general complex rather than real coefficients) the identity (11) continues to hold since it is a homogeneous polynomial of degree m in the components of $x_1, \cdots, x_m$, vanishing for all real values of these components. Of course it is not true that

$$\varphi(x, \cdots, x) > 0$$

continues to hold for nonzero $x \in \bar{V}$. Now define

$$e_1 = v_1 + iv_2 \in \bar{V}$$
$$e_2 = v_1 - iv_2 \in \bar{V} \tag{12}$$

and observe that e_1 and e_2 are linearly independent in $\bar{V}$ and satisfy

$$Te_1 = \gamma e_1$$
$$Te_2 = \bar{\gamma} e_2 \, .$$

Let $\omega = (\omega_1, \cdots, \omega_m)$ be a sequence for which each ω_i is either 1 or 2, $i = 1, \cdots, m$:

$$\varphi(Te_{\omega_1}, \cdots, Te_{\omega_m}) = \gamma^k \bar{\gamma}^{m-k} \varphi(e_{\omega_1}, \cdots, e_{\omega_m}) \, ,$$

where k of the ω_i are 1 and $m - k$ are 2. But by the above remarks

$$\varphi(Te_{\omega_1}, \cdots, Te_{\omega_m}) = \varphi(e_{\omega_1}, \cdots, e_{\omega_m})$$

and taking absolute values we have

$$(|\gamma|^m - 1) \, |\varphi(e_{\omega_1}, \cdots, e_{\omega_m})| = 0 \, .$$

Thus if $|\gamma| \neq 1$ it follows that

$$\varphi(e_{\omega_1}, \cdots, e_{\omega_m}) = 0 \tag{13}$$

for all ω for which ω_i is 1 or 2 for $i = 1, \cdots, m$. From (12) we have $v_1 = (e_1 + e_2)/2$ and hence using (13) we see that

$$\begin{aligned} \varphi(v_1, \cdots, v_1) &= \varphi\left(\frac{e_1 + e_2}{2}, \cdots, \frac{e_1 + e_2}{2}\right) \\ &= 0 \, . \end{aligned} \tag{14}$$

However $v_1 \in V$ and $\varphi \in P(H, T)$ and therefore (14) is a contradiction. Thus $|\gamma| = 1$ and the proof of Lemma 2 is complete.

LEMMA 3. *If m is even, and T has real eigenvalues $r_1, \cdots, r_n$, and every elementary divisor of T is linear then $P(H, T)$ is nonempty.*

Proof. Since T has linear elementary divisors there exists a basis for V of eigenvectors $e_1, \cdots, e_n$. Let $g_1, \cdots, g_n$ be a dual basis in V^*. Let g_t^m denote the multilinear functional whose value for any $x_1, \cdots, x_m$ in V is

$$\prod_{j=1}^{m} g_t(x_j) \, .$$

Clearly $g_t^m \in M_m(V, H, R)$. Set

$$\varphi = \sum_{t=1}^{n} g_t^m .$$

Then if $x_j = \sum_{k=1}^{n} \xi_{jk} e_k$, $j = 1, \cdots, m$, and $Te_k = r_k e_k$, $k = 1, \cdots, n$,

$$\begin{aligned}
\varphi(Tx_1, \cdots, Tx_m) &= \sum_{t=1}^{n} \prod_{j=1}^{m} g_t(Tx_j) \\
&= \sum_{t=1}^{n} \prod_{j=1}^{m} g_t\Big(\sum_{k=1}^{n} \xi_{jk} Te_k\Big) \\
&= \sum_{t=1}^{n} \prod_{j=1}^{m} \xi_{jt} r_t \\
&= \sum_{t=1}^{n} r_t^m \prod_{j=1}^{m} \xi_{jt} \\
&= \sum_{t=1}^{n} \prod_{j=1}^{m} \xi_{jt} \\
&= \sum_{t=1}^{n} \prod_{j=1}^{m} g_t(x_j) \\
&= \varphi(x_1, \cdots, x_m) .
\end{aligned}$$

Hence $\varphi \in \mathfrak{A}(H, T)$. Moreover, if $x = \sum_{t=1}^{n} c_t e_t$ then

$$\begin{aligned}
\varphi(x, \cdots, x) &= \sum_{t=1}^{n} g_t(x)^m \\
&= \sum_{t=1}^{n} c_t^m .
\end{aligned}$$

But m is even and hence $\varphi \in P(H, T)$. To complete the proof of Theorem 1 we note that if $\varphi \in P(H, T)$ and if $e_1, \cdots, e_n$ is any basis of V then $\varphi(x, x, \cdots, x)$ is a homogeneous polynomial of degree m in $c_1, \cdots, c_n$. Hence, on the compact hypersphere S defined by $\sum_{t=1}^{n} c_t^2 = 1$ in V, φ must assume a positive minimum value m_φ. By a similar argument for any $\psi \in \mathfrak{A}(H, T)$, $|\psi|$ must assume a maximum M_ψ for $\sum_{t=1}^{n} c_t^2 = 1$. Now let ψ be an arbitrary element of $\mathfrak{A}(H, T)$ and choose a positive constant a such that $a > M_\psi / m_\varphi$. If $0 \neq x \in V$ and $\|x\|^2 = \sum_{t=1}^{n} c_t^2$ then $(x/\|x\|) \in S$ and

$$\begin{aligned}
a\varphi(x, \cdots, x) - \psi(x, \cdots, x) &= a\,\|x\|^m \varphi\Big(\frac{x}{\|x\|}, \cdots, \frac{x}{\|x\|}\Big) \\
&\quad - \|x\|^m \psi\Big(\frac{x}{\|x\|}, \cdots, \frac{x}{\|x\|}\Big) \\
&\geqq \|x\|^m (a m_\varphi - M_\psi) \\
&> 0 .
\end{aligned}$$

In other words,

$$a\varphi - \psi \in P(H, T)$$

so that ψ is a linear combination of elements in $P(H, T)$.

To proceed to the proof of Theorem 2 we use Theorem 1 to conclude immediately that since T has real eigenvalues the elementary divisors are all linear and thus there exists a basis of eigenvectors of T:

$$Te_k = \gamma_k e_k\ , \qquad k = 1, \cdots, n\ .$$

It is not difficult to show [2] that the decomposable tensors

$$e_\omega^* = e_{\omega_1} * \cdots * e_{\omega_m}\ , \qquad \omega \in \Delta\ ,$$

constitute a basis for $V^m(H)$.

We compute that

$$\begin{aligned} K(T)e_\omega^* &= Te_{\omega_1} * \cdots * Te_{\omega_m} \\ &= \gamma_{\omega_1} e_{\omega_1} * \cdots * \gamma_{\omega_m} e_{\omega_m} \\ &= \prod_{t=1}^n \gamma_t^{m_t(\omega)} e_\omega^* \end{aligned} \tag{15}$$

where $m_t(\omega)$ denotes the multiplicity of occurrence of t in ω, $t = 1, \cdots, n$. The formula (15) shows that $(K(T) - I)e_\omega^*$ is 0 or a nonzero multiple of e_ω^* according as

$$\prod_{t=1}^n \gamma_t^{m_t(\omega)}$$

is 1 or -1. Now we can assume without loss of generality that the eigenvalues $\gamma_1, \cdots, \gamma_n$ are so organized that $\gamma_1 = \cdots = \gamma_p = 1$, $\gamma_{p+1} = \cdots = \gamma_n = -1$. (This is of course merely a notational convenience.) Then

$$\begin{aligned} \prod_{t=1}^n \gamma_t^{m_t(\omega)} &= \prod_{t=p+1}^n (-1)^{m_t(\omega)} \\ &= (-1)^{m - \sum_{t=1}^p m_t(\omega)} \\ &= (-1)^{\sum_{t=1}^p m_t(\omega)}\ . \end{aligned}$$

Thus $\prod_{t=1}^n \gamma_t^{m_t(\omega)} = 1$ if and only if $\sum_{t=1}^p m_t(\omega)$ is even. This last statement just means that $1, \cdots, p$ (i.e., $i_1, \cdots, i_p$) occur altogether an even number of times in ω.

The proof of the corollary is completed by first noting that if $H = S_m$ then the set Δ is the totality of nondecreasing sequences of length m chosen from $1, \cdots, n$. Thus by Theorem 2 if $P(H, T)$ is nonempty and T has real eigenvalues $\gamma_1, \cdots, \gamma_n$ then these eigenvalues are ± 1 and we lose no generality in assuming that $\gamma_1 = \cdots = \gamma_p = 1$, $\gamma_{p+1} = \cdots = \gamma_n = -1$. We want to count the total number of ω in Δ for which

$$\sum_{t=1}^{p} m_t(\omega) \equiv 0 \pmod 2 . \tag{16}$$

Now, a sequence satisfying (16) may be constructed as follows. Suppose that k is a fixed integer, $0 \leqq 2k \leqq m$, and we count the number of sequences in Δ in which $\sum_{t=1}^{p} m_t(\omega) = 2k$. The total number of nondecreasing sequences of length $2k$ using the integers $1, \cdots, p$ is

$$\binom{p + 2k - 1}{2k} = \binom{p - 1 + 2k}{p - 1}$$

and any one of these can be completed to a nondecreasing sequence of length m by adjoining a nondecreasing sequence of length $m - 2k$ using the integers $p + 1, \cdots, n$. There are a total of

$$\binom{n - p + m - 2k - 1}{m - 2k} = \binom{n - p - 1 + m - 2k}{n - p - 1}$$

ways of doing this. This completes the proof of the corollary.

To proceed to the proof of Theorem 3 we remark that Theorem 1 cannot be directly applied because we are not assuming that the eigenvalues of T are real; in general this is not the case. However the statement (b) does follow from Theorem 1. If E is any basis of V, A is the matrix representation of T, and $C = [\varphi]_E^E$, then to say that $\varphi \in \mathfrak{A}(H, T)$ is equivalent to the assertion that

$$A^T C A = C . \tag{17}$$

If $\varphi \in P(H, T)$ then C is a positive definite symmetric matrix and can therefore be written $C = K^2$, where K is also positive definite symmetric. Then (17) is immediately equivalent to the statement that KAK^{-1} is a real orthogonal matrix and (a) is evident. Conversely if (a) and (b) obtain then there exists a real nonsingular matrix S such that $S^{-1}AS$ is a direct sum of a diagonal matrix with ± 1 along the main diagonal together with certain 2-square matrices of the form

$$\begin{bmatrix} a_k & b_k \\ -b_k & a_k \end{bmatrix} . \tag{18}$$

Since $|\gamma_k| = 1$, $k = 1, \cdots, n$, the matrix (18) is orthogonal and hence $S^{-1}AS = U$ where U is an n-square real orthogonal matrix. If we set

$(S^T)^{-1}S^{-1} = C$ then C is a positive definite symmetric matrix and we compute that

$$\begin{aligned} A^TCA &= (S^{-1})^T U^T S^T (S^T)^{-1} S^{-1} S U S^{-1} \\ &= (S^{-1})^T S^{-1} \\ &= C\ . \end{aligned}$$

Thus if $[\varphi]_E^E = C$ then $\varphi \in P(H, T)$. The dimension of $\mathfrak{A}(H, T)$ can equally well be computed as in the general case by finding $\eta(K(T) - I)$ where $K(T)$ is the induced mapping on the complex space of 2-symmetric tensors over $\bar{V}$, i.e., $\bar{V}^2(S_2)$. The mapping $K(T)$ is just the 2nd Kronecker power of T restricted to the second symmetric space. This mapping is customarily denoted by $P_2(T)$[5]. Since T has a basis of eigenvectors $v_1, \cdots, v_n$, so does $P_2(T)$ and, for $1 \leqq i \leqq j \leqq n$,

$$P_2(T)v_i * v_j = \gamma_i\gamma_j v_i * v_j\ .$$

Thus $\dim \mathfrak{A}(H, T)$ is precisely the number of pairs of integers (i, j), $1 \leqq i \leqq j \leqq n$, for which

$$\gamma_i\gamma_j = 1\ . \tag{19}$$

But T has the distinct eigenvalues $a_k + ib_k$ of multiplicity e_k, $k = 1, \cdots, p$. This yields a total of

$$\sum_{t=1}^{p} e_t^2$$

pairs (i, j) for which (19) is satisfied. Also, T has 1 as an eigenvalue q times and -1 as an eigenvalue l times and this yields an additional

$$\frac{q(q+1)}{2} + \frac{l(l+1)}{2}$$

pairs (i, j) for which (19) is satisfied. This proves that

$$\dim \mathfrak{A}(H, T) = \frac{q(q+1)}{2} + \frac{l(l+1)}{2} + \sum_{j=1}^{p} e_j^2\ .$$

We now turn to the derivation of (6). First, we assert that since T has linear elementary divisors over the complex numbers [4] there exists a basis E of V such that the matrix representation of T has the following form:

$$A = \sum_{k=1}^{p}{}^{\boldsymbol{\cdot}}\, I_{e_k} \otimes \begin{bmatrix} a_k & b_k \\ -b_k & a_k \end{bmatrix} \dot{+}\, I_q \,\dot{+}\, -I_l \tag{20}$$

where I_s is the s-square identity matrix. We set $C = [\varphi]_E^E$ and partition C conformally with (20):

$$C = \left[\begin{array}{ccc|cc} C_{11} & \cdots & C_{1d} & & \\ \vdots & & \vdots & \multicolumn{2}{c}{Z} \\ C_{d1} & \cdots & C_{dd} & & \\ \hline & & & C_q & C_r \\ & Z^T & & & \\ & & & C_r^T & C_l \end{array}\right],$$

C_{ij} is 2-square, $i, j = 1, \cdots, d = \sum_{j=1}^{p} e_j$, C_q is q-square symmetric and C_l is l-square symmetric. Set $L = \dot{\sum}_{k=1}^{p} I_{e_k} \otimes (a_k I_2 + b_k F)$ and observe that for (17) to be satisfied Z must satisfy

$$L^T Z(I_q \dotplus -I_l) = Z\,. \tag{21}$$

Now, $L^T \otimes (I_q \dotplus -I_l)$ has eigenvalues $\pm(a_k \pm ib_k)$ [3, p. 9] and none of these is equal to 1. Hence (21) has only the zero matrix as a solution. Similarly we see that $C_r = 0$. Next, consider a typical C_{ij}, $j > i$, call it K. Then K must satisfy an equation of the form

$$(a_s I_2 - b_s F) K (a_r I_2 + b_r F) = K\,. \tag{22}$$

The matrix

$$(a_s I_2 - b_s F) \otimes (a_r I_2 + b_r F)$$

has eigenvalues

$$(a_s \pm ib_s)(a_r \pm ib_r). \tag{23}$$

If $r \neq s$, (23) cannot be 1 and in this case $K = 0$. If $r = s$ then precisely two of the four complex numbers (23) are 1. Thus the nullity of the matrix

$$(a_s I_2 - b_s F) \otimes (a_s I_2 + b_s F) - I_4 \tag{24}$$

is 2. But $K = I_2$ and $K = F$ are two linearly independent solutions to (22) for $r = s$. Also note that since C is symmetric C_{ii} must be a multiple of I_2. It follows that the submatrix

$$\begin{bmatrix} C_{11} & \cdots & C_{1d} \\ \vdots & & \vdots \\ C_{d1} & \cdots & C_{dd} \end{bmatrix}$$

is itself a direct sum of $2e_k$-square matrices of the form

$$\left[\begin{array}{cc|cc|c|cc|c|cc|c|cc}
x_{11} & 0 & x_{12} & y_{12} & & & & & & & & & \\
0 & x_{11} & -y_{12} & x_{12} & & & & & & & & & \\
\hline
x_{12} & -y_{12} & x_{22} & 0 & & & & & & & & & \\
y_{12} & x_{12} & 0 & x_{22} & & & & & & & & & \\
\hline
 & & & & \ddots & & & & & & & & \\
\hline
 & & & & & x_{rr} & 0 & \cdots & x_{rs} & y_{rs} & & & \\
 & & & & & 0 & x_{rr} & \cdots & -y_{rs} & x_{rs} & & & \\
\hline
 & & & & & & & \ddots & \vdots & & & & \\
\hline
 & & & & & & & & x_{ss} & 0 & & & \\
 & & & & & & & & 0 & x_{ss} & & & \\
\hline
 & & & & & & & & & & \ddots & & \\
\hline
 & & & & & & & & & & & x_{e_ke_k} & 0 \\
 & & & & & & & & & & & 0 & x_{e_ke_k}
\end{array}\right]$$

This matrix is of the form $X_k \otimes I_2 + Y_k \otimes F$ where $X_k = (x_{ij})$ is e_k-square symmetric and $Y_k = (y_{ij})$ is e_k-square skew-symmetric. This completes the proof of Theorem 3.

3. Some examples. Let $m = 2p$ and let S'_p be the symmetric group of degree p on $p+1, \cdots, m$. In general if V is a Euclidean space with inner product (x, y) then $V^m(H)$ is also a Euclidean space [2] in which the inner product of two symmetric products $x_1 * \cdots * x_m$ and $y_1 * \cdots * y_m$ is given by

$$(x_1 * \cdots * x_m, y_1 * \cdots * y_m) = \frac{1}{m!} \sum_{\sigma \in H} \prod_{i=1}^{m} (x_i, y_{\sigma(i)}) \,. \tag{25}$$

Set $H = S_p \times S'_p$ (direct product) and define $\varphi \in M_m(V, H, R)$ by

$$\varphi(x_1, \cdots, x_p, x_{p+1}, \cdots, x_m) = (x_1 * \cdots * x_p, x_{p+1} * \cdots * x_m) \,. \tag{26}$$

Clearly φ is symmetric with respect to H and

$$\begin{aligned}\varphi(x, \cdots, x, x, \cdots, x) &= \| x * \cdots * x \|^2 \\ &\geqq 0 \,.\end{aligned}$$

Moreover $x * \cdots * x = 0$ if and only if $x = 0$ [2]. Hence φ is positive definite. Now suppose that $\varphi \in P(H, T)$ where $T: V \to V$. Then

$$\varphi(Tx_1, \cdots, Tx_p, Tx_{p+1}, \cdots, Tx_m) = \varphi(x_1, \cdots, x_m)$$

and from (26) we have

(27) $(Tx_1 * \cdots * Tx_p, Tx_{p+1} * \cdots * Tx_m) = (x_1 * \cdots * x_p, x_{p+1} * \cdots * x_m)$.

It follows from (27) that

(28) $$K(T^* T) = I$$

where T^* is the adjoint of T and $K(T)$ is the induced transformation in the symmetry class $V^p(S_p)$. It is not difficult to show [7] that (28) implies that $T^* T = \omega I_v$ where $|\omega| = 1$. However, since $T^* T$ is positive definite, $T^* T = I_V$, and hence T is orthogonal. It follows that T must have linear elementary divisors over the complex numbers.

In Theorem 1 we proved only that if $P(H, T)$ is nonempty then T has linear elementary divisors corresponding to real eigenvalues. We conjecture that in fact the preceding example is typical in the sense that T always has linear elementary divisors over the complex numbers if $P(H, T)$ is assumed to be nonempty.

We now give an example to show that if $\varphi \in \mathfrak{A}(H, T)$, but φ is not positive definite, then the elementary divisors of T over the complex numbers need not be linear. Let $H = S_2$ and let $\dim V = 4$. Choose T to have

$$(\lambda^2 + 1)^2$$

as its only elementary divisor. Then there exists a real basis $E = \{e_1, \cdots, e_4\}$ of V so that

$$[T]_E^E = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{bmatrix} .$$

Let $A = [T]_E^E$. Then from (17) it suffices to determine a symmetric matrix C such that

(29) $$A^T C A = C .$$

Define C as follows:

$$C = \begin{bmatrix} 0 & 1 & 0 & -3 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -3 & 0 & 1 & 0 \end{bmatrix} .$$

Then C is symmetric (but not positive definite) and (29) is easily verified. This example also shows that $P(H, T)$ is empty. It is routine to verify that $\dim \mathfrak{A}(H, T) = 1$ in this case but the formula (5) produces the integer 4.

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Received September 7, 1967, and in revised form May 21, 1969. The work of both authors was supported under grant AFOSR 698-67 of the Air Force Office of Scientific Research.

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