PACIFIC JOURNAL OF MATHEMATICS
Vol. 31, No. 2, 1969

ON THE BRAUER SPLITTING THEOREM

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This paper presents a proof for the Brauer splitting theorem in the context of a commutative ring with no idempotents except 0 and 1 and continues this investigation. The main results in this paper are the Brauer splitting theorem and the classification of all finitely generated projective indecomposable modules over a separable group algebra.

Throughout this paper we assume that the ring R is a commutative ring with no idempotents except 0 and 1, that the group G has order n invertible in R, and that all RG-modules are unitary left RG-modules. We know that the order of G, n, is invertible in R if and only if RG is separable.

1. First, let us recall the following Brauer splitting theorem: Let K be a field and G be a group of order n invertible in K, then $K(\sqrt[m]{1})$ is a splitting field for G, where m is the exponent of G and $\sqrt[m]{1}$ is a primitive m^{th}-root of 1 ([6], Th. 41-1, p. 292 and Corollary 70-24, p. 475). In [8], G. J. Janusz defined a ring R to be a splitting ring for G if the group algebra RG is the direct sum of central separable R-algebras each equivalent to R in the Brauer group of R; that is, $RG \cong \bigoplus \sum_{i=1}^{s} \mathrm{Hom}_R(P_i, P_i)$, where $\{P_i\}$ are finitely generated projective faithful R-modules, the number of different conjugate classes in G is equal to s. He then proved the Brauer splitting theorem for a Noetherian regular domain, R. This section gives a proof for the above theorem when R is any commutative ring with no idempotents except 0 and 1.

LEMMA 1. *Let R_0 be a subring of R. If R_0 is a splitting ring for G, then R is a splitting ring for G.*

Proof. Because R_0 is a splitting ring for G, $R_0G \cong \bigoplus \sum_{i=1}^{s} \mathrm{Hom}_{R_0}(P_i, P_i)$ where $\{P_i\}$ are finitely generated projective faithful R_0-modules. Then we have

$$RG \cong R \otimes_{R_0} R_0G \cong R \otimes_{R_0} \left(\bigoplus \sum_{i=1}^{s} \mathrm{Hom}_{R_0}(P_i, P_i)\right)$$
$$\cong \bigoplus \sum_{i=1}^{s} R \otimes_{R_0} \mathrm{Hom}_{R_0}(P_i, P_i) \cong \bigoplus \sum_{i=1}^{s} \mathrm{Hom}_R(R \otimes_{R_0} P_i, R \otimes_{R_0} P_i)\,,$$

where $\{R \otimes_{R_0} P_i\}$ are finitely generated projective faithful R-modules. This follows since $\{P_i\}$ are finitely generated projective faithful R_0-

modules ([1], Proposition 5-5). Thus R is a splitting ring for G.

THEOREM 2. *If R is a commutative ring with no idempotents except 0 and 1 and RG is a separable group algebra, then $R[\sqrt[m]{1}]$ is a splitting ring for G where $\sqrt[m]{1}$ is a primitive m^{th}-root of 1.*

Proof. Let Z be the set of integers, Q be the set of rationals. The proof divides into two cases.

Case 1. The prime ring of R is finite. Let $\operatorname{Char}(R) = p^e$, where p is a prime integer and e is in Z.

$Z/(p^e)$ is a local ring with the maximal ideal $(p)/(p^e)$ which is also nilpotent. For $(Z/(p^e))[\theta]$ where $\theta = \sqrt[m]{1}$, we have

$$\frac{(Z/(p^e))[\theta]}{((p)/(p^e))[\theta]} \cong (Z/(p))(\bar{\theta})$$

where $\bar{\theta}$ is a primitive m^{th}-root of 1 over $Z/(p)$. Now $(Z/(p))(\bar{\theta})$ is a field; so $((p)/(p^e))[\theta]$ is a maximal ideal. On the other hand, since $(p)/(p^e)$ is nilpotent, $((p)/(p^e))[\theta]$ is also nilpotent. But then $((p)/(p^e))[\theta]$ is an unique maximal ideal and a nilpotent ideal of $(Z/(p^e))[\theta]$. Therefore, $(Z/(p^e))[\theta]$ is a complete local ring where the completion is in the sense of m-topology (see [9], p. 254). Then the Brauer group natural map

$$B((Z/(p^e))[\theta]) \longrightarrow B\left(\frac{(Z/(p^e))[\theta]}{((p)/(p^e))[\theta]}\right) \cong B((Z/(p))(\bar{\theta}))$$

is monomorphic ([1], Corollary 6-2). But $(Z/(p))(\bar{\theta})$ is a splitting field for G; so $(Z/(p^e))[\theta]$ is a splitting ring for G. Thus $R[\theta]$ is a splitting ring for G by the lemma.

Case 2. The prime ring of R is $Z(n)$ which is the quotient ring of Z with respect to the multiplicative closed set $\{n, n^2, \cdots\}$. Since $Z(n)[\theta]$ is a Dedekind domain, it is Noetherian and regular. Then the Brauer group natural map $B(Z(n)[\theta]) \to B(Q(\theta))$ is monomorphic ([1], Th. 7-2). But $Q(\theta)$ is the quotient field of $Z(n)[\theta]$ and a splitting field for G by the Brauer splitting theorem. Therefore, $Z(n)[\theta]$ is a splitting ring for G and so $R[\sqrt[m]{1}]$ is a splitting ring for G by the lemma. By combining Cases 1 and 2, the theorem is proved.

REMARK. The above theorem tells us the existence of a splitting ring, $R[\sqrt[m]{1}]$, for G, if RG is a separable group algebra. We also know that $R[\sqrt[m]{1}]$ is a finitely generated projective and separable R-algebra ([8], Corollary 2-4). But there exists a central separable R-

algebra without a finitely generated projective and separable splitting ring. The following example is due to 0. Goldman: Let R be $Z[\sqrt{2}]$, i, j, k be the usual quaternion basis. If $\alpha = (1 + i)/\sqrt{2}$ and $\beta = (1 + j)/\sqrt{2}$, then $R1 \oplus R\alpha \oplus R\beta \oplus R\alpha\beta$ is central separable over R. But R has no finitely generated projective and separable extension except direct sums of copies of R, and $R1 \oplus R\alpha \oplus R\beta \oplus R\alpha\beta$ cannot be split.

2. In this section, assume RG is a split group algebra,

$$RG \cong \oplus \sum_i \mathrm{Hom}_R (P_i, P_i) , \qquad i = 1, 2, \cdots, s .$$

When $\{P_i\}$ are considered as RG-modules ([3], p. 5), the classification of all finitely generated projective indecomposable RG-modules can be obtained. Observe that the order of the group G, n, is invertible in R if and only if RG is separable. Therefore, any RG-module M is finitely generated and projective over RG if and only if M is finitely generated and projective over R (see the proof of Proposition 1-5 in [8]).

Let RG be a separable R-algebra and M be a finitely generated projective RG-module; for any x in M there exist $X_1, X_2, \cdots X_q$ in M and $F_1, F_2, \cdots, F_q$ in $\mathrm{Hom}_R (M, R)$ so that $x = \sum_{i=1}^q F_i(x)X_i$. We call $\{F_i, X_i, i = 1, 2, \cdots, q\}$ a R-dual basis of M, and $T_M(x) = \sum_{i=1}^q F_i(xX_i)$ the character of M at x in RG ([4], Proposition 3-1). By a group character we mean the restriction of T_M to G. Obviously, a character T_M is completely determined by its restriction to G. In particular, let R be a splitting ring for G; then

$$RG \cong \oplus \sum_{i=1}^s \mathrm{Hom}_R (P_i, P_i) \cong \oplus \sum_{i=1}^s (RG)E_i ,$$

where E_i is the i^{th}-central primitive idempotent of RG. We let

$$T^i = T_{P_i} .$$

PROPOSITION 3. *If M and N are two isomorphic finitely generated projective RG-modules, then they have the same characters.*

Proof. Let M and N be two isomorphic finitely generated projective RG-modules and let α be the isomorphism. If $\{F_i, X_i, i = 1, 2, \cdots, q\}$ is a dual basis of M, then we claim that $\{F_i\alpha^{-1}, \alpha X_i, i = 1, 2, \cdots, q\}$ is a dual basis of N. In fact, for any a in N, there exists b in M such that $\alpha(b) = a$; so

$$\begin{aligned} a = \alpha\Big(\sum_{i=1}^q F_i(b)\, X_i\Big) &= \sum_i F_i(b)(\alpha X_i) \\ &= \sum_i F_i\alpha^{-1}\alpha(b)(\alpha X_i) = \sum_i ((F_i\alpha^{-1})\alpha(b))(\alpha X_i) \\ &= \sum_i (F_i\alpha^{-1}(a))(\alpha X_i) . \end{aligned}$$

This means that $\{F_i\alpha^{-1}, \alpha X_i, i = 1, 2, \cdots, q\}$ is a dual basis of N. But the character of any finitely generated projective RG-module is independent of the dual basis chosen; so $T_N(g) = \sum_i F_i\alpha^{-1}(g\alpha X_i) = \sum_i F_i\alpha^{-1}(\alpha g X_i)$, (for α is a RG-isomorphism), and so $= \sum_i F_i(gX_i) = T_M(g)$.

The following proposition will play an important role in our discussion.

PROPOSITION 4. *If N is a finitely generated projective faithful R-module and M a finitely generated projective left* $\mathrm{Hom}_R(N, N)$*-module, then $M \cong N \otimes_R N'$ with N' a finitely generated projective R-module.*

Proof. By the Morita Theorem on *p.* 9 in [3].

REMARK. Proposition 4 gives a counter-example to the converse statement of Proposition 3. Because of Proposition 4, let M and N be two finitely generated projective indecomposable RG-modules over the same central component of the split group algebra RG; that is, $\mathrm{Hom}_R(P_i, P_i)$, then $M \cong P_i \otimes_R N'$ and $N \cong P_i \otimes_R N''$, where N' and N'' are finitely generated projective indecomposable R-modules. Suppose N' and N'' are in $P(R)$, the class group of R, then

$$T_M(g) = T_{P_i}(g)T_{N'}(1) = T_{P_i}(g)\cdot 1 = T_N(g) .$$

But $P_i \otimes_R N' \cong P_i \otimes_R N''$ only if $N' \cong N''$.

LEMMA 5. *If RG is a split group algebra; that is,*

$$RG \cong \oplus \sum_{i=1}^{s} \mathrm{Hom}_R(P_i, P_i) \cong \oplus \sum_{i=1}^{s} (RG)E_i ,$$

then

$$E_i = \sum_g \frac{k_i T^i(g^{-1})}{n} g ,$$

where g is in G, $k_i =$ rank (P_i) and $T^i = T_{P_i}$.

Proof. Since

$$RG \cong \oplus \sum_{i=1}^{s} (RG)E_i \cong \oplus \sum_{i=1}^{s} \mathrm{Hom}_R(P_i, P_i),\ E_i = \sum_g E_i(g)g ,$$

for all g in G, $E_i(g)$ in R. We then have

$$E_i h^{-1} = \sum_g E_i(g)(gh^{-1})$$

for some h in G. Taking the character afforded by RG, we have

$$T_{RG}(E_i h^{-1}) = \sum_g E_i(g) T_{RG}(gh^{-1}) .$$

But $T_{RG}(gh^{-1}) = 0$ in case $gh^{-1} \neq 1$, and $= n$ in case $gh^{-1} = 1$ or $g = h$. Hence $T_{RG}(E_i h^{-1}) = E_i(h)n, E_i(h) = T_{RG}(E_i h^{-1})/n$ (for n is invertible in R).

Next, we find $T_{RG}(E_i h^{-1})$. Because P_i is a finitely generated projective R-module, $\mathrm{Hom}_R(P_i, P_i) \cong P_i \otimes_R \mathrm{Hom}_R(P_i, R)$ ([3], Morita Theorem I). Noting that $\mathrm{rank}(P_i) = \mathrm{rank}(\mathrm{Hom}_R(P_i, R))$, we have

$$T_{(RG)E_i}(g) = T^i(g)k_i \qquad \text{for all } i = 1, 2, \cdots, s .$$

Therefore,

$$T_{RG}(E_i h^{-1}) = \sum_{j=1}^{s} T_{(RG)E_j}(E_i h^{-1}) = \sum_j k_j T^j(E_i h^{-1}) .$$

But $T^j(E_i h^{-1}) = 0$ in case $i \neq j$, so

$$T_{RG}(E_i h^{-1}) = k_i T^i(E_i h^{-1}) = k_i T^i(h^{-1}) .$$

Hence,

$$E_i(h) = \frac{T_{RG}(E_i h^{-1})}{n} = \frac{k_i T^i(h^{-1})}{n} .$$

By substituting $E_i(h)$ in E_i, we have

$$E_i = \sum_g E_i(g)g = \sum_g \frac{k_i T^i(g^{-1})}{n} g .$$

This completes the proof.

LEMMA 6. *For* $i = 1, 2, \cdots, s$, $\mathrm{rank}(P_i)$ *is neither* 0 *nor a zero divisor in* R.

Proof. First, $\mathrm{rank}(P_i)$ is not 0, otherwise E_i is 0 by Lemma 5. This is impossible.

Next, let $\mathrm{rank}(P_i)$ be k_i, and suppose that k_i is a zero divisor in R. We then have a nonzero element, k', in R such that $k'k = 0$. But by Lemma 5,

$$E_i = k_i \sum_g \frac{T^i(g^{-1})g}{n} ;$$

so,

$$k'E_i = k'k_i \sum_g \frac{T^i(g^{-1})g}{n} = (k'k_i) \sum_g \frac{T^i(g^{-1})g}{n} = 0 .$$

Noting that $(RG)E_i \cong \mathrm{Hom}_R(P_i, P_i)$, we have

$$k' \mathrm{Hom}_R(P_i, P_i) \cong k'(RG)E_i = k'E_i(RG) = 0 .$$

On the other hand, P_i is a faithful R-module; so $\mathrm{Hom}_R(P_i, P_i)$ is a faithful R-module. Therefore, $k' \mathrm{Hom}_R(P_i, P_i) = 0$ implies $k' = 0$. This is a contradiction. Thus we have proved that k_i is not a zero divisor in R.

THEOREM 7, *Suppose R is a splitting ring for G and all finitely generated projective indecomposable R-modules are of rank 1. Then for any two finitely generated projective indecomposable RG-modules M and N, we have $E_iM \neq 0$ and $E_iN \neq 0$ if and only if $T_M(g) = T_N(g)$ for all g in G, where E_i is the i^{th}-central primitive idempotent of RG.*

Proof. If $E_iM \neq 0$ and $E_iN \neq 0$, then $M \cong E_iM \oplus (1 - E_i)M$ and $N \cong E_iN \oplus (1 - E_i)N$. Since M and N are indecomposable, $(1-E_i)M = 0$ and $(1 - E_i)N = 0$. We have $N = E_iN$ and $M = E_iM$ as left $\mathrm{Hom}_R(P_i, P_i)$-modules. Therefore, by Proposition 4, $M \cong P_i \otimes_R N'$ and $N \cong P_i \otimes_R N''$ where N' and N'' are finitely generated projective R-modules. Since M and N are indecomposable RG-modules, N' and N'' are in $P(R)$. Therefore,

$$\begin{aligned} T_M(g) &= T_{P_i \otimes_R N'}(g) = T_{P_i}(g) \cdot 1 \\ &= T_{P_i}(g) T_{N''}(1) = T_N(g) . \end{aligned}$$

Conversely, if $T_M(g) = T_N(g)$ for all g in G, then $T_M(a) = T_N(a)$ for all a in RG. Suppose $E_iM \neq 0$ and $E_iN = 0$ for some i; then there exists a $j \neq i$ such that $E_jN \neq 0$. Thus M is a $(RG)E_i$-module and N is a $(RG)E_j$-module, and so we have

$$T_M(E_i) = T_{P_i}(E_i) = T_{P_i}(1) = \mathrm{rank}\,(P_i) .$$

By Lemma 6, $\mathrm{rank}(P_i) \neq 0$ in R, so $T_M(E_i) \neq 0$. Obviously, $T_N(E_i) = 0$. Thus $T_M \neq T_N$ on RG. Consequently, $T_M(g) \neq T_N(g)$ for some g in G. This is a contradiction to $T_M(g) = T_N(g)$ for all g in G, and hence the proof is completed.

COROLLARY 8. *If R is a splitting ring for G, and all finitely generated projective indecomposable R-modules are of rank 1; then there are exactly s-classes of finitely generated projective indecomposable RG-modules over different central components each uniquely determined up to an element in $P(R)$.*

Proof. Let M be a finitely generated projective indecomposable

RG-module. From the theorem, we have $M = E_iM \cong P_i \otimes_R N'$ where N' is in $P(R)$. On the other hand, P_i, $i = 1, 2, \cdots, s$, is a finitely generated projective indecomposable RG-module over the i^{th}-central component. Therefore, there are exactly s-classes of finitely generated projective indecomposable RG-modules each uniquely determined up to an element in $P(R)$.

From the above result, we have computed the first Grothendieck group of RG, $K^0(RG)$, in the sense of [2], p. 31.

COROLLARY 9. *If R is a splitting ring for G, then*

$$K^0(RG) = (Z \oplus P(R)) \oplus (Z \oplus P(R)) \oplus \cdots \oplus (Z \oplus P(R)) .$$

A natural question to ask is whether the classification of all finitely generated projective indecomposable RG-modules can be obtained for a nonsplit group algebra. The answer is not known. But for some special rings, we have a definite answer.

For a separable group algebra RG, we have the decomposition, $RG \cong \oplus \sum_{i=1}^{t} A_i$, where A_i has no proper central idempotents and t is an integer.

THEOREM 10. *If R is local or semi-local,* then there are *exactly t-isomorphic classes of finitely generated projective indecomposable RG-modules.*

Proof. From the decomposition of RG, A_i is a central separable C_i-algebra for each A_i, where C_i is the center of A_i ([1], Th. 2–3). Since R is local or semi-local, C_i is semi-local by the lemma on p. 25 in [5]. Therefore any two finitely generated projective indecomposable RG-modules over the i^{th}-component A_i are in an isomorphic class of finitely generated projective indecomposable RG-modules ([7], Th. 1).

COROLLARY 11. *If R is local or semi-local, then*

$$K^0(RG) = Z \oplus Z \oplus \cdots \oplus Z ,$$

t-copies of Z.

This paper forms a part of the author's Doctoral Dissertation at Purdue University written under the guidance of Professor F. R. DeMeyer. The author wishes to thank Professor DeMeyer for his guidance, criticism, and encouragement throughout this study.

The author wishes to thank the referee for many helpful suggestions.

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Received October 14, 1968.

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