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ON THE INVERSION FORMULA FOR THE CHARACTERISTIC FUNCTION

T. KAWATA

In the inversion formula

$$F(x) - F(0) = \lim_{T\to\infty} \frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-ixt}-1}{-it} f(t)dt$$

for the characteristic function $f(t)$ of a distribution function $F(x)$, the limit of the symmetric integral is used. The purpose of this paper is to give a necessary and sufficient condition for the existence of the asymmetric improper integral $\lim_{T,T'\to\infty}\int_{-T'}^{T}$ on the right of the above formula.

Let $F(x)$ be a probability distribution function and $f(t)$ the corresponding characteristic function,

$$f(t) = \int_{-\infty}^{\infty} e^{itx} dF(x) . \tag{1.1}$$

We assume in this note that $F(x)$ is standardized so that

$$F(x) = \tfrac{1}{2}[F(x+0) + F(x-0)] . \tag{1.2}$$

The well known inversion formula states that

$$F(x) - F(0) = \lim_{T\to\infty} \frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-ixt}-1}{-it} f(t)dt \tag{1.3}$$

for every $-\infty < x < \infty$.

It is also known that the symmetric integral of the right hand side cannot be replaces by the improper integral $\lim_{T,T'\to\infty}\int_{-T'}^{T'}$ (T, T' going to infinity independently).

Actually we may easily see that

$$\mathrm{Re}\Big(\frac{1}{2\pi}\int_{0}^{T} \frac{e^{-ixt}-1}{-it} f(t)dt\Big) = \mathrm{Re}\Big(\frac{1}{2\pi}\int_{-T}^{0} \frac{e^{-ixt}-1}{-it} f(t)dt\Big) , \tag{1.4}$$

and

$$\mathrm{Im}\Big(\frac{1}{2\pi}\int_{0}^{T} \frac{e^{-ixt}-1}{-it} f(t)dt\Big) = -\mathrm{Im}\Big(\frac{1}{2\pi}\int_{-T}^{0} \frac{e^{-ixt}-1}{-it} f(t)dt\Big) , \tag{1.5}$$

and hence $\frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-ixt}-1}{-it} f(t)dt$ cancels out its imaginary part.

The real part (1.4) always converges to $\frac{1}{2}[F(x) - F(0)]$. This gives the proof of (1.3). (See [1], pp. 263–264).

However the imaginary part (1.5) does not necessarily converge without some condition on $F(x)$. This is why the limit of the symmetric integral in (1.3) cannot be replaced by the general improper integral.

2. **The condition for the existence of the improper integral.** We shall give the necessary and sufficient condition for the existence of the limit of (1.5).

THEOREM 1. *In order that the limit of* (1.5) *when* $T \to \infty$ *exists, it is necessary and sufficient that the integral*

$$\lim_{\varepsilon \to 0+} \int_\varepsilon^1 \frac{G(u, x) - G(u, 0)}{u} du \tag{2.1}$$

exists where,

$$G(u, x) = F(u + x) - F(-u + x) \tag{2.2}$$

and if (2.1) *exists*

$$\lim_{T\to\infty} \operatorname{Im}\left(\frac{1}{2\pi}\int_0^T \frac{e^{-ixt} - 1}{-it} f(t)dt\right) = \int_0^\infty \frac{G(u, x) - G(u, 0)}{u} du\,. \tag{2.3}$$

It must be noted that the integral of the right hand side of (2.3) exists in the neighborhood of the infinity. In fact $G(u, x) - G(u, 0) = [F(u + x) - F(u)] - [F(-u + x) - F(-u)]$ and $F(u + x) - F(u) \in L$, $(-\infty, \infty)$ for every fixed x.

We shall now prove the theorem.

Let

$$I(x, T) = \operatorname{Im}\left(\frac{1}{2\pi}\int_0^T \frac{e^{-ixt} - 1}{-it} f(t)dt\right).$$

We then easily see that

$$\begin{aligned} I(x, T) &= \frac{1}{2\pi}\int_{-\infty}^\infty dF(u)\int_0^T \frac{\sin xt \sin ut - (1 - \cos xt)\cos ut}{t} dt \\ &= \frac{1}{2\pi}\int_{-\infty}^\infty dF(u)\int_0^T \frac{\cos (u - x)t - \cos ut}{t} dt \\ &= \frac{1}{2\pi}\int_{-\infty}^\infty dF(u)\int_0^T dt\int_{u-x}^u \sin vt dv \\ &= \frac{1}{2\pi}\int_{-\infty}^\infty dF(u)\int_{u-x}^u \frac{1 - \cos vT}{v} dv \end{aligned}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1-\cos vT}{v} dv \int_{v}^{v+x} dF(u)$$
$$= \frac{1}{2\pi}\int_{0}^{\infty} [G(v, x) - G(v, 0)] \frac{1-\cos vT}{v} dv \,.$$

As was mentioned before, $G(v, x) - G(v, 0) \in L_1(-\infty, \infty)$. Hence the Riemann-Lebesgue lemma shows that

$$\lim_{T\to\infty}\int_{\varepsilon}^{\infty}[G(v, x) - G(v, 0)]\frac{\cos vT}{v}dv = 0$$

for any $\varepsilon > 0$. Therefore we may write

$$\begin{aligned} I(x, T) = & \int_0^{\varepsilon} \frac{G(v, x) - G(v, 0)}{v}(1 - \cos vT)dv \\ & + \int_{\varepsilon}^{\infty} \frac{G(v, x) - G(v, 0)}{v} dv + o(1) \end{aligned} \tag{2.4}$$

as $T \to \infty$, for a fixed $\varepsilon > 0$.

Now we shall show the sufficiency of the condition of the theorem. Let $\varepsilon > 0$ be arbitrary but fixed. Write

$$\begin{aligned} & \int_0^{\varepsilon} \frac{G(v, x) - G(v, 0)}{v}(1 - \cos vT)dv \\ & = \int_0^{1/T} + \int_{1/T}^{\varepsilon} = K_1 + K_2 \,, \end{aligned} \tag{2.5}$$

say. We have

$$\begin{aligned} |K_1| & \leqq \int_0^{1/T} |G(v, x) - G(v, 0)| \frac{1 - \cos vt}{v} dv \\ & \leqq CT \int_0^{1/T} |G(v, x) - G(v, 0)| \, dv \,, \end{aligned} \tag{2.6}$$

for some constant C.

$\lim_{v\to 0+} [G(v, x) - G(v, 0)]$ exists since F is nondecreasing and it must be zero, otherwise (2.1) does not exist. Hence the last expression converges to zero.

$$K_1 = o(1), \quad \text{as} \quad T \to \infty \,. \tag{2.7}$$

Next write

$$\chi(v) = G(v, x) - G(v, 0) \,. \tag{2.8}$$

Choose ε such that $|\chi(v)| < \delta$ for $|v| \leqq \varepsilon$ for an arbitrary chosen δ. Since $\chi(v)/v$ is of bounded variation in $[1/T, \varepsilon]$, we have, using the second mean value theorem,

$$K_2 = \int_{1/T}^{\varepsilon} \frac{\chi(v)}{v} dv - \int_{1/T}^{\varepsilon} \frac{\chi(v)}{v} \cos vT dv$$
$$= \int_{1/T}^{\varepsilon} \frac{\chi(v)}{v} dv - T\chi\left(\frac{1}{T}\right)\int_{1/T}^{\xi} \cos vT dv - \frac{\chi(\varepsilon)}{\varepsilon}\int_{\xi}^{\varepsilon} \cos vT dv$$

for some $1/T < \xi < \varepsilon$. Thus

$$\left| K_2 - \int_{1/T}^{\varepsilon} \frac{\chi(v)}{v} dv \right| \leqq 2\chi\left(\frac{1}{T}\right) + 2\chi(\varepsilon) \leqq 4\delta .$$

Therefore from (2.4) and (2.5)

$$\left| I(x, T) - \frac{1}{2\pi}\int_{1/T}^{\infty} \frac{\chi(v)}{v} dv \right| \leqq \frac{2\delta}{\pi} + o(1) . \tag{2.9}$$

This shows the sufficiency of the condition of the theorem and gives (2.3).

We shall next show the necessity. Define $\chi(v)$ as before. We see that $\chi(v)$ has the limit c as $v \to +0$. If $c \neq 0$, then from (2.4)

$$I(x, T) - c\int_0^{\varepsilon} \frac{1 - \cos vT}{v} dv$$
$$= \int_0^{\varepsilon} \frac{[\chi(v) - c](1 - \cos vT)}{v} dv + \int_{\varepsilon}^{\infty} \frac{\chi(v)}{v} dv + o(1) .$$

The first integral of the right hand side is handled in the same way as in deriving (2.6) and (2.9) with $\chi(v) - c$ in place of

$$\chi(v) = G(v, x) - G(v, 0) .$$

Actually instead of (2.6) we see that K_1 with $\phi(v) - c$ is bounded by $CT\int_0^{1/T} |\chi(v) - c| dv$ which is $o(1)$. In place of (2.9) we have

$$\left| I(x, T) - \frac{c}{2\pi}\int_0^{\varepsilon} \frac{1 - \cos vT}{v} dv - \frac{1}{2\pi}\int_{1/T}^{\varepsilon} \frac{\chi(v) - c}{v} dv \right.$$
$$\left. - \frac{1}{2\pi}\int_{\varepsilon}^{\infty} \frac{\chi(v)}{v} dv \right| \leqq C_1\delta + o(1) , \tag{2.10}$$

where C_1 is some constant.

$$\int_0^{\varepsilon} \frac{1 - \cos vT}{v} dv = 2\int_0^{\varepsilon T} \frac{\sin^2 v/2}{v} dv \geqq C_2 \log \varepsilon T , \tag{2.11}$$

where C_2 is an absolute constant. Choose ε for an arbitrary given $\eta < C_2$ so that $|\chi(v) - c| < \eta$ for $0 < v < \varepsilon$. Then

$$\left| \int_{1/T}^{\varepsilon} \frac{\chi(v) - c}{v} dv \right| \leqq \eta \log \varepsilon T . \tag{2.12}$$

Hence if $I(x, T)$ has a limit as $T \to \infty$, then in view of (2.11) and (2.12), (2.10) implies a contradiction. Hence we have that $c = 0$. Using (2.10), this yields

$$\left| I(x, T) - \frac{1}{2\pi}\int_{1/T}^{\infty} \frac{\chi(v)}{v} dv \right| \leqq C_1\delta + o(1) \, .$$

This proves the necessity of the condition.

3. Remarks. From Theorem 1, we immediately obtain

THEOREM 2. *In order that*

$$\lim_{T, T' \to \infty} \frac{1}{2\pi}\int_{-T'}^{T} \frac{e^{-ixt} - 1}{-it} f(t) dt$$

exists, it is necessary and sufficient that (2.1) *exists for* $\varepsilon > 0$. (*The limit is* $F(x) - F(0)$).

Similar arguments apply to the integral

$$(3.1) \quad J_1(x, T) = \int_1^T \frac{f(t)e^{-ixt}}{it} dt \quad \text{and} \quad J_2(x, T) = \int_{-T}^{-1} \frac{f(t)e^{-ixt}}{it} dt \, .$$

We easily see that J_1 and J_2 are conjugate complex. We may show that *in order for* $J_1(x, T)$ *or* $J_2(x, T)$ *to converge as* $T \to \infty$, *it is necessary and sufficient that*

$$(3.2) \qquad \int_0^{\varepsilon} \frac{F(u + x) - F(-u + x)}{u} du < \infty$$

for some $\varepsilon > 0$.

(3.1) implies that

$$(3.3) \qquad \lim_{T \to \infty} \frac{1}{2\pi}\int_{-T}^{T} f(t)e^{-ixt} dt = 0$$

which is very well known when $F(x)$ is continuous at x. (3.1) says more than this about the improper integrability of $f(t)$ near infinity with the additional condition (3.2) on $F(x)$.

The sufficiency of (3.2) for the existence of the limits of (3.1) was proved in [2] before.

REFERENCES

1. T. Kawata, *The characteristic function of a probability distribution*, Tohoku Math. J. **48** (1941).

2. A. Rényi, *Wahrscheinlichkeitsrechnung*, Berlin, 1962.

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THE CATHOLIC UNIVERSITY OF AMERICA